Algebraic Topology

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Abstract

Lecture notes for the course Algebraic Topology lectured in the academic year 2018-2019.

Topology and Breakfast.

LECTURE 1

The Brouwer Fixed Point Theorem

Even the simplest problems in topology – for instance, whether two topological spaces X and Y are homeomorphic – are oftentimes very hard to answer. In order to show X and Y are homeomorphic, it suffices to find a single homeomorphism $f\colon X \to Y$. But in order to show that they are *not* homeomorphic, one needs to prove no such homeomorphism can exist. And how on earth are you meant to do that? Even if *you* can't find one, how do you know that tomorrow some really smart mathematician isn't going to magically come up with one? This is where *algebraic topology* comes in. The idea is to associate *algebraic invariants* of a topological space. Here "invariants" means that two homeomorphic spaces should have the same invariants. Thus to show two spaces are *not* homeomorphic, it suffices to show they have different invariants.

So, to summarise the entire course:

- Topology is hard.
- Algebra is easy.
- Algebraic topology converts topological problems into algebraic problems.
- Profit.

We illustrate this philosophy with an example. Let $B^n \subset \mathbb{R}^n$ denote the *closed* n-dimensional unit ball

$$B^n = \{x \in \mathbb{R}^n \mid |x| \le 1\}.$$

The boundary of B^n is the $(n-1)$-dimensional unit sphere S^{n-1}:

$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The following famous theorem is due to Brouwer.

THEOREM 1.1 (The Brouwer Fixed Point Theorem). *For all $n \ge 1$, every continuous map $f\colon B^n \to B^n$ has a fixed point.*

In the case $n = 1$, this theorem has a simple proof using connectivity:

Proof of Theorem 1.1 in the case $n = 1$. Suppose $f(-1) = a$ and $f(1) = b$. If $a = -1$ or $b = 1$ we are done, so assume that $a > -1$ and $b < 1$. Consider the graph of f:

$$\mathrm{Gr}(f) := \{(x, f(x)) \mid x \in [-1, 1]\}.$$

A fixed point of f is the same thing as a point of intersection between $\mathrm{Gr}(f)$ and the diagonal

$$\Delta := \{(x, x) \mid x \in [-1, 1]\}.$$

Since f is continuous, $\mathrm{Gr}(f)$ is connected[1]. Let

$$A := \{(x, f(x)) \mid f(x) > x\}, \qquad B := \{(x, f(x)) \mid f(x) < x\}.$$

Then $(-1, a) \in A$ and $(1, b) \in B$, so in particular A and B are both non-empty. If $\mathrm{Gr}(f) \cap \Delta = \emptyset$ then $\mathrm{Gr}(f) = A \cup B$. Since f is continuous, A and B are open[2] in $\mathrm{Gr}(f)$. This contradicts the fact that $\mathrm{Gr}(f)$ is connected. ■

Interestingly, it is not known how to extend this simple argument to deal with the case $n > 1$. Nevertheless there are several different complicated arguments. For instance, there is an analytical argument that goes as follows: first approximate f by a sequence of *differentiable* functions g_k with the property that f has a fixed point if and only if all the g_k do for large k. Then prove directly that any differentiable function must have a fixed point.

The "cutest" proof uses methods from algebraic topology. Later on in the course we will construct a **homology functor** H_n for each $n \geq 0$, which associates to any topological space X an abelian group $H_n(X)$, and to any continuous map $f \colon X \to Y$ a homomorphism

$$H_n(f) : H_n(X) \to H_n(Y).$$

The induced maps $H_n(f)$ have the property that if $f \colon X \to Y$ and $g : Y \to Z$ then

$$H_n(g \circ f) = H_n(g) \circ H_n(f) \colon H_n(X) \to H_n(Z), \tag{1.1}$$

and

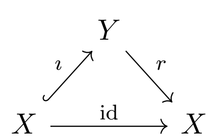
$$H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)} \colon H_n(X) \to H_n(X). \tag{1.2}$$

Moreover the homology functor H_n vanishes on the ball B^{n+1} but not on the sphere S^n:

$$H_n(B^{n+1}) = 0, \qquad H_n(S^n) \neq 0, \qquad \forall\, n \geq 1. \tag{1.3}$$

The construction of H_n and the verification of (1.1), (1.2) and (1.3) will take some time. Nevertheless, armed with these only these properties, it is easy to prove Theorem 1.1 in all dimensions.

Definition 1.2. Suppose X is a subspace of a topological space Y. We say that X is a **retract** of Y if there exists a continuous map $r : Y \to X$ such that $r(x) = x$ for all $x \in X$. Equivalently, denoting by $\imath : X \hookrightarrow Y$ the inclusion, this means that the following diagram *commutes*:

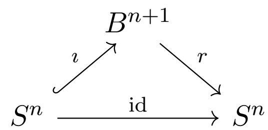


[1] It is the image of the continuous map $B^1 \to B^1 \times B^1$ given by $x \mapsto (x, f(x))$.

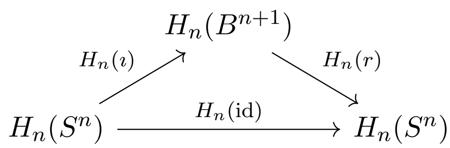
[2] Consider the map $g : \mathrm{Gr}(f) \to \mathbb{R}$ given by $g(x, f(x)) = x - f(x)$. Then $A = g^{-1}\big((-\infty, 0)\big)$ and $B = g^{-1}\big((0, \infty)\big)$.

Lemma 1.3. *For all $n \geq 1$, S^n is not a retract of B^{n+1}.*

Proof. Suppose for contradiction that there exists a retraction $r\colon B^{n+1} \to S^n$, so that the following diagram commutes:



Equation (1.1) means that we can "apply the homology functor H_n" to this commutative diagram to obtain another one:



Note this diagram is a commutative diagram of group homomorphisms between abelian groups, rather than a commutative diagram of continuous maps between topological spaces. Since $H_n(B^{n+1}) = 0$ by (1.3) the map $H_n(r)\colon H_n(B^{n+1}) \to H_n(S^n)$ is the zero map. But since $H_n(\mathrm{id}) = \mathrm{id}$ by (1.2) and $H_n(S^n) \neq 0$, this is a contradiction. ∎

Remark 1.4. In fact, Lemma 1.3 is also true for $n = 0$. The 0-dimensional sphere is just $\{-1, 1\}$, which is disconnected. Since $[-1, 1]$ is connected and the image of a connected subset under a continuous map is connected, it follows there does not exist *any* continuous surjective map $r\colon B^1 \to S^0$ (and thus in particular there does not exist a retraction.)

We now show how Theorem 1.1 follows from Lemma 1.3.

Proof of Theorem 1.1. Take $n \geq 0$. Suppose $f\colon B^{n+1} \to B^{n+1}$ has no fixed points. Then for every point $x \in B^{n+1}$, there is a unique line that starts at $f(x)$, goes through x, and then hits a point on the boundary S^n of B^{n+1}. Let us denote by $r\colon B^{n+1} \to S^n$ the map that sends x to the point on S^n that this line hits. See Figure 1.1. Since f is continuous, the map r is also continuous[3]. If $x \in S^n$ then clearly $r(x) = x$. Thus r is a retraction. This contradicts[4] Lemma 1.3. ∎

Let us now formalise the notion of a "homology functor", by introducing elements of a field of mathematics called **category theory**. In this course, we will only ever use category theory as a convenient "language" to phrase theorems from algebraic topology in—we will never actually use any genuine theorems in category theory.

[3]This is an easy exercise.

[4]If $n = 0$, apply Remark 1.4 instead of Lemma 1.3.

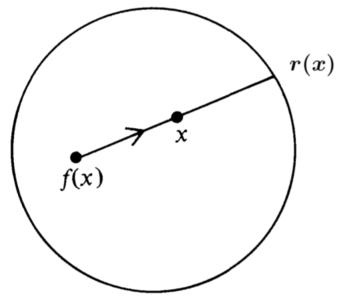


Figure 1.1: The retract r.

Remark 1.5. A word of warning: category theory is often (lovingly) referred to as abstract nonsense. But fear not: nothing we will do will ever be *that* abstract!

Definition 1.6. A **category** C consists of three ingredients. The first is a *class* $\mathrm{obj}(\mathsf{C})$ of **objects**. Secondly, for each ordered pair of objects (A, B) there is a *set* $\mathrm{Hom}(A, B)$ of **morphisms** from A to B. Sometimes instead of $f \in \mathrm{Hom}(A, B)$ we write $f : A \to B$ or $A \xrightarrow{f} B$. Finally, there is a rule, called **composition**, which associates to every ordered triple (A, B, C) of objects a map

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C),$$

written

$$(f, g) \mapsto g \circ f,$$

which satisfies the following three axioms:

1. The Hom sets are pairwise disjoint; that is, each $f \in \mathrm{Hom}(A, B)$ has a unique **domain** A and a unique **target** B.
2. Composition is associative whenever defined, i.e. given

   $$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

   one has

   $$(h \circ g) \circ f = h \circ (g \circ f).$$

3. For each $A \in \mathrm{obj}(\mathsf{C})$ there is a unique morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ called the *identity* which has the property that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$ for every $f : A \to B$.

Remark 1.7. Note that we said that $\mathrm{obj}(\mathsf{C})$ was a *class* and $\mathrm{Hom}(A, B)$ was a *set*. There is (an important, but technical) difference between a class and a set. If you've ever taken a class on logic/set theory, you'll know that not every "collection" of objects is formally a set. For instance, the collection of all sets is itself not a set! A class is a more general concept (the collection of all sets is a class). Nevertheless, as far as this course is concerned, the distinction is irrelevant, and you are free to ignore this remark!

Here are four examples of categories:

EXAMPLE 1.8. The category Sets of **sets**. The objects of Sets are all the sets, and $\mathrm{Hom}(A, B)$ is just the set of all functions from A to B, and composition is just the usual composition of functions.

EXAMPLE 1.9. The category Top of **topological spaces**. The objects of Top are all the topological spaces, and $\mathrm{Hom}(X, Y)$ is just the set $C(X, Y)$ of all *continuous* functions from X to Y, and composition is just the usual composition of functions.

EXAMPLE 1.10. The category Groups of **groups**. The objects of Groups are just groups, and $\mathrm{Hom}(G, H)$ is just the set $\mathrm{Hom}(G, H)$ of all *homomorphisms* from G to H, and composition is just the usual composition of homomorphisms.

EXAMPLE 1.11. The category Ab of **abelian groups**. The objects of Ab are just abelian groups, and $\mathrm{Hom}(G, H)$ is again just the set $\mathrm{Hom}(G, H)$ of all *homomorphisms* from G to H, and composition is just the usual composition of homomorphisms.

REMARK 1.12. The fact that we require the morphism sets to be pairwise disjoint has several pedantic consequences. For example, suppose $A \subsetneq B$ are two sets. Then the inclusion $\imath \colon A \hookrightarrow B$ and the identity map $\mathrm{id}_A \colon A \to A$ are different morphisms, since they have different targets. One should be aware that we only allow the composition $g \circ f$ when the range of f is *exactly* the same as the domain of g. Suppose X, Y, Y' and Z are topological spaces with $Y \subsetneq Y'$. From the point of view of analysis, say, if $f \colon X \to Y$ and $g \colon Y' \to Z$ are continuous functions then the composition $g \circ f \colon X \to Z$ is clearly a well-defined continuous function. But from the point of view of category theory, the composition $g \circ f$ *does not exist!* Rather, one must first take the inclusion $\imath \colon Y \hookrightarrow Y'$ and then consider the composition $g \circ \imath \circ f$, which is a well-defined element of the morphism space $C(X, Z)$.

A **functor** is a map from one category to another:

DEFINITION 1.13. Suppose C and D are two categories. A **functor** $T \colon \mathsf{C} \to \mathsf{D}$ associates to each $A \in \mathrm{obj}(\mathsf{C})$ an object $T(A) \in \mathrm{obj}(\mathsf{D})$, and to each morphism $A \xrightarrow{f} B$ in C a morphism $T(A) \xrightarrow{T(f)} T(B)$ in D which satisfies the following two axioms:

1. If $A \xrightarrow{f} B \xrightarrow{g} C$ in C then $T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$ in D and
$$T(g \circ f) = T(g) \circ T(f).$$
2. $T(\mathrm{id}_A) = \mathrm{id}_{T(A)}$ for every $A \in \mathrm{obj}(\mathsf{C})$.

The easiest example of a functor is a **forgetful functor**:

EXAMPLE 1.14. The forgetful functor $\mathsf{Top} \to \mathsf{Sets}$ simply "forgets" the topological structure. Thus it assigns to each topological space its underlying set, and to each continuous function it assigns the same function, considered now simply as a map between two *sets* (i.e. it "forgets" the function is continuous).

We can now make sense of the homology functor mentioned earlier.

THEOREM 1.15. *For each $n \geq 0$ there exists a functor $H_n : \mathsf{Top} \to \mathsf{Ab}$ called a* **homology functor** *with the property that for all $n \geq 0$,*

$$H_n(B^{n+1}) = 0, \qquad H_n(S^n) \neq 0.$$

I say "a" homology functor since H_n is not (quite) unique (we will construct several different ones eventually). In fact, before constructing homology functors we will first construct an "easier" functor called the **fundamental group**. This will (almost[5]) be a functor

$$\pi_1 \colon \mathsf{Top} \to \mathsf{Groups},$$

and its construction will take us up to the end of Lecture 4.

[5]Strictly speaking π_1 will be a functor from the category of *pointed topological spaces*, more on this later.

LECTURE 2

The notion of homotopy

In this lecture we introduce the notion of "deforming" one function (or one space) into another, which mathematically is known as **homotopy**. Throughout this course, we denote by

$$I := [0,1], \qquad \partial I = \{0,1\}.$$

DEFINITION 2.1. Suppose X and Y are topological spaces and $f_0, f_1 \colon X \to Y$ are two continuous functions. A **homotopy** from f_0 to f_1 is a continuous function

$$F \colon X \times I \to Y$$

such that

$$F(x,0) = f_0(x), \qquad F(x,1) = f_1(x).$$

We write $F \colon f_0 \simeq f_1$ to indicate F is a homotopy from f_0 to f_1, and we write $f_0 \simeq f_1$ to indicate there exists such an F.

Given a homotopy $F \colon f_0 \simeq f_1$, setting $f_t(x) := F(x,t)$, we obtain a family f_t of continuous functions which deforms f_0 at time $t = 0$ into f_1 at time $t = 1$. Since F is continuous on $X \times I$, the family f_t depends continuously on t.

The following lemma will be used time and time again. We will refer to it as "the gluing lemma".

LEMMA 2.2 (The gluing lemma). *Let X be a topological space. Assume X can be written as a finite union*

$$X = \bigcup_{i=1}^{N} X_i,$$

where each X_i is a closed subspace of X. Assume we are given a topological space Y and continuous functions

$$f_i \colon X_i \to Y,$$

with the property that

$$f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}, \qquad \forall\, i,j \text{ such that } X_i \cap X_j \neq \emptyset.$$

Then there exists a unique continuous function $f \colon X \to Y$ such that

$$f|_{X_i} = f_i, \qquad \forall\, i = 1, \dots, N. \tag{2.1}$$

Proof. We need only check that the function f defined as in (2.1) is continuous (it is clearly unique). Suppose $C \subseteq Y$ is a closed set. Then

$$\begin{aligned} f^{-1}(C) &= \left(\bigcup_{i=1}^{N} X_i \right) \cap f^{-1}(C) \\ &= \bigcup_{i=1}^{N} (X_i \cap f^{-1}(C)) \\ &= \bigcup_{i=1}^{N} (X_i \cap f_i^{-1}(C)) \\ &= \bigcup_{i=1}^{N} f_i^{-1}(C). \end{aligned}$$

Since each f_i is continuous, this is the finite union of closed sets and hence is closed. Since C was arbitrary, f is continuous. ■

On Problem Sheet A you will enjoy proving the following minor variation of Lemma 2.2.

LEMMA 2.3 (Another gluing lemma). *Let X be a topological space. Assume X can be written as an arbitrary union*

$$X = \bigcup_i X_i,$$

where each X_i is an open subspace of X. Assume we given a topological space Y and continuous functions

$$f_i \colon X_i \to Y,$$

with the property that

$$f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}, \qquad \forall\, i, j \text{ such that } X_i \cap X_j \neq \emptyset.$$

Then there exists a unique continuous function $f \colon X \to Y$ such that

$$f|_{X_i} = f_i, \qquad \forall\, i \in \mathbb{N}.$$

Our first application of the gluing lemma is to show that homotopy is an equivalence relation on the space of continuous maps.

PROPOSITION 2.4. *Let X and Y denote two topological spaces. Then homotopy is an equivalence relation on the space $C(X, Y)$ of all continuous maps from X to Y.*

Proof. We check the three properties:

- *Reflexivity*: if $f \in C(X, Y)$ define $F(x, t) := f(x)$. Then clearly $F \colon f \simeq f$.
- *Symmetry*: if $F \colon f \simeq g$ then define $G(x, t) := F(x, 1 - t)$. Then $G \colon g \simeq f$.

- *Transitivity*: if $F\colon f \simeq g$ and $G\colon g \simeq h$, define

$$H(x,t) := \begin{cases} F(x,2t), & 0 \le t \le \frac{1}{2}, \\ G(x,2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

  Since F and G agree on their overlap $X \times \{\frac{1}{2}\} \subset X \times I$ the gluing lemma implies that H is continuous. Since $H(x,0) = f$ and $H(x,1) = h$, this shows that $f \simeq h$.

This completes the proof. ■

Definition 2.5. We denote by $[f]$ the equivalence class of f under homotopy, and we denote by $[X, Y]$ the space of equivalence classes.

We now show that composition of equivalence classes makes sense.

Proposition 2.6. *Suppose $f_0, f_1\colon X \to Y$ and $g_0, g_1\colon Y \to Z$ are continuous functions with $f_0 \simeq f_1$ and $g_0 \simeq g_1$. Then $g_0 \circ f_0 \simeq g_1 \circ f_1$, that is*

$$[f_0] = [f_1] \text{ and } [g_0] = [g_1] \qquad \Rightarrow \qquad [g_0 \circ f_0] = [g_1 \circ f_1].$$

Proof. Suppose $F\colon f_0 \simeq f_1$ and $G\colon g_0 \simeq g_1$. Define

$$H\colon X \times I \to Z, \qquad H(x,t) = G(f_0(x), t).$$

Then H is continuous and since

$$H(x,0) = G(f_0(x), 0) = g_0(f_0(x))$$

and

$$H(x,1) = G(f_0(x), 1) = g_1(f_0(x)),$$

this shows that

$$g_0 \circ f_0 \simeq g_1 \circ f_0. \tag{2.2}$$

Next, consider

$$K\colon X \times I \to Z, \qquad K(x,t) = g_1(F(x,t)).$$

Then K is continuous, and since

$$K(x,0) = g_1(F(x,0)) = g_1(f_0(x))$$

and

$$K(x,1) = g_1(F(x,1)) = g_1(f_1(x)),$$

this shows that

$$g_1 \circ f_0 \simeq g_1 \circ f_1. \tag{2.3}$$

Combining (2.2) and (2.3) and using transitivity, we see that $g_0 \circ f_0 \simeq g_1 \circ f_1$ as required. ■

Now for this lecture's serving of abstract nonsense. Let us explain how to take quotients of categories.

Definition 2.7. Suppose C is a category. A **congruence** on C is an equivalence relation $\sim$ on the union

$$\bigcup_{(A,B)\in \mathrm{obj}(\mathsf{C})\times \mathrm{obj}(\mathsf{C})} \mathrm{Hom}(A,B)$$

such that:

1. If $f \in \mathrm{Hom}(A,B)$ and $f \sim g$ then $g \in \mathrm{Hom}(A,B)$.
2. If $f_0 : A \to B$ and $g_0 : B \to C$ and $f_0 \sim f_1$ and $g_0 \sim g_1$ then $g_0 \circ f_0 \sim g_1 \circ f_1$.

A congruence allows us to form the quotient category:

Proposition 2.8. *Suppose C is a category and $\sim$ is a congruence on C. Denote by $[f]$ the equivalence class of a morphism under $\sim$. Then there is a well-defined* **quotient category** C' *given as follows: the objects of C' are simply* $\mathrm{obj}(\mathsf{C})$ *again, and*

$$\mathrm{Hom}_{\mathsf{C}'}(A,B) = \{[f] \mid f \in \mathrm{Hom}(A,B)\},$$

and composition in C' is given by

$$[g] \circ [f] := [g \circ f].$$

Proof. Property (1) of Definition 2.7 shows that the morphism sets of C' are well-defined sets that are pairwise disjoint. Property (2) of Definition 2.7 shows that the composition in C' is well-defined. It is clear that this composition is associative, and $[\mathrm{id}_A]$ is the identity morphism in $\mathrm{Hom}_{\mathsf{C}'}(A,A)$. This completes the proof. ■

It will not surprise you to learn we have just constructed a congruence.

Example 2.9. The **homotopy category** hTop is the category whose objects are topological spaces, with morphism spaces $[X,Y]$ the equivalence class of continuous maps under homotopy. This is the quotient category of Top under the congruence obtained via homotopy.

Definition 2.10. Let C be a category. An **isomorphism** in C is a morphism $f \in \mathrm{Hom}(A,B)$ for which there exists another morphism $g \in \mathrm{Hom}(B,A)$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Thus in Sets, isomorphisms are just bijections. In Groups, isomorphisms are group isomorphisms, and in Top, isomorphisms are homeomorphims. Let us unravel what an isomorphism in hTop is.

Definition 2.11. A continuous map $f \colon X \to Y$ between two topological spaces is called a **homotopy equivalence** if there exists a continuous map $g \colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Thus an isomorphism in hTop is a morphism $[f]$, where f is a homotopy equivalence.

Definition 2.12. We say that two spaces X and Y have the **same homotopy type** if there exists a homotopy equivalence from $f \colon X \to Y$.

Clearly homeomorphic spaces have the same homotopy type, but it is *not* necessarily the case that spaces with the same homotopy type are homeomorphic. We shall see an example of this next lecture. Let us now look at the "opposite" notion to a homotopy equivalence.

DEFINITION 2.13. A continuous map $f\colon X \to Y$ is said to be **nullhomotopic** if there exists a **constant map** $c\colon X \to Y$ (i.e. map such that there exists $q \in Y$ such that $c(x) = q$ for all $x \in X$) such that $f \simeq c$.

When $X = S^n$, there is an easy criterion for deciding whether a map is nullhomotopic. Before stating the result, we need one more definition.

DEFINITION 2.14. Suppose X and Y are topological spaces and X' is a subset of X. We say that two continuous map $f_0, f_1\colon X \to Y$ such that $f_0|_{X'} = f_1|_{X'}$ are **homotopic relative to** X' or **homotopic rel** X' for short if there exists a homotopy $F\colon f_0 \simeq f_1$ such that

$$F(x,t) = f_0(x) = f_1(x), \qquad \forall\, x \in X', \forall\, t \in I.$$

If such an F exists we write $F\colon f_0 \simeq f_1$ rel X'.

This generalises Definition 2.1, since taking $X' = \emptyset$ recovers our original notion of homotopy. For fixed X being homotopic rel X' is an equivalence relation – you will prove a more general version statement on Problem Sheet B (a special case of this is given in Proposition 3.13 next lecture.) By a slight abuse of notation if X' is a single point $\{p\}$ we will write "rel p" instead of "rel $\{p\}$".

PROPOSITION 2.15. *Let Y be a topological space and $n \geq 0$. The following are equivalent for a continuous map $f\colon S^n \to Y$:*

1. *f is nullhomotopic.*
2. *There exists a continuous map $g\colon B^{n+1} \to Y$ such that $g|_{S^n} = f$.*
3. *If $p \in S^n$ and $c\colon S^n \to Y$ is the constant map $c(x) = f(p)$ then f is homotopic to c rel p.*

Proof. We first show (1) implies (2). Suppose f is nullhomotopic, i.e. there exists $F\colon f \simeq c$, where c is the constant map $c(x) = q$. Define $g\colon B^{n+1} \to Y$ by

$$g(x) := \begin{cases} q, & 0 \leq |x| \leq \frac{1}{2}, \\ F\big(\frac{x}{|x|}, 2 - 2|x|\big), & \frac{1}{2} \leq |x| \leq 1. \end{cases}$$

This makes sense: if $x \neq 0$ then $\frac{x}{|x|}$ belongs to S^n, and if $\frac{1}{2} \leq |x| \leq 1$ then $2 - 2|x| \in I$. If $|x| = \frac{1}{2}$ then

$$F\big(\tfrac{x}{|x|}, 1\big) = c\big(\tfrac{x}{|x|}\big) = q.$$

The gluing lemma shows that g is continuous. Moreover g does extend f since if $x \in S^n$ then $|x| = 1$ and hence $g(x) = F(x, 0) = f(x)$.

To show (2) implies (3), suppose $g\colon B^{n+1} \to Y$ extends f. Define $F\colon S^n \times I \to Y$ by

$$F(x,t) = g((1-t)x + tp).$$

This makes sense as $(1-t)x + tp$ belongs to B^{n+1}. F is clearly continuous, and $F(x,0) = g(x) = f(x)$ (since g extends f) and $F(x,1) = g(p) = f(p) = c(x)$. Thus $F\colon f \simeq c$. Moreover $F(p,t) = g(p) = f(p)$ for all $t \in I$, and hence $F\colon f \simeq c$ rel p.

Finally, it is obvious that (3) implies (1). ■

Definition 2.16. A space X is said to be **contractible** if the identity map id_X is nullhomotopic.

Corollary 2.17. *For all $n \geq 0$, the sphere S^n is not contractible.*

This proof uses Lemma 1.3, which we have not yet properly proved (we haven't constructed the homology functor yet!). You will be relieved to note that we will not use the proof of Corollary 2.17 in the construction of the homology functor.

Proof. Take $Y = S^n$ and $f = \mathrm{id}_{S^n}$. Then by Proposition 2.15, if f is nullhomotopic then there exists a continuous map $g\colon B^{n+1} \to S^n$ which extends f. The map g is then a retraction, and this contradicts Lemma 1.3. ■

Paths and the fundamental groupoid

In this lecture we define a rather pathetic functor, called π_0. We then define the *fundamental groupoid*. In the next lecture we will use the fundamental groupoid to define a much more interesting functor, the *fundamental group* π_1.

DEFINITION 3.1. A **path** u in a topological space X is a continuous map $u\colon I \to X$. If $u(0) = x$ and $u(1) = y$ we say u is a **path from** x **to** y. If $x = y$ then we say that u is a **loop**.

We will always use the letters u, v and w to denote paths (in contrast to f, g and h for arbitrary continuous maps). Moreover we will parametrise a path with the letter s, so u is the map $s \mapsto u(s)$, thus keeping the letter t for a homotopy parameter. This will hopefully help to keep the notation clear. Paths gives us a new notion of connectivity.

DEFINITION 3.2. A topological space X is **path connected** if for all $x, y \in X$ there exists a path from x to y.

Hopefully you are all easily able to prove the following result[1].

LEMMA 3.3. *Let X and Y be topological spaces. Then:*

1. *If X is path connected then X is connected (but the converse is not necessarily true).*
2. *If X and Y are path connected then so is $X \times Y$.*
3. *If $f\colon X \to Y$ is continuous and X is path connected then so is $f(X)$.*

Here we prove the following equally easy result:

PROPOSITION 3.4. *If X is a topological space then the binary relation $\sim$ on X defined by "$x \sim y$ if there exists a path from x to y" is an equivalence relation.*

Proof. A constant path based at x shows that $x \sim x$ for all $x \in X$. If u is a path from x to y then the path $\bar{u}(s) := u(1-s)$ is a path from y to x, and hence $x \sim y$ implies $y \sim x$. Finally if u is a path from x to y and v is a path from y to z then

$$w(s) := \begin{cases} u(2s), & 0 \le s \le \frac{1}{2}, \\ v(2s-1), & \frac{1}{2} \le s \le 1, \end{cases} \tag{3.1}$$

is a well-defined path from x to z (the gluing lemma shows that w is continuous.) Thus $x \sim y$ and $y \sim z$ implies $x \sim z$. ∎

[1] I debated putting this on Problem Sheet B but decided it was too easy ...

Definition 3.5. The equivalence classes of X under the equivalence relation $\sim$ are called the **path components** of X.

We now construct the functor π_0.

Definition 3.6. Given a topological space X, let $\pi_0(X)$ denote the set of path components of X. If $f\colon X \to Y$ is a continuous map, define $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ to be the map that send a path component X' of X to the unique path component of Y containing $f(X')$ (this is well-defined due to Lemma 3.3.)

We then have:

Proposition 3.7. *$\pi_0\colon \mathsf{Top} \to \mathsf{Sets}$ is a functor. Moreover if $f \simeq g$ then $\pi_0(f) = \pi_0(g)$.*

Proof. The fact that π_0 is a functor is easy to check (i.e. that π_0 preserves identities and composition). Let us check that homotopic maps have the same image under π_0. Suppose $F\colon f \simeq g$. If X' is a path component of X then $X' \times I$ is path connected and hence so is $F(X' \times I)$ (here we are using Proposition 3.3 twice). Since

$$f(X') = F(X' \times \{0\}) \subseteq F(X' \times I)$$

and

$$g(X') = F(X' \times \{1\}) \subseteq F(X' \times I),$$

we see that the unique path component of Y containing $F(X' \times I)$ contains both $f(X')$ and $g(X')$. Thus $\pi_0(f) = \pi_0(g)$. ■

Corollary 3.8. *If X and Y have the same homotopy type then they have the same number of path components.*

Corollary 3.8 can be proved directly, but let us give an "abstract" proof using Problem A.2 and Problem A.3 from Problem Sheet A.

Proof. By the last part of Proposition 3.7 and Problem A.3, we may regard π_0 as a functor $\mathsf{hTop} \to \mathsf{Sets}$. If X and Y have the same homotopy type then there exists a continuous map $f\colon X \to Y$ such that $[f]$ is an isomorphism in hTop. Then by Problem A.2, $\pi_0([f])$ is an isomorphism in Sets. An isomorphism in Sets is a bijection; thus $\pi_0(X)$ and $\pi_0(Y)$ have the same cardinality. ■

Corollary 3.8 is about as interesting as it gets when it comes to the functor π_0. This is because π_0 has the misfortune of taking values in Sets, and there is not much one can with a set other than count it (i.e. the only obstruction to two sets being isomorphic is that they should have the same cardinality) Next lecture we will introduce another functor π_1 which takes values in Groups. As groups have many obstructions to being isomorphic, this functor will be considerably more interesting.

The basic idea behind π_1 is that one can "multiply" paths if one ends where the other begins, via (3.1). Let us formalise this as a definition.

Definition 3.9. Let u and v be paths in X with $u(1) = v(0)$. Then we define

$$(u * v)(s) := \begin{cases} u(2s), & 0 \leq s \leq \frac{1}{2}, \\ v(2s-1), & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Remark 3.10. Note that the ordering here is the *opposite* to composition: $u * v$ means "first do u, then do v", meanwhile $g \circ f$ means "first do f, then do g".

Our aim is to construct a group whose elements are certain homotopy classes of paths in X with binary operation given by multiplying paths as above. However by Problem B.1 on Problem Sheet B, if X is path connected then since I is contractible, all paths $u\colon I \to X$ are homotopic, and thus if we tried to construct a group from homotopy classes of paths this group would have precisely one element (and so would be just as uninteresting as $\pi_0(X)$!) To recify this problem we use relative homotopy classes.

Definition 3.11. We define the **path class** of a path $u\colon I \to X$ to be the equivalence class $[u]$ of u, where the equivalence relation is being homotopic relative to $\partial I = \{0, 1\}$.

Remark 3.12. There is a potential for confusion here, in that we are using the same notation $[\cdot]$ to denote both the homotopy class and the relative homotopy class. However, it should always be clear from the context which is intended. In particular, for paths we will only ever talk about their path class, not their homotopy class, and thus the notation $[u]$ *always* means the path class.

The next result is similar to Proposition 2.6.

Proposition 3.13. *Suppose $u_0, u_1\colon I \to X$ and $v_0, v_1\colon I \to X$ are paths with*

$$u_0(1) = u_1(1) = v_0(0) = v_1(0).$$

Assume that

$$[u_0] = [u_1] \qquad \text{and} \qquad [v_0] = [v_1].$$

Then

$$[u_0 * v_0] = [u_1 * v_1].$$

Proof. If $U\colon u_0 \simeq u_1$ rel ∂I and $V\colon v_0 \simeq v_1$ rel ∂I then the map $W\colon I \times I \to X$ given by

$$W(s,t) := \begin{cases} U(2s,t), & 0 \leq s \leq \frac{1}{2}, \\ V(2s-1,t), & \frac{1}{2} \leq s \leq 1, \end{cases}$$

is a continuous map (the gluing lemma applies because functions agree on $\{\frac{1}{2}\} \times I$) which determines a homotopy from $u_0 * v_0$ to $u_1 * v_1$ rel ∂I. ■

If u is a path from x to y, then running backwards along u gives a path from y to x. Let us fix some notation for this:

DEFINITION 3.14. Given a path $u : I \to X$, we denote by $\bar{u} : I \to X$ the path u parametrised backwards:

$$\bar{u}(s) = u(1-s).$$

Next, let us given a name to the constant path:

DEFINITION 3.15. Given a point $p \in X$, we denote by e_p the constant path $e_p(s) = p$. By a slight abuse of notation we denote by $[p]$ the path class $[e_p]$.

We now use this data to define a category. We will phrase this is as "definition" and then prove afterwards that it really is well-defined.

DEFINITION 3.16. Let X be a topological space. We define the **fundamental groupoid** of X to be the category $\Pi(X)$ where:

- $\mathrm{obj}(\Pi(X)) = X$, that is, the objects of $\Pi(X)$ are the points in X themselves,
- $\mathrm{Hom}(x, y)$ is the set of path classes of paths from x to y:

$$\mathrm{Hom}(x, y) := \{[u] \mid u \text{ is a path from } x \text{ to } y\},$$

- and finally the composition

$$\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z)$$

  is given by

$$\big([u], [v]\big) \mapsto [u * v]$$

  (note by assumption this concatenation makes sense as $u(1) = y = v(0)$).

Let us prove this really does form a category.

PROPOSITION 3.17. *Let X be a topological space. Then $\Pi(X)$ is a well-defined category. The identity element of $\mathrm{Hom}(p, p)$ is $[p]$.*

Proof. From Definition 1.6, there are three things we need to verify:

1. the Hom sets are pairwise disjoint,
2. that composition is associative when defined,
3. that there exists an identity element in each Hom set.

Here (1) is obvious. Let us first prove (3). We claim that $[p]$ (i.e. the path class of e_p) is the identity element in $\mathrm{Hom}(p, p)$. For this we must prove that for any path u with $u(0) = p$ we have $e_p * u \simeq u$ rel ∂I, and similarly for any path v with $v(1) = p$ we have $v * e_p \simeq v$ rel ∂I. We will prove the first statement only, as the second is similar. Consider Figure 3.1. The shaded triangle is the set $\{(s, t) \mid 2s \leq 1 - t\}$. For fixed t, consider the horizontal line L_t that runs from the start of the shaded region to the right-hand edge (the point $(1, t)$). The function

$$l_t(s) := \frac{s - \frac{1}{2}(1-t)}{1 - \frac{1}{2}(1-t)}$$

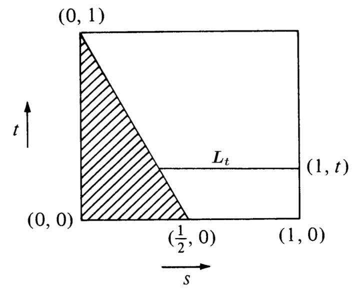


Figure 3.1: Proving $e_p * u \simeq u$ rel ∂I.

maps L_t onto $[0,1]$. Now consider the map $U\colon I \times I \to X$ given by

$$U(s,t) := \begin{cases} p, & 2s \leq 1-t, \\ u(l_t(s)), & 2s \geq 1-t. \end{cases}$$

The gluing lemma shows that U is continuous, and by construction $U\colon e_p * u \simeq u$ rel ∂I.

Now let us prove associativity. Suppose u, v and w are three paths with $u(1) = v(0)$ and $v(1) = w(0)$. This is a similar but slightly trickier argument, and we will not write out the formulae precisely. Consider Figure 3.2. Draw two slanted lines, one

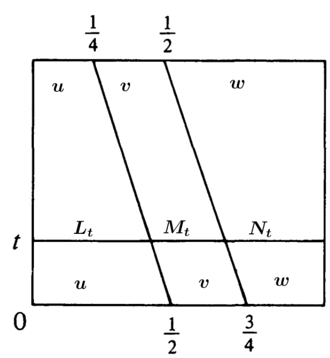


Figure 3.2: Proving $(u * v) * w \simeq u * (v * w)$ rel ∂I.

that starts at $(1/4, 1)$ and runs to $(1/2, 0)$, and one that starts at $(1/2, 1)$ and runs to $(3/4, 0)$. Now let L_t, M_t and N_t denote the three horizontal lines as marked that come from intersecting the horizontal line with t fixed. Then let l_t, m_t and n_t denote reparametrisations that map L_t, M_t and N_t onto $[0,1]$ respectively. The desired homotopy is U is obtained by setting $U(t,s) = u(l_t(s))$ on the left-hand region, setting $U(s,t) = v(m_t(s))$ on the middle region and finally setting $U(t,s) = w(n_t(s))$ on the right-hand region. The gluing lemma shows that U is continuous, and by construction we have $U\colon (u*v)*w \simeq u*(v*w)$ rel ∂I. This completes the proof. ■

In fact, the category $\Pi(X)$ has an additional special property:

Proposition 3.18. *Every morphism in* $\Pi(X)$ *is an isomorphism. More precisely, for any path* u *from* x *to* y*, one has*

$$[u] * [\bar{u}] = [x], \qquad [\bar{u}] * [u] = [y].$$

Proof. We must show $u * \bar{u} \simeq e_x$ rel ∂I and $\bar{u} * u \simeq e_y$ rel ∂I. Again, I will prove only the first statement. Moreover this time round, I will give the formulae but not the picture[2]. To this end consider the function $U \colon I \times I \to X$ given by

$$U(s,t) := \begin{cases} u\big(2s(1-t)\big), & 0 \le s \le \frac{1}{2}, \\ u\big(2(1-s)(1-t)\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

The gluing lemma shows that U is continuous, and one checks that $U \colon u * \bar{u} \simeq e_x$ rel ∂I. This completes the proof. ■

Categories with this property have a special name.

Definition 3.19. Let C be a category. We say that C is a **groupoid category** if:

- C is a **small** category[3], which by definition means that $\mathrm{obj}(\mathsf{C})$ is a *set* and not just a class, cf. Remark 1.7.
- Every morphism $f : A \to B$ in C is an isomorphism.

Thus Proposition 3.18 can alternatively be rephrased as: the fundamental groupoid is a groupoid cateogory.

[2]And thus you should draw the picture out!

[3]Nothing we ever do in this course will ever need to worry about the distinction between a set and a class, so you are free to ignore this part of the definition if you want ...

LECTURE 4

The fundamental group

In this lecture we define our first genuinely exciting functor, π_1. The idea is simply to restrict to loops.

Definition 4.1. Let X be a topological space and fix a point $p \in X$, which we call the **basepoint**. The **fundamental group** of X with basepoint p is

$$\pi_1(X,p) := \mathrm{Hom}_{\Pi(X)}(p,p) = \{[u] \mid u \text{ is a loop in } X \text{ based at } p\}.$$

An immediate corollary of Proposition 3.17 is the following result.

Corollary 4.2. *For any topological space X and any $p \in X$, the set $\pi_1(X,p)$ is a group with multiplication given by*

$$[u] * [v] := [u * v]$$

and identity element $[p]$. The inverse of an element $[u]$ is given by $[\bar{u}]$:

$$[u]^{-1} = [\bar{u}].$$

Since the fundamental group $\pi_1(X,p)$ involves a choice of basepoint p, in order to make π_1 into a proper functor we need to work with a slightly different category. Before introducing this, let us explain how a smaller category can sit inside a larger one.

Definition 4.3. Suppose C and D are two categories. We say that C is a **subcategory** of D if:

1. $\mathrm{obj}(\mathsf{C}) \subseteq \mathrm{obj}(\mathsf{D})$,
2. $\mathrm{Hom}_{\mathsf{C}}(A,B) \subseteq \mathrm{Hom}_{\mathsf{D}}(A,B)$ for all $A, B \in \mathrm{obj}(\mathsf{C})$, where we denote Hom sets in C by $\mathrm{Hom}_{\mathsf{C}}(\square, \square)$,
3. if $f \in \mathrm{Hom}_{\mathsf{C}}(A,B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B,C)$ then the composite $g \circ f \in \mathrm{Hom}_{\mathsf{C}}(A,C)$ is equal to the composite $g \circ f \in \mathrm{Hom}_{\mathsf{D}}(A,C)$,
4. if $C \in \mathrm{obj}(\mathsf{C})$ then $\mathrm{id}_C \in \mathrm{Hom}_{\mathsf{C}}(C,C)$ is equal to $\mathrm{id}_C \in \mathrm{Hom}_{\mathsf{D}}(C,C)$.

If for every pair $A, B \in \mathrm{obj}(\mathsf{C})$ one always has $\mathrm{Hom}_{\mathsf{C}}(A,B) = \mathrm{Hom}_{\mathsf{D}}(A,B)$ then we say that C is a **full subcategory** of D.

As an example, the category Ab is a full subcategory of Groups.

Example 4.4. The category Top^2 has as objects all pairs (X, X') where X is a topological space and $X' \subseteq X$ is a subspace. A morphism $(X, X') \to (Y, Y')$ is a pair (f, f') of continuous maps $f : X \to Y$ and $f' \colon X' \to Y'$ such that the following diagram commutes, where the horizontal maps are inclusions:

$$\begin{array}{ccc} X' & \hookrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \hookrightarrow & Y \end{array}$$

The composition law is the usual one. Slightly less pedantically, we can think of a morphism in $\mathrm{Hom}((X, X'), (Y, Y'))$ simply as a continuous map $f \colon X \to Y$ with the property that $f(X') \subseteq Y'$.

We can regard Top as a subcategory of Top^2 if we identify a space X with the pair $(X, \emptyset)$. For us, it is also useful to consider the case where both X' and Y' are a single point in X and Y respectively. This gets its own name.

Example 4.5. The category Top_* of **pointed topological spaces** has as objects all ordered pairs (X, p) where X is a topological space and p is a point in X, referred to as the **basepoint**. Given two objects (X, p) and (Y, q), the morphism space is simply the set of continuous maps $f \colon X \to Y$ which send the basepoint p in X to the basepoint $q \in Y$:

$$\mathrm{Hom}((X, p), (Y, q)) := \{f \in C(X, Y) \mid f(p) = q\}.$$

We call such a map a **pointed map**.

We will write $f \colon (X, p) \to (Y, q)$ as shorthand to indicate that f is a continuous map from X to Y satisfying $f(p) = q$ (and hence a morphism in Top_*.) Note that Top_* is again a subcategory of Top^2. Let us now show that π_1 is a functor from Top_* to Groups.

Definition 4.6. Suppose $f \colon (X, p) \to (Y, q)$ is a pointed map. Define

$$\pi_1(f) \colon \pi_1(X, p) \to \pi_1(Y, q), \qquad [u] \mapsto [f \circ u].$$

This is well-defined: firstly $f \circ u \colon I \to Y$ is a continuous path that starts and ends at q, and hence $[f \circ u]$ is indeed an element of $\pi_1(Y, q)$. Moreover if $u \simeq v$ rel ∂I then $f \circ u \simeq f \circ v$ rel ∂I by Problem B.2 on Problem Sheet B.

Proposition 4.7. *$\pi_1 \colon \mathsf{Top}_* \to \mathsf{Groups}$ is a functor. Moreover if $f, g \colon (X, p) \to (Y, q)$ are continuous maps with $f \simeq g$ rel p then $\pi_1(f) = \pi_1(g)$.*

Proof. Suppose $f \colon (X, p) \to (Y, q)$ is a pointed map. To show that $\pi_1(f)$ is a homomorphism, observe that if u and v are closed paths based at $p \in X$ then

$$f \circ (u * v) = (f \circ u) * (f \circ v)$$

(this is an actual pointwise equality, not just homotopy!). The fact that π_1 preserves composition and identities in Top_* is clear. Finally, if $f \simeq g$ rel p then by Problem B.2 on Problem Sheet B again, one obtains $f \circ u \simeq g \circ u$ rel ∂I for any closed curve u in X based at p. Thus $\pi_1(f) = \pi_1(g)$. ∎

Remark 4.8. It follows from Problem B.2 that being homotopic rel p defines an equivalence relation on pointed maps $(X,p) \to (Y,q)$, and hence a congruence on the category Top_*. The associated quotient category is denoted by hTop_*. By analogy with the category hTop, we write $[(X,p),(Y,q)]$ for the morphism set between two objects (X,p) and (Y,q) in hTop_*. The last statement of Proposition 4.7 together with Problem A.3 implies that π_1 induces a functor $\pi_1 \colon \mathsf{hTop}_* \to \mathsf{Groups}$.

Remark 4.9. We won't need this (so I won't spell out the relevant definitions), but the association $[u] \mapsto [f \circ u]$ makes perfect sense for arbitrary path classes, and hence given a continuous map $f \colon X \to Y$, we obtain a map $\Pi(f) \colon \Pi(X) \to \Pi(Y)$ defined by

$$\Pi(f) \colon \mathrm{Hom}_{\Pi(X)}(x,y) \to \mathrm{Hom}_{\Pi(Y)}(f(x),f(y)), \qquad [u] \mapsto [f \circ u].$$

This makes the fundamental groupoid into a functor

$$\Pi : \mathsf{Top} \to \mathsf{Groupoids},$$

where $\mathsf{Groupoids}$ is the category of groupoids[1].

The next result shows that when studying fundamental groups, we may as well assume that our spaces are path connected.

Proposition 4.10. *Let X be a topological space, let $p \in X$, and let X' denote the path component containing p. Then the inclusion $\imath \colon X' \hookrightarrow X$ induces an isomorphism on π_1:*

$$\pi_1(X',p) \cong \pi_1(X,p).$$

Proof. Suppose $[u] \in \ker \pi_1(\imath)$. This means that $\imath \circ u \simeq e_p$ rel ∂I, where as usual $e_p \colon I \to X$ is the constant path at p. Let $U \colon I \times I \to X$ denote the homotopy. Then $U(0,0) = p$. Since $U(I \times I)$ is path connected (cf. Problem B.2) we have $U(I \times I) \subseteq X'$. Thus u is nullhomotopic in X', and hence $\pi_1(\imath)$ is injective. To see $\pi_1(\imath)$ is surjective, suppose $u \colon I \to X$ is a closed path at p. Then $u(I) \subseteq X'$. Thus we can pedantically define $u' \colon I \to X'$ by $u'(s) := u(s)$. Then clearly $\imath \circ u' = u$, and surjectivity follows. ■

Now let us investigate what happens when the basepoint is changed.

Proposition 4.11. *Suppose X is path connected and $p_0, p_1 \in X$. Then any path w from p_0 to p_1 induces an isomorphism*

$$\lambda_w \colon \pi_1(X,p_0) \cong \pi_1(X,p_1)$$

.

Proof. Define $\lambda_w \colon \pi_1(X,p_0) \to \pi_1(X,p_1)$ by

$$\lambda_w \colon [v] \mapsto [\bar{w} * v * w] \tag{4.1}$$

(note that the multiplication takes places in the fundamental groupoid $\Pi(X)$.) From Proposition 3.18 one sees that λ_w is an isomorphism; the inverse is given by $\lambda_{\bar{w}}$. ■

[1]Since a groupoid is a type of category, this is a category of categories!

Remark 4.12. Thus provided we work with path connected spaces only, we can write simply $\pi_1(X)$ to denote $\pi_1(X,p)$ for any $p \in X$. However should be aware that there is no *canonical* isomorphism between $\pi_1(X,p_0)$ and $\pi_1(X,p_1)$. Thus $\pi_1(X)$ is really a family of isomorphic groups. Moreover, as you will see in Problem Sheet C (cf. Problem C.5), sometimes simply knowing two groups are isomorphic is not enough – one really needs an explicit isomorphism.

We now discuss a subtle (and often tedious) point. To define the fundamental group we were forced to pick a basepoint, and thus the natural category to work with is Top_*. But "most" homotopic maps that crop up "in nature" are *not* pointed maps (i.e. given two homotopic maps, it is typically too much to hope for that they just so happen to preserve the basepoint.) Thus we need to investigate how the fundamental group behaves under a free[2] homotopy.

Proposition 4.13. *Suppose $f_0, f_1 \colon X \to Y$ are continuous maps and $F \colon f_0 \simeq f_1$ is a free homotopy from f_0 to f_1. Choose $p \in X$ and let w denote the path in Y given by $w(t) = F(p,t)$. Then there is a commutative diagram:*

$$\begin{array}{ccc} \pi_1(X,p) & \xrightarrow{\pi_1(f_0)} & \pi_1(Y, f_0(p)) \\ & \searrow^{\pi_1(f_1)} & \downarrow \lambda_w \\ & & \pi_1(Y, f_1(p)), \end{array}$$

where λ_w is the isomorphism given in (4.1).

Proof. Take $[u] \in \pi_1(X,p)$. Consider the homotopy[3] $V \colon I \times I \to Y$ given by

$$V(s,t) := \begin{cases} F\big(u\big(2(1-t)s\big), 2st\big), & 0 \le s \le \frac{1}{2}, \\ F\big(u\big(1 + 2t(s-1)\big), t + (1-t)(2s-1)\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

Then

$$V(s,0) = \begin{cases} F\big(u(2s), 0\big), & 0 \le s \le \frac{1}{2}, \\ F\big(u(1), 2s-1\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since $F(u(2s),0) = f_0(u(2s))$ and $F(u(1), 2s-1) = w(2s-1)$, we have

$$V(s,0) = (f_0 \circ u) * w(s).$$

Similarly

$$V(s,1) = \begin{cases} F\big(u(0), 2s\big), & 0 \le s \le \frac{1}{2}, \\ F\big(u(2s-1), 1\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since $F(u(0), 2s) = w(2s)$ and $F(u(2s-1), 1) = f_1(u(2s-1))$, we have

$$V(s,1) = w * (f_1 \circ u)(s).$$

[2] I will call a homotopy $F \colon f \simeq g$ between two maps a *free* homotopy when it is important to emphasise that it is *not* a relative homotopy.

[3] As ever, I encourage you to draw a picture here!

The gluing lemma shows that V is continuous, and

$$V(0,t) = F(u(0),0) = f_0(p)$$

and

$$V(1,t) = F(u(1),1) = f_1(p).$$

Thus V is a homotopy from $(f_0 \circ u) * w$ to $w * (f_1 \circ u)$ relative to ∂I. Thus in the fundamental groupoid $\Pi(Y)$, we have

$$[f_0 \circ u] * [w] = [w] * [f_1 \circ u],$$

or alternatively

$$[f_1 \circ u] = [\bar{w} * (f_0 \circ u) * w],$$

which implies that

$$\pi_1(f_1)[u] = \lambda_w \circ \pi_1(f_0)[u]$$

as maps $\pi_1(X,p) \to \pi_1(Y, f_1(p))$. ■

COROLLARY 4.14. *Suppose $f_0, f_1 \colon X \to Y$ are continuous maps and $F \colon f_0 \simeq f_1$ is a free homotopy from f_0 to f_1. Suppose $p \in X$ has the property that $f_0(p) = f_1(p)$. Set $q := f_0(p)$. Then $\pi_1(f_0)$ and $\pi_1(f_1)$ are* **conjugate** *group homomorphisms, that is, there exists $[w] \in \pi_1(Y,q)$ such that*

$$\pi_1(f_1)[u] = [w]^{-1} * \pi_1(f_0)[u] * [w], \qquad \forall\, [u] \in \pi_1(X,p). \tag{4.2}$$

In particular, if $\pi_1(Y,q)$ is abelian then $\pi_1(f_0) = \pi_1(f_1)$.

Proof. Using the notation of Proposition 4.13, the path w is now a *closed* path in Y, and hence $[w] \in \pi_1(Y,q)$. Thus the path class $[\bar{w} * (f_0 \circ u) * w]$, which can always be factored in the fundamental groupoid of Y can now be factored in $\pi_1(Y,q)$:

$$[\bar{w} * (f_0 \circ u) * w] = [\bar{w}] * [f_0 \circ u] * [w] = [w]^{-1} * [f_0 \circ u] * [w].$$

Thus (4.2) follows. Finally, the last statement is immediate, since if $\pi_1(Y,q)$ is abelian then we can write

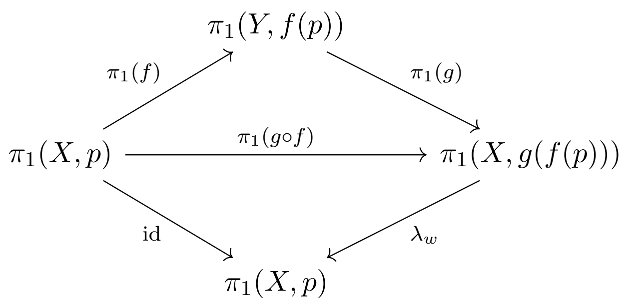
$$[w]^{-1} * \pi_1(f_0)[u] * [w] = [w]^{-1} * [w] * \pi_1(f_0)[u] = \pi_1(f_0)[u].$$

■

We now show that for path connected spaces X and Y, having the same homotopy type is enough to ensure that the fundamental groups $\pi_1(X,p)$ and $\pi_1(Y,q)$ coincide for any $p \in X$ and $q \in Y$. This will follow from the following result.

PROPOSITION 4.15. *Suppose $f \colon X \to Y$ is a homotopy equivalence. Then for any $p \in X$ the induced map $\pi_1(f) \colon \pi_1(X,p) \to \pi_1(Y, f(p))$ is an isomorphism.*

Proof. Choose a continuous map $g\colon Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. If $F\colon g\circ f \simeq \mathrm{id}_X$ is a free homotopy, let $w(s) := F(p, s)$, so that w is a path from $g(f(p))$ to p. By Proposition 4.13, the lower triangle of the following diagram commutes:



The top triangle also commutes because π_1 is a functor. Since λ_w is an isomorphism, $\pi_1(g \circ f)$ is also an isomorphism. Thus $\pi_1(f)$ is injective and $\pi_1(g)$ is surjective. A similar diagram starting from $f \circ g \simeq \mathrm{id}_Y$ shows that $\pi_1(g)$ is injective and $\pi_1(f)$ is surjective. ∎

An immediate corollary of this result (together with the fact that a space consisting of one point obviously has a trivial fundamental group) we have:

Corollary 4.16. *Suppose X has the same homotopy type as a path connected space Y. Then for all $p \in X$ and $q \in Y$ one has $\pi_1(X, p) \cong \pi_1(Y, q)$. If X is a contractible space then $\pi_1(X, p) = \{1\}$ for all $p \in X$.*

We conclude by giving the property of having a trivial fundamental group a name:

Definition 4.17. A topological space X is called **simply connected** if it is path connected and has $\pi_1(X, p) = \{1\}$ for some (and hence all) $p \in X$.

LECTURE 5

The fundamental group of the circle and pushouts

We have yet to give an example of a path-connected space that is *not* simply connected! If it turned out that every such space had trivial fundamental group then as a mathematical tool the fundamental group would be as useful as a dead cat. Luckily this is not the case: many spaces are not simply connected. In this lecture we will perform our very first computation: the fundamental group of the circle S^1. At the end of the lecture we flip back into "abstract nonsense" mode and define the notion of a *pushout* in a category. We will need this next lecture when we prove the *Seifert-van Kampen Theorem*.

In this lecture it is convenient to regard S^1 as the unit circle in $\mathbb{C}$, i.e

$$S^1 = \{z \in \mathbb{C} \mid |z| = 1\}.$$

This is of course consistent with our previous definition under the identification of $\mathbb{C}$ with $\mathbb{R}^2$.

Definition 5.1. Define $\exp\colon \mathbb{R} \to S^1$ by

$$\exp(s) := e^{2\pi i s}.$$

We will always take 1 as the basepoint in S^1. We will use the fact that S^1 is itself a group: Given two points $z = e^{2\pi i s}$ and $w = e^{2\pi i t}$ in S^1, their product is $z \cdot w = e^{2\pi i (s+t)}$, and the inverse of z is just $e^{-2\pi i s}$. We say z and w are *antipodal* if $z \cdot w^{-1} = -1$.

Proposition 5.2. *Let $n \geq 0$ and let X be a compact convex subset of $\mathbb{R}^n$ and let $p \in X$. Suppose $f\colon (X, p) \to (S^1, 1)$ is a continuous map, and let $m \in \mathbb{Z}$. Then there exists a unique continuous map $\tilde{f}\colon (X, p) \to (\mathbb{R}, m)$ such that $\exp \circ \tilde{f} = f$:*

$$\begin{array}{ccc} & & (\mathbb{R}, m) \\ & \overset{\tilde{f}}{\nearrow} & \big\downarrow \exp \\ (X, p) & \xrightarrow[f]{} & (S^1, 1) \end{array}$$

Remark 5.3. We call $\tilde{f}$ a **lift** of f. Note that the requirement that m be an integer is forced: if $\tilde{f}(p) = s$ then asking that $\exp(s) = f(p) = 1$ implies that s is an integer.

Proof. Since X is a compact metric space, f is uniformly continuous, and hence there exists $\varepsilon > 0$ such that if $|x - y| < \varepsilon$ then $f(x)$ and $f(y)$ are not antipodal points. Since X is bounded there exists an integer N such that $|x - y| < N\varepsilon$ for all $x, y \in X$. Now for each $x \in X$, subdivide the line segment with endpoints p and x (which is contained in X by convexity) into N intervals of equal length. Call the endpoints of these endpoints $p = l_0(x), l_1(x), \dots, l_N(x) = x$. The functions $l_i \colon X \to X$ are continuous[1] and for each $0 \le i \le N-1$, the points $f(l_i(x))$ and $f(l_{i+1}(x))$ are not antipodal. This means that for each $0 \le i \le N-1$, the map

$$g_i \colon X \to S^1 \setminus \{-1\}, \qquad g_i(x) = f(l_i(x))^{-1} \cdot f(l_{i+1}(x))$$

is continuous (here we are using multiplication in S^1 as above). Note that $g_i(p) = 1$ for all i. Moreover since $l_N(x) = x$ for all x,

$$\begin{aligned} f(x) &= f(p) \cdot f(p)^{-1} \cdot f(l_1(x)) \cdot f(l_1(x))^{-1} \cdots f(l_{N-1}(x))^{-1} \cdot f(l_N(x)) \\ &= \underbrace{f(p)}_{=1} \cdot g_0(x) \cdot g_1(x) \cdots g_{N-1}(x). \end{aligned} \tag{5.1}$$

If we restrict the map exp to $(-\frac{1}{2}, \frac{1}{2})$ then it is a homeomorphsim onto $S^1 \setminus \{-1\}$; let us denote its inverse by Λ (actually $\Lambda = \frac{1}{2\pi i} \log$.) Then $\Lambda(1) = 0$. Since $g_i(x) \ne -1$ for all x, the function $\Lambda \circ g_i$ is defined and continuous. Now consider the function $\tilde{f} \colon X \to \mathbb{R}$ given by

$$\tilde{f}(x) := m + \sum_{i=0}^{N-1} \Lambda(g_i(x)).$$

Then $\tilde{f}$ is continuous, with $\tilde{f}(p) = m$ and from (5.1) one has

$$\exp \circ \tilde{f} = f.$$

It remains to prove $\tilde{f}$ is unique. Suppose $\tilde{g}$ was another such map with $\exp \circ \tilde{g} = f$ and $\tilde{g}(p) = m$. Define $\tilde{h}(x) = \tilde{f}(x) - \tilde{g}(x)$. Then $\tilde{h}$ is continuous and $\exp \circ \tilde{h}$ is identically equal to 1. Since $\exp \colon \mathbb{R} \to S^1$ is a homomorphism (thinking of both $\mathbb{R}$ and $\mathbb{Z}$ as groups) with kernel equal to $\mathbb{Z}$, the function $\tilde{h} \colon X \to \mathbb{R}$ is integer-valued. Since X is connected (as X is convex), $\tilde{h}$ must be constant. Since $\tilde{h}(p) = \tilde{f}(p) - \tilde{g}(p) = m - m = 0$, the constant must be zero. Thus $\tilde{f} = \tilde{g}$ and the proof is complete. ■

COROLLARY 5.4. *Let $u \colon I \to S^1$ be a loop with $u(0) = u(1) = 1$. Then there exists a unique lift $\tilde{u} \colon I \to \mathbb{R}$ (i.e. $\exp \circ \tilde{u} = u$) with $\tilde{u}(0) = 0$. Moreover if $v \colon (I, \partial I) \to (S^1, 1)$ is another path with $u \simeq v$ rel ∂I then if $\tilde{v}$ is the unique lift of v with $\tilde{v}(0) = 0$ then $\tilde{u} \simeq \tilde{v}$ rel ∂I. In particular, $\tilde{u}(1) = \tilde{v}(1)$.*

Proof. The first statement follows from Proposition 5.2 since I is a compact convex subset of $\mathbb{R}$. To prove the second statement, suppose $U \colon u \simeq v$ rel ∂I. Since $I \times I$ is a compact convex subset of $\mathbb{R}^2$, Proposition 5.2 provides us with a unique map $\tilde{U} \colon I \times I \to \mathbb{R}$ such that $\exp \circ \tilde{U} = U$ with $\tilde{U}(0,0) = 0$. We claim that

$$\tilde{U} \colon \tilde{u} \simeq \tilde{v} \qquad \text{rel } \partial I.$$

[1]Exercise: Check they really are continuous!

For this first note that if $\tilde{u}_1(s) := \tilde{U}(s,0)$ then $\tilde{u}_1$ is a lift of u with $\tilde{u}_1(0) = 0$; thus by the uniqueness part of Proposition 5.2, we must have $\tilde{u}_1 = \tilde{u}$. Similarly $\tilde{U}(s,1) = \tilde{v}(s)$. This shows that $\tilde{U}\colon \tilde{u} \simeq \tilde{v}$, and it remains to show that this homotopy is a homotopy relative to ∂I.

For this consider $\tilde{w}(t) := \tilde{U}(0,t)$. Then $\exp(\tilde{w}(t)) = \exp(\tilde{U}(0,t)) = U(0,t) = 1$, and hence arguing as above we see that $\tilde{w}(t) = 0$ for all t. Now consider $\tilde{w}_1(t) := \tilde{U}(1,t)$. Then $\exp \circ \tilde{w}_1 = 1$, and thus by uniqueness $\tilde{w}_1$ is a constant function. The constant is equal to $\tilde{U}(1,0) = \tilde{u}(1)$ (and also to $\tilde{U}(1,1) = \tilde{v}(1)$.) This completes the proof. ■

Definition 5.5. Given a loop $u\colon (I, \partial I) \to (S^1, 1)$, we define the **degree** of u to be the integer $\deg(u) = \tilde{u}(1)$, where $\tilde{u}$ is the unique lift of u with $\tilde{u}(0) = 0$.

The last statement of Corollary 5.4 tells us that deg induces a well defined map

$$\deg\colon \pi_1(S^1, 1) \to \mathbb{Z}, \qquad \deg\big([u]\big) := \deg(u),$$

where u is any representative of $[u]$.

We can now prove the main result of this lecture.

Theorem 5.6. *The function* $\deg\colon \pi_1(S^1,1) \to \mathbb{Z}$ *is an isomorphism. In particular,*

$$\deg\big([u] * [v]\big) = \deg\big([u]\big) + \deg\big([v]\big). \tag{5.2}$$

Proof. Firstly, deg is surjective since the loop $s \mapsto \exp(ms)$ has degree m (note this is the function $z \mapsto z^m$ if we think of u as a map from $S^1 \subset \mathbb{C}$ to itself). Secondly, if $\deg(u) = 0$ then $\tilde{u}$ is a loop in $\mathbb{R}$ based at 0. Now $\pi_1(\mathbb{R}, 0) = \{1\}$ by the last statement of Corollary 4.16. Since $\pi_1(\exp)\colon \pi_1(\mathbb{R}, 0) \to \pi_1(S^1, 1)$ is a homomorphism and

$$[u] = \pi_1(\exp)[\tilde{u}],$$

we thus have $[u] = 1$ in $\pi_1(S^1, 1)$.

To complete the proof we will show that deg is a homomorphism from $\pi_1(S^1, 1)$ to $\mathbb{Z}$, that is, that (5.2) holds. Then from the above its kernel is trivial, whence it follows that deg is injective, and hence an isomorphism. So suppose u and v are loops in S^1 based at 1 of degree m and n respectively. To compute $\deg(u * v)$, we must find a path $\tilde{w}\colon I \to \mathbb{R}$ with $\exp \circ \tilde{w} = u * v$. Let $\tilde{u}$ be the unique lift of u with $\tilde{u}(0) = 0$, and similarly for $\tilde{v}$. Then consider the path $\tilde{v}'\colon I \to \mathbb{R}$ given by $\tilde{v}'(s) = m + \tilde{v}(s)$. Then $\tilde{v}'$ is a path from m to $m + n$. Then $\tilde{w} := \tilde{u} * \tilde{v}'$ is a path in $\mathbb{R}$ with $\tilde{w}(0) = 0$ and $\tilde{w}(1) = m + n$. Note that

$$\exp(\tilde{w}(s)) = \begin{cases} \exp\big(\tilde{u}(2s)\big) & 0 \le s \le \frac{1}{2}, \\ \exp\big(\tilde{v}'(2s-1)\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

Since $\exp \circ \tilde{u} = u$ and since

$$\exp\big(\tilde{v}'(s)\big) = \exp\big(m + \tilde{v}(s)\big) = e^{2\pi i m} \exp(\tilde{v}(s)) = v(s),$$

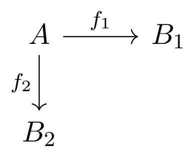
it follows that $\tilde{w}$ is a lift of $u * v$. Since $\tilde{w}(1) = m + n$, we thus have

$$m + n = \deg(u * v) = \deg(u) + \deg(v)$$

as required. This completes the proof. ■

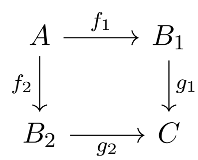
As promised, we conclude this lecture with some more abstract nonsense that we will need next time.

Definition 5.7. Suppose C is a category and A, B_1, B_2 are three objects in C, and $f_1\colon A \to B_1$ and $f_2\colon A \to B_2$ are two morphisms. A **diagram** in C is a picture[2] of the form:

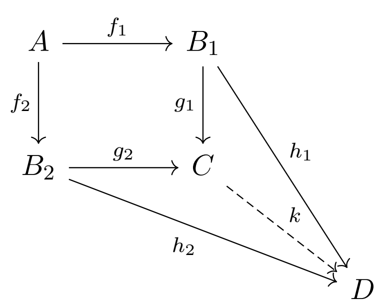


(δ)

A **solution** to the diagram (δ) is an object C together with two morphisms $g_1\colon B_1 \to C$ and $g_2\colon B_2 \to C$ such that the following diagram commutes:



A **pushout**[3] of the diagram (δ) is a solution (C, g_1, g_2) which satisfies the following **universal property**: if (D, h_1, h_2) is any other solution to (δ) then there is a *unique* morphism $k\colon C \to D$ such that the following diagram commutes:



(Δ)

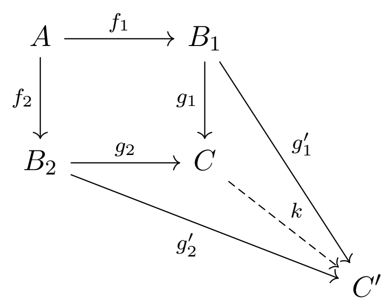
A pushout may or may not exist (it depends on the category C), but if it does then it is unique up to isomorphism.

Lemma 5.8. *If a pushout exists then it is unique up to isomorphism.*

Proof. Suppose (C, g_1, g_2) and (C', g_1', g_2') are two pushouts. Then applying diagram

[2] In Lecture 16 we will define a more general notion of a diagram which allows for pictures of a different shapes.
[3] We will generalise this in Lecture 16 when we introduce **colimits**.

(Δ) with $D = C'$ gives a morphism $k \colon C \to C'$ such that $k \circ g_1 = g_1'$ and $k \circ g_2 = g_2'$:



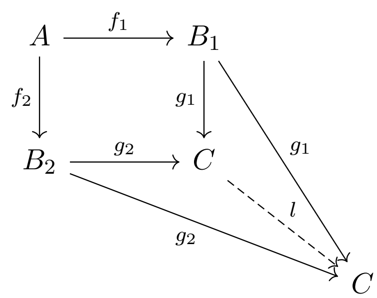
Reversing the roles of C and C' gives another morphism $k' \colon C' \to C$ such that $k' \circ g_1' = g_1$ and $k' \circ g_2' = g_2$. To complete the proof we claim that $k' \circ k = \mathrm{id}_C$ and $k \circ k' = \mathrm{id}_{C'}$. For this, first note that

$$k' \circ k \circ g_1 = k' \circ g_1' = g_1, \tag{5.3}$$

and similarly

$$k' \circ k \circ g_2 = k' \circ g_2' = g_2. \tag{5.4}$$

Now take $D = C$: then the universal property means there is a unique map $l \colon C \to C$ such that the following commutes:



By (5.3) and (5.4) taking $l = k' \circ k$ makes this diagram commute: hence by uniqueness this must be the map l:

$$l = k' \circ k.$$

But of course there is another map that also works: take $l = \mathrm{id}_C$! By uniqueness, it thus follows that

$$k' \circ k = \mathrm{id}_C.$$

Now, repeating this but with C' in both the two bottom right slots shows that $k \circ k' = \mathrm{id}_{C'}$. The proof is complete. ∎

Remark 5.9. Morally (we shall see many other examples of this throughout the course), whenever something is defined via a universal property[4] then uniqueness comes "for free". However, one still always needs to prove existence.

[4]Before you ask: Yes, it is possible to give a formal definition of exactly what a "universal property" is, but I'm not going to do so since it requires more category theory than one should use in polite company. Instead, just think of a universal property as meaning "making a diagram commute in the most efficient manner possible".

LECTURE 6

The Seifert-van Kampen Theorem

In this lecture we prove our first genuinely difficult result, called the *Seifert-van Kampen Theorem*. Roughly speaking, the Seifert-van Kampen Theorem allows us to "decompose" a topological space into smaller pieces, and compute the fundamental group of the full space in terms of the fundamental groups of the smaller pieces.

Let us begin by proving that in the category Groups, a pushout always exists.

DEFINITION 6.1. Let G and H be groups (not necessarily abelian). A **word** of length n in G and H is an expression of the form

$$s_1 s_2 \cdots s_n$$

where each s_i belongs to either G or H. A word can be **reduced** in two different ways:

1. If any s_i is equal to the identity element 1_G or 1_H, remove it.
2. If two consecutive elements s_i and s_{i+1} both belong to G (or both belong to H), then replace them by their product $s_i \cdot s_{i+1}$ as a single element of G (or H). This produces a word of length $n-1$.

After performing these operations as many times as possible, the word is necessarily an alternating product

$$g_1 h_1 g_2 h_2 \cdots g_m h_m,$$

where $g_i \in G$ and $h_i \in H$, and only g_1 or h_m is allowed to be the identity element. Such a word is then called a **reduced word**. The **free product** of G and H, written $G * H$, is the group whose elements are reduced words, and the product is given by concatenating followed by reduction.

In Problem Sheet C, you will show that the free product can also be characterised by a universal property.

PROPOSITION 6.2. *The pushout exists for the diagram (δ) in* Groups*. Indeed, suppose we are given groups G, H_1, H_2 and group homomorphisms ϕ_1, ϕ_2 as in diagram (δ):*

$$\begin{array}{ccc} G & \xrightarrow{\phi_1} & H_1 \\ {\scriptstyle \phi_2}\downarrow & & \\ H_2 & & \end{array}$$

Let N denote the normal subgroup of the free product $H_1 * H_2$ generated by all elements of the form $\phi_1(g^{-1}) \cdot \phi_2(g)$ for $g \in G$. Then the quotient group $K := (H_1 * H_2)/N$ is a pushout.

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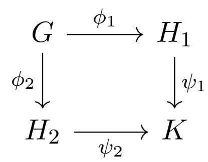
Proof. Define $\psi_i \colon H_i \to K$ by $\psi_i(h_i) = h_i \cdot N$. We claim that (K, ψ_1, ψ_2) is a solution, i.e. the following commutes:



For this, we must show that for all $g \in G$, as cosets in K, one has

$$\phi_1(g) \cdot N = \phi_2(g) \cdot N,$$

or equivalently that

$$\phi_1(g)^{-1} \cdot \phi_2(g) \cdot N = N.$$

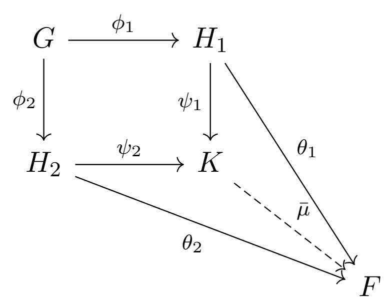
Since ϕ_1 is a homomorphism,

$$\phi_1(g)^{-1} = \phi_1(g^{-1}).$$

Since $\phi_1(g^{-1}) \cdot \phi_2(g) \in N$ for all $g \in G$, the claim follows. Now suppose (F, θ_1, θ_2) is another solution. The definition of the free product provides a unique homomorphism[1]

$$\mu \colon H_1 * H_2 \to F$$

such that $\mu|_{H_i} = \theta_i$. Since $\theta_2 \circ \phi_2 = \theta_1 \circ \phi_1$, it follows that $N \leq \ker \mu$, and hence μ induces a unique homomorphism $\bar{\mu} \colon K \to F$ such that the diagram (Δ) commutes:



Thus K is a pushout, as claimed. ∎

Definition 6.3. We call the group K the **free product with amalgamation** of $\phi_1 \colon G \to H_1$ and $\phi_2 \colon G \to H_2$ and write $K = H_1 *_G H_2$. This notation is a little imprecise, since K depends on the homomorphisms ϕ_1 and ϕ_2.

Corollary 6.4. *If $H_2 = \{1\}$ is the trivial group then the free product with amalgamation is given by H_1/N, where N is the normal subgroup generated by $\phi_1(G)$.*

With these group-theoretic preliminaries out of the way, we can finally state the main result of today's lecture.

[1] If you are confused exactly how μ is defined, I invite you to look at Problem C.1.

Theorem 6.5 (The Seifert-van Kampen Theorem). *Let $X = X_1 \cup X_2$ with X_1 and X_2 open subsets. Assume X_1, X_2 and $X_0 := X_1 \cap X_2$ are all non-empty and path connected. Let*

$$\imath_i \colon X_0 \hookrightarrow X_i, \qquad \jmath_i \colon X_i \hookrightarrow X$$

denote inclusions for $i = 1, 2$. Let $p \in X_0$. Then the fundamental group $\pi_1(X, p)$ is the free product with amalgamation of the group homomorphisms $\pi_1(\imath_1) \colon \pi_1(X_0, p) \to \pi_1(X_1, p)$ and $\pi_1(\imath_2) \colon \pi_1(X_0, p) \to \pi_1(X_2, p)$:

$$\pi_1(X, p) \cong \pi_1(X_1, p) *_{\pi_1(X_0,p)} \pi_1(X_2, p).$$

Before giving the proof, let us note three useful special cases of Theorem 6.5.

Corollary 6.6. *Under the assumptions of Theorem 6.5, one has:*

1. *If X_2 is simply connected then*

$$\pi_1(\jmath_1) \colon \pi_1(X_1, p) \to \pi_1(X, p)$$

is a surjection with kernel the normal subgroup generated by $\pi_1(\imath_1)\big(\pi_1(X_0, p)\big)$.

2. *If X_0 is simply connected then $\pi_1(X, p)$ is the free product of $\pi_1(X_1, p)$ and $\pi_1(X_2, p)$.*
3. *If X_2 and X_0 are simply connected then*

$$\pi_1(\jmath_1) \colon \pi_1(X_1, p) \to \pi_1(X, p)$$

is an isomorphism.

We will need the following piece of point-set topology in the course of the proof of Theorem 6.5.

Lemma 6.7 (The Lebesgue Number Lemma[2]). *Let (X, d) be a compact metric space. Suppose $\mathcal{U}$ is an open cover of X. Then there exists $\delta > 0$, called a* **Lebesgue number** *for $\mathcal{U}$ such that every subset A of X with diameter less than δ is contained in some element of $\mathcal{U}$.*

Proof of Theorem 6.5. Consider the diagram:

$$\begin{array}{ccc} \pi_1(X_0, p) & \xrightarrow{\pi_1(\imath_1)} & \pi_1(X_1, p) \\ {\scriptstyle \pi_1(\imath_2)} \big\downarrow & & \\ \pi_1(X_2, p) & & \end{array}$$

We will show that $(\pi_1(X, p), \pi_1(\jmath_1), \pi_1(\jmath_2))$ is a pushout. Since we already know that a pushout is unique by Lemma 5.8, and since in Groups the pushout is given by the

[2]See here for a proof.

free product with amalgamation by Proposition 6.2, it thus follows that $\pi_1(X,p)$ must be this product:

$$\pi_1(X,p) \cong \pi_1(X_1,p) *_{\pi_1(X_0,p)} \pi_1(X_2,p).$$

It is clear that $(\pi_1(X,p), \pi_1(\jmath_1), \pi_1(\jmath_2))$ is a solution, i.e. that the following commutes:

$$\begin{CD} \pi_1(X_0,p) @>{\pi_1(\imath_1)}>> \pi_1(X_1,p) \\ @V{\pi_1(\imath_2)}VV @VV{\pi_1(\jmath_1)}V \\ \pi_1(X_2,p) @>>{\pi_1(\jmath_2)}> \pi_1(X,p) \end{CD}$$

So suppose (G,ϕ_1,ϕ_2) is another solution:

$$\begin{CD} \pi_1(X_0,p) @>{\pi_1(\imath_1)}>> \pi_1(X_1,p) \\ @V{\pi_1(\imath_2)}VV @VV{\phi_1}V \\ \pi_1(X_2,p) @>>{\phi_2}> G \end{CD} \tag{6.1}$$

We will construct a unique homomorphism $\psi\colon \pi_1(X,p) \to G$ such that the following diagram commutes.

$$\begin{tikzcd} \pi_1(X_0,p) \arrow[r, "\pi_1(\imath_1)"] \arrow[d, "\pi_1(\imath_2)"'] & \pi_1(X_1,p) \arrow[d, "\pi_1(\jmath_1)"'] \arrow[ddr, "\phi_1"] & \\ \pi_1(X_2,p) \arrow[r, "\pi_1(\jmath_2)"] \arrow[drr, "\phi_2"'] & \pi_1(X,p) \arrow[dr, dashed, "\psi"] & \\ & & G \end{tikzcd} \tag{6.2}$$

Suppose u is a loop in X at p. Since X_1 and X_2 are open, we can find a finite set of points $p = x_0, x_1, \ldots, x_n = p$ along u with the property that each x_i lies in X_0 and each segment of u from x_{i-1} to x_i lies in either X_1 or X_2. Let $v_i\colon I \to X$ denote the path obtained by reparametrising the segment of u_i from x_{i-1} to x_i. Explicitly, if $s_{i-1} < s_i$ are such that $u(s_{i-1}) = x_{i-1}$ and $u(s_i) = x_i$, then

$$v_i(s) = u\big((1-s)s_{i-1} + s\,s_i\big).$$

Now, for each i, select a path w_i in X_0 from p to x_i (take w_0 and w_n to be the constant path e_p.) Then by concatenating we obtain loops

$$w_{i-1} * v_i * \bar{w}_i$$

which lies entirely in either X_1 or X_2. See Figure 6.1. Each loop therefore defines an

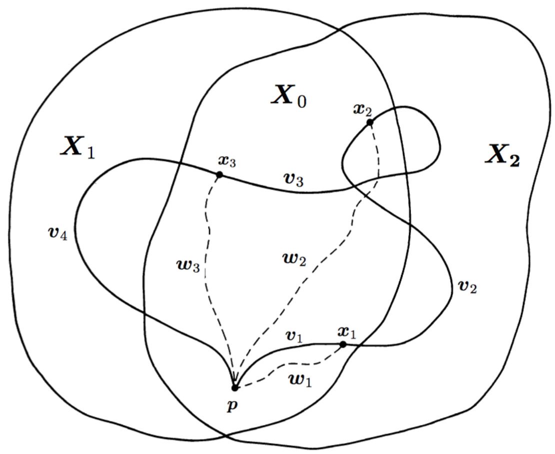


Figure 6.1: Breaking u into smaller pieces.

element either of $\pi_1(X_1, p)$ or $\pi_1(X_2, p)$. Note that

$$u \simeq \prod_{i=1}^{n} \big(w_{i-1} * v_i * \bar{w}_i\big), \qquad \text{rel } \partial I. \tag{6.3}$$

We now define our desired map ψ by

$$\psi[u] := \phi_*[w_0 * v_1 * \bar{w}_1] \cdot \phi_*[w_1 * v_2 * \bar{w}_2] \cdots \phi_*[w_{n-1} * v_n * \bar{w}_n],$$

where ϕ_* means either ϕ_1 or ϕ_2 depending on whether $[w_{i-1} * v_i * \bar{w}_i]$ belongs to $\pi_1(X_1, p)$ or $\pi_1(X_2, p)$. Note that there is a potential ambiguity if $w_{i-1} * v_i * \bar{w}_i$ lies in both X_1 and X_2, because we could choose either ϕ_1 or ϕ_2. But in this case $[w_{i-1} * v_i * \bar{w}_i]$ lies in $\pi_1(X_0, p)$, and commutativity of the diagram (6.1) shows that we get the same result.

This defines the map ψ, but there are still many things we need to check before the proof is complete:

1. Is the definition of ψ independent of the points x_i and the paths w_i?
2. Does ψ make sense on $\pi_1(X, p)$? That is, if u_1 and u_2 are two homotopic loops rel p do we get the same answer if we use u_1 and u_2 in the definition of ψ?
3. Is ψ a homomorphism?
4. Does the diagram (6.2) commute?
5. Is ψ unique with respect to these properties?

Let's pretend for a second that we've already proved (1) and (2). The last three properties are then easy: (3) and (4) follow by construction, and uniqueness comes from the fact that if a loop v is entirely contained in X_1 then requiring (6.2) to commutes means we are *forced* to define $\psi[v] = \phi_1[v]$ (and similarly for X_2), and thus (6.3) means that if we want ψ to be a group homomorphism we have no choice but to define it as we have done.

So now let's get on with proving the first two parts, starting with (1). Consider a point x_i, and suppose w_i' is another path from p to x_i. See Figure 6.2. Then we have

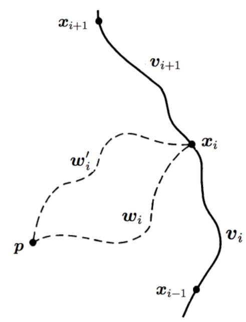


Figure 6.2: w_i and w_i'.

$$\phi_*[w_i * v_{i+1} * \bar{w}_{i+1}] = \phi_*[w_i * \bar{w}_i' * w_i' * v_{i+1} * \bar{w}_{i+1}] = \phi_*[w_i * \bar{w}_i'] \cdot \phi_*[w_i' * v_{i+1} * \bar{w}_{i+1}].$$

On the other hand,

$$\begin{aligned}\phi_*[w_{i-1} * v_i * \bar{w}_i] &= \phi_*[w_{i-1} * v_i * \bar{w}_i' * w_i' * \bar{w}_i] \\ &= \phi_*[w_{i-1} * v_i * \bar{w}_i'] \cdot \phi_*[w_i' * \bar{w}_i] \\ &= \phi_*[w_{i-1} * v_i * \bar{w}_i'] \cdot \left(\phi_*[w_i * \bar{w}_i']\right)^{-1}\end{aligned}$$

Since $\bar{w}_i * w_i'$ is a loop based in X_0 and since the diagram (6.1) commutes, the value of ϕ_* on this loop will be the same whether ϕ_* is ϕ_1 or ϕ_2. Thus

$$\phi_*[w_{i-1} * v_i * \bar{w}_i] \cdot \phi_*[w_i * v_{i+1} * \bar{w}_{i+1}] = \phi_*[w_{i-1} * v_i * \bar{w}_i'] \cdot \phi_*[w_i' * v_{i+1} * \bar{w}_{i+1}]$$

This means that the product won't change when w_i' is used instead of w_i. Repeating this argument at each point x_i shows that $\psi[u]$ does not depend on the choice of the paths w_i.

Now we show independence of the points x_i. Suppose another point $y \in X_0$ is added along v_i separating the path v_i into two new paths v_i' and v_{i-1}'. See Figure 6.3. Let w' denote a path from p to y in X_0. Suppose for definiteness that the loop $w_{i-1} * v_i * \bar{w}_i$ is contained in X_1. Then the same is true of the two new loops $w_{i-1} * v_{i-1}' * \bar{w}'$ and $w' * v_i' * \bar{w}_i$, and we have

$$\begin{aligned}\phi_1[w_{i-1} * v_{i-1}' * \bar{w}'] \cdot \phi_1[w' * v_i' * \bar{w}_i] &= \phi_1[w_{i-1} * v_{i-1}' * \bar{w}' * w' * v_i' * \bar{w}_i] \\ &= \phi_1[w_{i-1} * v_i * \bar{w}_i].\end{aligned}$$

This shows that adding an additional point to the set of $\{x_i\}$ does not change the value of $\psi[u]$. More generally, the same is true if we add a finite number of points, that is, refining the $\{x_i\}$ leads to the same result. Now suppose we are given two different sets $\{x_i\}$ and $\{y_i\}$ of points. Their union is a common refinement of both the $\{x_i\}$ and the $\{y_i\}$, and we have just shown that the value of $\psi[u]$ doesn't change

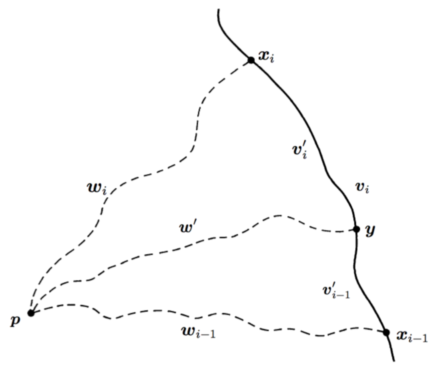


Figure 6.3: The new point y.

when refining the set of points. Thus the value of $\psi[u]$ is the same if we use either the $\{x_i\}$ or the $\{y_i\}$. This proves (1).

Now let us move onto the proof of (2). Suppose u and u' are two homotopic loops and $U\colon u \simeq u'$ is a homotopy rel ∂I. We subdivide the square $I \times I$ into lots of little squares in such a way that each smaller square is mapped by U into either X_1 or X_2. Such a decomposition exists by Lemma 6.7, where we take the open cover of $I \times I$ given by the connected components of $U^{-1}(X_1)$ and $U^{-1}(X_2)$. See Figure 6.4. Proceeding one small rectangle at a time, this deforms u into u' through a finite

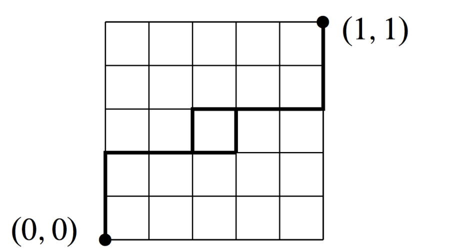


Figure 6.4: Subdividing $I \times I$.

sequence of paths such that each step involves a homotopy in which the only change occurs within either X_1 or X_2. For such a restricted deformation, the points $\{x_i\}$ may be chosen so that the value of ψ is unchanged. This proves (2), and thus completes the proof.

■

LECTURE 7

Singular homology

In this lecture we finally get started on defining the *homology functors* H_n referred to in Lecture 1. Let us begin with some preliminaries on free abelian groups.

Definition 7.1. Let B be a subset of an abelian group F. We say F is **free abelian** with **basis** B if the subgroup generated by b is infinite cyclic for each $b \in B$ and $F = \bigoplus_{b \in B} \langle b \rangle$ as a direct sum.

Thus a free abelian group is a (possibly uncountable) direct sum of copies of $\mathbb{Z}$. A typical element $x \in F$ has a unique expression

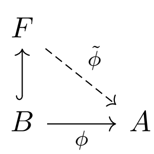
$$x = \sum_{b \in B} m_b \, b, \qquad m_b \in \mathbb{Z}$$

where **almost all** (meaning all but a finite number) of the m_b are zero. The following trivial lemma will be crucial in all that follows.

Lemma 7.2. *Let F be a free abelian group with basis B. If A is an abelian group and $\phi\colon B \to A$ is a function then there exists a unique group homomorphism $\tilde{\phi}\colon F \to A$ such that*

$$\tilde{\phi}(b) = \phi(b), \qquad \forall\, b \in B,$$

that is, the following diagram commutes:



Moreover any abelian group A is isomorphic to a quotient group of the form F/R, where F is a free abelian group.

Proof. Define $\tilde{\phi}$ by

$$\tilde{\phi}\left(\sum_{b \in B} m_b \, b \right) := \sum_{b \in B} m_b \, \phi(b).$$

Then $\tilde{\phi}$ is well-defined since any element of F has a unique expression of this form, and it is obviously a homomorphism. Moreover $\tilde{\phi}$ is unique since any two homomorphisms that agree on a set of generators (in this case B) must coincide. The last statement is on Problem Sheet D. ■

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We refer to the extension $\phi \mapsto \tilde{\phi}$ given in Lemma 7.2 as **extending by linearity**. By an abuse of notation, we will typically continue to write ϕ for the extension, rather than $\tilde{\phi}$.

LEMMA 7.3. *Given any set B, there exists a free abelian group having B as basis.*

Proof. If $B = \emptyset$, take $F = 0$. Otherwise, for each $b \in B$, let $\mathbb{Z}_b$ be a group whose elements are all symbols mb with $m \in \mathbb{Z}$ and addition defined by $mb+nb = (m+n)b$. Then $\mathbb{Z}_b$ is infinite cyclic with generator b. Now set

$$F := \bigoplus_{b \in B} \mathbb{Z}_b.$$

This is a free abelian group with basis given by the set $\{e_b \mid b \in B\}$, where e_b has a zero in each entry apart from the bth entry, where it is a 1. Identifying e_b with b, we see that F has basis B. ■

DEFINITION 7.4. The **rank** of a free abelian group F is the cardinality of any basis of B of F.

This is well-defined thanks to Problem D.1. Moreover two free abelian groups are isomorphic if and only if they have the same rank.

REMARK 7.5. One can extend the notion of rank to any abelian group: if G is an arbitrary abelian group then we say G has (possibly infinite) **rank** r if there exists a free abelian subgroup F of G such that F has rank r and G/F is torsion. Such subgroups F always exist (this is also part of Problem D.1). However it is not obvious that this definition is well-defined. Indeed, F is not unique, and it is by no means clear that the rank of F only depends on G. At the very end of the course we will develop one way of proving this.

Now let us define the notion of a simplex.

DEFINITION 7.6. An ordered tuple $(z_0, z_1, \dots, z_n)$ of points in $\mathbb{R}^m$ is said to be **affinely independent** if the set $\{z_1 - z_0, z_2 - z_0, \dots, z_n - z_0\}$ is linearly independent (thus necessarily $n \leq m$). Given an affinely independent tuple $(z_0, z_1, \dots, z_n)$ of vectors in $\mathbb{R}^m$, we denote by $[z_0, z_1, \dots, z_n]$ the n**-simplex spanned by** $(z_0, z_1, \dots, z_n)$, namely the set

$$[z_0, z_1, \dots, z_n] := \left\{ x \in \mathbb{R}^m \mid x = \sum_{i=0}^{n} s_i\, z_i, \text{ where } 0 \leq s_i \leq 1, \ \sum_{i=0}^{n} s_i = 1 \right\}.$$

We call the points z_i the **vertices** of the n-simplex $[z_0, z_1, \dots, z_n]$. The expression $x = \sum_{i=0}^{n} s_i\, z_i$ of any point $x \in [z_0, z_1, \dots, z_n]$ is unique[1]. We call the $(n+1)$-tuple $(s_0, s_1, \dots, s_m)$ the **barycentric coordinates** of x. The **barycentre** of the n-simplex $[z_0, z_1, \dots, z_n]$ is the unique point where all the s_i are equal, namely

$$\frac{1}{n+1}(z_0 + z_1 + \cdots + z_n). \tag{7.1}$$

[1]Exercise: Why?

Definition 7.7. Let $[z_0, z_1, \dots, z_n]$ be an n-simplex. The **face opposite to** z_i is the $(n-1)$-simplex[2] $[z_0, \dots \hat{z_i}, \dots, z_n]$. Here the circumflex $\hat{}$ means[3] "delete". Equivalently

$$[z_0, \dots \hat{z_i}, \dots, z_n] := \{x \in [z_0, z_1, \dots, z_n] \mid s_i = 0.\}.$$

An n-simplex thus has $n+1$ faces. The **boundary** of an n-simplex is the union of its faces.

Definition 7.8. The **standard n-simplex** in $\mathbb{R}^{n+1}$ is the n-simplex $[e_0, e_1, \dots, e_n]$, where e_i is the vector coordinates are all zero, apart from the $i+1$st position, which is 1. We denote the standard n-simplex by Δ^n.

So much for a simplex in $\mathbb{R}^{n+1}$. What about in an arbitrary topological space X?

Definition 7.9. Let X be a topological space. A **singular n-simplex in X** is a continuous map $\sigma \colon \Delta^n \to X$.

Since Δ^0 is a point, a 0-simplex in X is simply a point in X. Since Δ^1 is a closed interval, a 1-simplex is[4] the same thing as a path in X. The adjective "singular" is added to emphasis that the image $\sigma(\Delta^n)$ does not need to "look" anything like Δ^n, i.e. we do *not* require σ to be a homeomorphism. In particular, there is nothing stopping σ being a constant map.

Definition 7.10. Let X be a topological space and $n \geq 0$. Let $C_n(X)$ denote the free abelian group with basis the singular n-simplices in X (cf. Lemma 7.3.) We call an element of $C_n(X)$ a **singular n-chain**. It is convenient for notational reasons to also define $C_{-1}(X) = 0$.

Note that (as a group), $C_n(X)$ is typically huge: if X is an uncountable set then $C_n(X)$ is itself uncountable for all $n \geq 0$. We will shortly replace $C_n(X)$ with a (usually smaller) abelian group $H_n(X)$. First, let us explain how to obtain a singular $(n-1)$-simplex from a singular n-simplex.

Definition 7.11. Let $\sigma \colon \Delta^n \to X$ be a singular n-simplex in X. If we restrict σ to one of the faces of Δ^n, we get a continuous map from an $n-1$-simplex into X.

Actually this definition is cheating a little bit; whilst any face of Δ^n is an $(n-1)$-simplex, it is not the *standard* $(n-1)$-simplex, since the domain is wrong. Thus strictly speaking, the restriction of a n-simplex σ in X to a face is not actually a singular $(n-1)$-simplex in X, since it is not a continuous map from Δ^{n-1} into X. There are two ways round this tedious pedantry:

1. Ignore it. After all, it's clear what we mean.
2. Fix it by making the notation more complicated.

[2] This is clearly an $(n-1)$-simplex as a subset of a linearly independent set is also linearly independent.
[3] This is a convention we will use throughout the course.
[4] Not quite! We will come back to this in Lecture 9.

We shall go[5] for option (2). To this end, let us define the i**th face map**

$$\varepsilon_i \colon \Delta^{n-1} \to \Delta^n, \qquad i = 0, 1, \dots, n$$

that maps the standard $(n-1)$-simplex Δ^{n-1} homeomorphically onto the ith face of Δ^n. Explicitly,

$$\varepsilon_0(s_0, s_1, \dots, s_{n-1}) = (0, s_0, s_1, \dots, s_{n-1}),$$

for $i = 0$, and for $1 \leq i \leq n-1$,

$$\varepsilon_i(s_0, s_1, \dots, s_{n-1}) = (s_0, s_1, \dots, s_{i-1}, 0, s_i, \dots, s_{n-1}),$$

and finally

$$\varepsilon_n(s_0, s_1, \dots, s_{n-1}) = (s_0, s_1, \dots, s_{n-1}, 0).$$

Where necessary we will write $\varepsilon_i^n \colon \Delta^{n-1} \to \Delta^n$ (this is needed for instance in (7.2) below).

We can now "improve" Definition 7.11:

Definition 7.12. Let $\sigma \colon \Delta^n \to X$ be a singular n-simplex in X and let $0 \leq i \leq n$. The composition $\sigma \circ \varepsilon_i \colon \Delta^{n-1} \to X$ is then a singular $(n-1)$-simplex in X, which we call the **restriction of σ to the ith face**.

We can now define the boundary of a singular n-simplex.

Definition 7.13. Let $\sigma \colon \Delta^n \to X$ be a singular n-simplex in X. The **boundary** of σ is the alternating sum of the restriction of σ to the faces:

$$\partial\sigma := \sum_{i=0}^{n} (-1)^i \sigma \circ \varepsilon_i.$$

Thus the boundary of σ is *not* a singular $(n-1)$-simplex, but rather a formal *sum* of singular $(n-1)$-simplices, and hence (by definition) a singular $(n-1)$-chain: $\partial\sigma \in C_{n-1}(X)$. We define the boundary of a singular 0-simplex to be zero.

Remark 7.14. If we omit the face maps (which we will occasionally do, cf. in Proposition 8.5 next lecture), the formula is slightly more intuitive (albeit formally incorrect):

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_n]}.$$

Applying Lemma 7.2 we obtain a well defined map on the free abelian group $C_n(X)$.

[5] Being pedantic is an important quality for a mathematician to have (or at least, to pretend to have when teaching others ...)

Definition 7.15. The **singular boundary operator**

$$\partial\colon C_n(X) \to C_{n-1}(X)$$

is the unique homomorphism extending the operator from Definition 7.13. Occasionally for clarity we will write $\partial_n\colon C_n(X) \to C_{n-1}(X)$.

Thus for each $n \geq 0$ we have constructed a sequence of free abelian groups and homomorphisms. We illustrate this pictorially as

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial} C_{n-1}(X) \longrightarrow \cdots \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \longrightarrow 0.$$

Anticipating the category Comp of *chain complexes* that we will introduce in Lecture 10, we will bundle all the groups $C_n(X)$ together and write $(C_\bullet(X), \partial)$ to denote all the groups and maps at once.

Proposition 7.16. *$\partial^2 = 0$, that is, for any $n \geq 0$ the composition*

$$C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X)$$

is always zero.

Proof. Since $C_{n+1}(X)$ is generated by all the $(n+1)$-simplices, by Lemma 7.2 it suffices to show that if $\sigma\colon \Delta^{n+1} \to X$ is a singular $(n+1)$-simplex then $\partial^2\sigma = 0$. As you can probably guess, the point it that since the boundary operator was defined via an alternating sum, when you apply it twice things cancel. Indeed, if $k < j$ then one has the following *face relation*:

$$\varepsilon_j^{n+1} \circ \varepsilon_k^n = \varepsilon_k^{n+1} \circ \varepsilon_{j-1}^n \colon \Delta^{n-1} \to \Delta^{n+1}. \tag{7.2}$$

To prove (7.2), it suffices to observe that both sides give the same answer when fed a vertex e_i for $i = 0, 1, \dots, n-1$. Now we compute:

$$\begin{aligned}
\partial^2\sigma &= \partial\left(\sum_{j=0}^{n+1}(-1)^j \sigma \circ \varepsilon_j^{n+1}\right) \\
&= \sum_{k=0}^{n}\sum_{j=0}^{n+1}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n \\
&= \underbrace{\sum_{j \leq k}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n}_{(*)} + \underbrace{\sum_{k<j}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n}_{(\dagger)}.
\end{aligned}$$

We claim that the two terms $(*)$ and $(\dagger)$ cancel. Indeed, to see this first apply (7.2) to $(**)$ and change variables by setting $l = k$ and $m = j - 1$ to obtain:

$$\sum_{k<j}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n = \sum_{k<j}(-1)^{j+k}\sigma \circ \varepsilon_k^{n+1} \circ \varepsilon_{j-1}^n = \sum_{l \leq m}(-1)^{l+m+1}\sigma \circ \varepsilon_l^{n+1} \circ \varepsilon_m^n.$$

The last expression is the same as $(*)$, only every term appears with the opposite sign. This completes the proof. ■

Definition 7.17. A **singular n-cycle** in X is a singular n-chain that lies in the kernel of ∂. We denote by $Z_n(X)$ the set of all singular n-cycles. A **singular n-boundary** in X is a singular n-chain that lies in the image of ∂. We denote by $B_n(X)$ the set of all singular n-boundaries[6]. Both $Z_n(X)$ and $B_n(X)$ are subgroups of $C_n(X)$. Moreover since $\partial^2 = 0$, we have

$$B_n(X) \subseteq Z_n(X) \subseteq C_n(X).$$

We can therefore form the quotient group. This will be the eponymous *singular homology*.

Definition 7.18. We define the n**-singular homology** group of X, written $H_n(X)$, to be the quotient group

$$H_n(X) = Z_n(X) \Big/ B_n(X).$$

Thus $H_n(X)$ is an abelian (not free abelian!) group for each n. Given a singular n-cycle c, we denote[7] by $\langle c \rangle$ the coset $c + B_n(X) \in H_n(X)$ and call $\langle c \rangle$ the **homology class** determined by c.

We will conclude this lecture by showing that H_n is a functor. This means that we need to associate to each continuous map $f \colon X \to Y$ a homomorphism $H_n(f) \colon H_n(X) \to H_n(Y)$.

Definition 7.19. If $f \colon X \to Y$ is a continuous map and $\sigma \colon \Delta^n \to X$ is a singular n-simplex in X then $f \circ \sigma \colon \Delta^n \to Y$ is a singular n-simplex in Y. We therefore obtain an induced map $f_\# \colon C_n(X) \to C_n(Y)$ by extending this by linearity (Lemma 7.2:

$$f_\# \left(\sum m_\sigma \, \sigma \right) := \sum m_\sigma f \circ \sigma.$$

We can think of $f_\#$ as defining being a map $f_\# \colon C_\bullet(X) \to C_\bullet(X)$. You will not be surprised to learn that the operation $X \mapsto C_\bullet(X)$ and $f \mapsto f_\#$ defines a functor[8]. We will study this in Lecture 10 (it's a functor $\mathsf{Top} \to \mathsf{Comp}$.) For now though, let us prove that $f_\#$ descends to the quotient to define a map on $H_n(X) \to H_n(Y)$. This is the content of the following proposition.

Proposition 7.20. *If $f \colon X \to Y$ is continuous, then the following diagrams commutes for every n:*

$$\begin{array}{ccc} C_n(X) & \xrightarrow{\partial} & C_{n-1}(X) \\ {\scriptstyle f_\#}\downarrow & & \downarrow{\scriptstyle f_\#} \\ C_n(Y) & \xrightarrow[\partial]{} & C_{n-1}(Y) \end{array}$$

[6]Boundary begins with a "b", hence the notation $B_n(X)$. Similarly cycle begins with a "c", hence the notation ... wait a second ... Damnit, we already used $C_n(X)$ for the chain groups! Next best option: *Zykel* begins with a "z" ...

[7]We use angle brackets $\langle \cdot \rangle$ rather than square brackets $[\cdot]$ to distinguish between homotopy and homology classes.

[8]It would be more logical to write $C_\bullet(f)$ instead of $f_\#$, but this is too messy.

Proof. It suffice to evaluate both $f_\# \circ \partial$ and $\partial \circ f_\#$ on a singular n-simplex σ in X. Now

$$f_\# \circ \partial\sigma = f_\# \left(\sum_{i=0}^{n} (-1)^i \sigma \circ \varepsilon_i \right) = \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ \varepsilon_i.$$

Similarly

$$\partial \circ f_\#\sigma = \partial(f \circ \sigma) = \sum_{i=0}^{n} (-1)^i f \circ \sigma \circ \varepsilon_i.$$

■

Corollary 7.21. *If $f\colon X \to Y$ is continuous then both $f_\#(Z_n(X)) \subseteq Z_n(Y)$ and $f_\#(B_n(X)) \subseteq B_n(Y)$. Thus $f_\#$ induces a map $H_n(f)\colon H_n(X) \to H_n(Y)$.*

Proof. If $\partial c = 0$ then $\partial(f_\# c) = f_\#(\partial c) = 0$, so that $f_\# c \in Z_n(Y)$. Similarly if $b = \partial c$ then $f_\# b = f_\#(\partial c) = \partial(f_\# c)$, so that $f_\# b \in B_n(Y)$. ■

Thus $f_\#$ induces a map $H_n(f)\colon H_n(X) \to H_n(Y)$, given by

$$H_n(f)\langle c\rangle := \langle f_\# c\rangle.$$

Corollary 7.22. *For each $n \geq 0$, $H_n\colon \mathsf{Top} \to \mathsf{Ab}$ is a functor.*

Proof. We need only check that $H_n(g \circ f) = H_n(g) \circ H_n(f)$ and that $H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$. Both of these are immediate from the definitions. ■

Corollary 7.23. *If X and Y are homeomorphic then $H_n(X) \cong H_n(Y)$ for all $n \geq 0$.*

Proof. Immediate from Problem A.2. ■

Thinking back to Lecture 1, we have now constructed the singular homology functors from Theorem 1.15. In order for our proof of the Brouwer Fixed Point Theorem 1.1 to be complete, we need to verify that $H_n(B^{n+1}) = 0$ and $H_n(S^n) \neq 0$. We will prove that $H_n(B^{n+1}) = 0$ next lecture; the fact that $H_n(S^n) \neq 0$ will take much longer (Lecture 15).

LECTURE 8

The homotopy axiom

In this lecture we prove that if $f, g\colon X \to Y$ are homotopic maps then the induced maps $H_n(f)$ and $H_n(g)$ coincide for every $n \geq 0$:

$$[f] = [g] \qquad \Rightarrow \qquad H_n(f) = H_n(g) \qquad \forall n \geq 0. \tag{8.1}$$

we will prove this in Theorem 8.9 below. By Problem A.2, this means that H_n may be regarded as a functor $H_n\colon \mathsf{hTop} \to \mathsf{Ab}$. This should be compared to π_1: we initially defined π_1 as a functor $\mathsf{Top}_* \to \mathsf{Groups}$ and then later showed that π_1 induces a functor from hTop_* to Groups. The property (8.1) is usually called the **homotopy axiom**. This terminology will be explained at the end of the course when we cover the *Eilenberg-Steenrod axioms*.

We begin by stating two elementary properties of singular homology, both of which appear on Problem Sheet D.

Proposition 8.1 (The dimension axiom). *Let X be a one-point space $\{*\}$. Then $H_n(X) = 0$ for all $n > 0$.*

Just as with the homotopy axiom (8.1), the meaning of the name "dimension axiom" in Proposition 8.1 will get explained later. For the next result, let us recall that if $\{G_\lambda \mid \lambda \in \Lambda\}$ is a collection of groups, an element of $\bigoplus_{\lambda\in\Lambda} G_\lambda$ is a tuple (g_λ) where all but finitely many of the g_λ are equal to the identity.

Proposition 8.2. *Let X be a topological space. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ denote the path components of X. Then for every $n \geq 0$ one has*

$$H_n(X) \cong \bigoplus_{\lambda\in\Lambda} H_n(X_\lambda).$$

Thus, just as with the fundamental group, for computational purposes we may always assume our spaces are path connected. In general, it is very hard to compute $H_n(X)$ for $n > 0$, but it is always possible to compute $H_0(X)$.

Proposition 8.3. *If X is a non-empty path connected space then $H_0(X) = \mathbb{Z}$. A generator is given by $\langle x\rangle$ for any point $x \in X$, and if $x, y \in X$ then $\langle x\rangle = \langle y\rangle$. Moreover, if $\langle c\rangle$ is any generator then $\langle c\rangle = \langle x\rangle$ for some (and hence every) $x \in X$.*

Proof. We identify a 0-singular simplex in X with a point in X. Since $\partial\colon C_0(X) \to 0$ is the zero map, every point in X is a 0-cycle: $Z_0(X) = C_0(X)$. Thus each point

$x \in X$ determines a class $\langle x \rangle \in H_0(X)$. Let us now identify $B_0(X)$. A typical element of $c \in C_0(X)$ is of the form

$$c = \sum_{x \in X} m_x\, x, \qquad m_x \in \mathbb{Z},$$

where all but finitely many of the m_x are equal to zero. We claim that:

$$B_0(X) = \left\{ \sum m_x\, x \mid \sum m_x = 0 \right\}. \tag{8.2}$$

Firstly, suppose $c = \sum_{i=1}^n m_i\, x_i$ satisfies $\sum_{i=1}^n m_i = 0$. We wish to build a singular 1-chain a such that $\partial a = c$. For this, choose a point $p \in X$ and for each $i = 1, \dots, n$, let $u_i \colon I \to X$ denote a path starting at p and ending at x_i. After identifying $I = [0, 1]$ with $\Delta^1 = [e_0, e_1]$, we may regard each u_i as a singular 1-simplex $\sigma_i \colon \Delta^1 \to X$ such that $\sigma_i(e_0) = p$ and $\sigma_i(e_1) = x_i$. Note that

$$\partial \sigma_i = \sigma_i(e_1) - \sigma_i(e_0) = x_i - p \in C_0(X).$$

Now set $a := \sum_{i=1}^n m_i \sigma_i$. Then

$$\begin{aligned} \partial a &= \partial \left(\sum_{i=1}^n m_i\, \sigma_i \right) \\ &= \sum_{i=1}^n m_i\, \partial \sigma_i \\ &= \sum_{i=1}^n m_i x_i - \left(\sum_{i=1}^n m_i \right) p \\ &= c - 0 = c. \end{aligned}$$

Conversely, suppose $d \in B_0(X)$. Then there exists $b \in C_1(X)$ such that $\partial b = d$. Write $b = \sum_{j=1}^k l_j\, \tau_j$, where $\tau_j \colon \Delta^1 \to X$ and $l_i \in \mathbb{Z}$. Then

$$d = \sum_{j=1}^k l_j \big(\tau_j(e_1) - \tau_j(e_0) \big).$$

Thus in the expansion of d, each coefficient l_j appears twice and with the opposite sign. Thus the sum of the coefficients of d is zero. This proves (8.2).

Thus by (8.2), the map

$$\phi \colon Z_0(X) = C_0(X) \to \mathbb{Z}, \qquad \phi \left(\sum_{x \in X} m_x\, x \right) := \sum_{x \in X} m_x \tag{8.3}$$

is a surjection whose kernel is precisely $B_0(X)$. Thus $H_0(X) \cong \operatorname{im} \phi = \mathbb{Z}$.

Now suppose $x, y \in X$. A path from x to y determines a singular 1-simplex σ with $\partial \sigma = y - x$. Thus $\langle x \rangle = \langle y \rangle \in H_0(X)$. Finally suppose $a = \sum_i m_i\, x_i$ is a 0-cycle such that $\langle a \rangle$ is a generator of $H_0(X)$. Then we must have $\phi(a) = \pm 1$. Replacing a with $-a$ if necessary, we may assume that $\phi(a) = 1$. Thus $\sum_i m_i = 1$. Then for any point $x \in X$, we have $a = x + (a - x)$. Since $a - x \in B_0(X)$ by (8.2) we therefore have $\langle x \rangle = \langle a \rangle$. This completes the proof. ■

An immediate corollary of Proposition 8.3 is:

Corollary 8.4. *Let X and Y be path connected spaces and $f\colon X \to Y$ continuous. Then $H_0(f)\colon H_0(X) \to H_0(Y)$ maps a generator of $H_0(X)$ to a generator of $H_0(Y)$.*

The main step in the proof of the homotopy axiom is the following innocuous looking statement.

Proposition 8.5. *Let X be a topological space and define inclusions $\imath, \jmath\colon X \hookrightarrow X \times I$ by*

$$\imath(x) := (x, 0), \qquad \jmath(x) := (x, 1).$$

Then

$$H_n(\imath) = H_n(\jmath), \qquad \forall\, n \geq 0.$$

Remark 8.6. There is a very cute three line proof which uses an abstract result in homological algebra called the **Acyclic Models Theorem**. We will prove this right at the end of the course, and we will then come back to Proposition 8.5 and give a second proof. Therefore don't worry if you find the proof of Proposition 8.5 below horrible; we will eventually see a nicer one.

In order to prove Proposition 8.5 we introduce the idea of a **chain homotopy**.

Lemma 8.7. *Let $f, g\colon X \to Y$ be continuous maps. Assume for each*[1] *$n \geq -1$ there is a homomorphism*

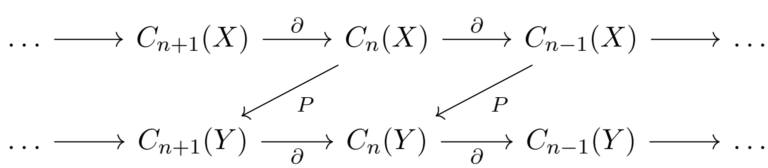
$$P\colon C_n(X) \to C_{n+1}(Y)$$

with

$$f_\# - g_\# = \partial P + P\partial.$$

Then $H_n(f) = H_n(g)$ for all $n \geq 0$.

The maps P_n look like this:



Beware though, this diagram is *not* commutative! Just as with the ∂ maps, sometimes for clarity we will include the subscript and write $P_n\colon C_n(X) \to C_{n+1}(Y)$. In Lecture 10 we will define an abstract version of the operators P, which will then be called **chain homotopies**.

Proof. Take $c \in Z_n(X)$. Then

$$(f_\# - g_\#)c = (\partial P + P\partial)c = \partial P c \in B_n(Y).$$

Thus $H_n(f)\langle c \rangle = H_n(g)\langle c \rangle$. ■

[1] The map $P_{-1}\colon 0 = C_{-1}(X) \to C_0(Y)$ is necessarily the zero map.

Let us now prove Proposition 8.5. The first step is the following lemma.

LEMMA 8.8. *$\Delta^n \times I$ is the union of $n+1$ copies of Δ^{n+1}.*

Proof. For $i = -1, 0, 1, \dots, n-1$, let $g_i \colon \Delta^n \to I$ denote the map

$$g_i(s_0, s_1, \dots, s_n) = \sum_{j>i} s_j.$$

Note g_i is indeed a map to I, as $x = \sum s_i e_i$ implies $\sum s_i = 1$. Let $G_i \subset \Delta^n \times I$ denote the graph of g_i. Then G_i is homeomorphic to Δ^n via the projection $\Delta^n \times I \to \Delta^n$ onto the first factor. Let us now label the vertices at the "bottom" (i.e. $\Delta^n \times \{0\}$) of $\Delta^n \times I$ by $e_0', e_1', \dots, e_n'$ and those at the "top" by $e_0'', e_1'', \dots, e_n''$. Then G_i is the n-simplex

$$G_i = [e_0', \dots, e_i', e_{i+1}'', \dots, e_n''].$$

Since G_i lies below G_{i-1} as $g_i \leq g_{i-1}$, it follows that the region between G_i and G_{i-1} is the $(n+1)$-simplex $[e_0', \dots, e_i', e_i'', \dots, e_n'']$; this is indeed an $(n+1)$-simplex as e_i'' is not in G_i and hence not in the n-simplex $[e_0', \dots, e_i', e_{i+1}'', \dots, e_n'']$. Since $0 = g_n \leq g_{n-1} \leq \cdots \leq g_0 \leq g_{-1} = 1$, we see that $\Delta^n \times I$ is the union of the regions between the G_i, and hence the union of $n+1$ different $(n+1)$-simplices $[e_0', \dots, e_i', e_i'', \dots, e_n'']$, each intersecting the next in an n-simplex face. ∎

We now prove Proposition 8.5.

Proof of Proposition 8.5. We will break with our convention here since otherwise the notation becomes too messy. If $\sigma : \Delta^n \to X$ is a singular n-simplex, then we denote by

$$(\sigma \times \mathrm{id}_I)|_{[e_0', \dots, e_i', e_i'', \dots, e_n'']}$$

the singular $(n+1)$-simplex obtained from the previous lemma by restricting $\sigma \times \mathrm{id}_I \colon \Delta^n \times I \to X \times I$ to the $(n+1)$-simplex $[e_0', \dots, e_i', e_i'', \dots, e_n'']$. Of course, we should really precompose with an appropriate face map $\Delta^{n+1} \to [e_0', \dots, e_i', e_i'', \dots, e_n'']$ in order to make $\sigma \times \mathrm{id}_I|_{[e_0', \dots, e_i', e_i'', \dots, e_n'']}$ into a genuine singular $(n+1)$-simplex, however this is too cumbersome (the notation is already bad enough as it is!)

With this in mind, we define a homomorphism

$$P \colon C_n(X) \to C_{n+1}(X \times I)$$

by requiring that

$$P\sigma = \sum_{i=0}^{n} (-1)^i (\sigma \times \mathrm{id}_I)|_{[e_0', \dots, e_i', e_i'', \dots, e_n'']}$$

and then extending by linearity (cf. Lemma 7.2.)

Now let us look at $\partial P\sigma$, remembering that the $\hat{}$ notation over a vertex means "delete" (see also Remark 7.14):

$$\begin{aligned} \partial P\sigma = &\sum_{j \leq i} (-1)^{i+j} (\sigma \times \mathrm{id}_I)|_{[e_0', \dots, \hat{e}_j', \dots, e_i', e_i'', \dots, e_n'']} \\ &+ \sum_{i \leq j} (-1)^{i+j+1} (\sigma \times \mathrm{id}_I)|_{[e_0', \dots, e_i', e_i'', \dots, \hat{e}_j'', \dots e_n'']}. \end{aligned}$$

The terms with $i = j$ all cancel[2] apart from the first and last ones:

$$(\sigma \times \mathrm{id}_I)|_{[\hat{e}'_0, e''_0, \dots, e''_n]} \qquad \text{and} \qquad -(\sigma \times \mathrm{id}_I)|_{[e'_0, \dots, e'_n, \hat{e}''_n]},$$

which are precisely $\jmath_\# \sigma$ and $-\imath_\# \sigma$ respectively. Meanwhile the terms with $i \neq j$ are precisely $-P\partial\sigma$, since

$$\begin{aligned} P\partial\sigma &= \sum_{i=0}^{n} (-1)^i P\sigma|_{[e_0, \dots, \hat{e}_i, \dots, e_n]} \\ &= \sum_{j<i} (-1)^{i+j-1} (\sigma \times \mathrm{id}_I)|_{[e'_0, \dots, \hat{e}'_j, \dots, e'_i, e''_i, \dots, e''_n]} \\ &\quad + \sum_{i<j} (-1)^{i+j} (\sigma \times \mathrm{id}_I)|_{[e'_0, \dots, e'_i, e''_i, \dots, \hat{e}''_j, \dots e''_n]}. \end{aligned}$$

Putting this altogether we obtain

$$\partial P\sigma = \jmath_\# \sigma - \imath_\# \sigma - P\partial\sigma.$$

The same is true for any chain $c \in C_n(X)$ by Lemma 7.2, and hence Lemma 8.7 completes the proof. ∎

We can now prove (8.1).

THEOREM 8.9 (The homotopy axiom). *Let $f, g\colon X \to Y$ be two homotopic maps. Then $H_n(f) = H_n(g)$ for all $n \geq 0$. Thus for each $n \geq 0$, $H_n\colon \mathsf{hTop} \to \mathsf{Ab}$ is a functor.*

Proof. Let $F\colon f \simeq g$ be a homotopy. Then using the maps $\imath$ and $\jmath$ from Proposition 8.5, we have

$$f = F \circ \imath, \qquad g = F \circ \jmath.$$

Thus as H_n is a functor, we have

$$H_n(f) = H_n(F \circ \imath) = H_n(F) \circ H_n(\imath) = H_n(F) \circ H_n(\jmath) = H_n(g).$$

∎

In the same way as Corollary 7.23, we obtain the following result.

COROLLARY 8.10. *If X and Y have the same homotopy type then $H_n(X) \cong H_n(Y)$ for all $n \geq 0$, where the isomorphism is induced by any homotopy equivalence.*

Proof. Immediate from Problem A.2. ∎

We also have:

COROLLARY 8.11. *If X is contractible then $H_n(X) = 0$ for all $n > 0$.*

Proof. Immediate from the previous corollary and the dimension axiom (Proposition 8.1.) ∎

[2] Remember we are "hiding" the face maps with this notation ...

LECTURE 9

The Hurewicz Theorem

In this lecture we investigate the relationship between $H_1(X)$ and $\pi_1(X, p)$. It should come as no surprise that there is one, since Δ^1 is homeomorphic to an interval, but at the same time they cannot be identical, since $\pi_1(X, p)$ is not necessarily abelian (cf. Problem C.2 and Problem C.3), meanwhile by definition $H_1(X)$ always is.

We begin by defining a function $h\colon \pi_1(X, p) \to H_1(X)$ called the **Hurewicz map**. We will then prove that h is a surjective homomorphism and identify its kernel.

REMARK 9.1. We have already implicitly used the fact that Δ^1 and I are homeomorphic, and thus a singular 1-simplex is the same thing as a path. However in this lecture it is important to keep track of whether we are working with a singular 1-simplex or a path. To this end, let $\theta\colon \Delta^1 \to I$ denote the homeomorphism that sends

$$\Delta^1 \ni \sum_{i=0}^{1} s_i\, e_i \mapsto s_1 \in I.$$

We will use the following slightly imprecise convention: if $u\colon I \to X$ is a path, then $u' := u \circ \theta\colon \Delta^1 \to X$ is a singular 1-simplex. Explicitly,

$$u'(s_0, s_1) = u(s_1).$$

With this convention, if u and v are two paths such that $u(1) = v(0)$ then the concatenated path $u * v$ becomes:

$$(u * v)'(s_0, s_1) = \begin{cases} u'(2s_0 - 1, 2s_1), & 0 \le s_1 \le \frac{1}{2}, \\ v'(2s_0, 2s_1 - 1), & \frac{1}{2} \le s_1 \le 1. \end{cases} \tag{9.1}$$

Conversely if $\sigma\colon \Delta^1 \to X$ is a singular 1-simplex then $\sigma' := \sigma \circ \theta^{-1}\colon I \to X$ is a path. Explicitly

$$\sigma'(s) = \sigma(1 - s, s).$$

The imprecise bit is that the $'$ denotes either composition with θ or θ^{-1} depending on whether we start with a path or a simplex). Since we always use u, v, w for paths and σ, τ for simplices, this should not be too confusing.

PROPOSITION 9.2. *Let $p \in X$. There is a well defined function $h\colon \pi_1(X, p) \to H_1(X)$ given by*

$$[u] \mapsto \langle u' \rangle,$$

where u is a loop in X based at p.

Where necessary, we will write h_p instead of h to indicate the dependence on p.

Proof. Clearly $u' = u \circ \theta$ is a singular 1-simplex in X, and thus in particular belongs to $C_1(X)$. In fact, $u' \in Z_1(X)$, since

$$\partial u' = u(\theta(e_1)) - u(\theta(e_0)) = u(1) - u(0) = 0.$$

Thus $\langle u' \rangle$ is a well defined element in $H_1(X)$. Now recall the map $\omega : I \to S^1$ from the solution to Problem B.5 given by $\omega(s) = e^{2\pi i s}$. From u we obtain the map $\hat{u} : S^1 \to X$ given by $\hat{u} = u \circ \omega^{-1}$. The map $\hat{u}$ then induces a map $H_1(\hat{u}) : H_1(S^1) \to H_1(X)$. Then using the fact that H_1 is a functor, we have

$$\langle u' \rangle = \langle \hat{u} \circ \omega \circ \theta \rangle = H_1(\hat{u}) \langle \omega \circ \theta \rangle,$$

as elements of $H_1(X)$. Here we view $\omega \circ \theta : \Delta^1 \to S^1$ as a singular 1-simplex in S^1. Now if v is another closed path in X based at p with $u \simeq v$ rel ∂I then by the solution to Problem B.5 the corresponding maps $\hat{u}$ and $\hat{v}$ are homotopic rel 1. Thus by the homotopy axiom of singular homology, $H_1(\hat{u}) = H_1(\hat{v})$. Thus

$$\langle u' \rangle = H_1(\hat{u}) \langle \omega \circ \theta \rangle = H_1(\hat{v}) \langle \omega \circ \theta \rangle = \langle v' \rangle,$$

which shows that the class $\langle u' \rangle$ only depends on $[u]$. This completes the proof. ■

Now let us prove that h is a homomorphism of groups. Let us emphasise (we have already been doing this, but you may not have noticed) that we are using *additive* notation for homology classes (this makes sense because the homology groups are always abelian).

PROPOSITION 9.3. *The Hurewicz map $h : \pi_1(X, p) \to H_1(X)$ is a group homomorphism: for all $[u], [v] \in \pi_1(X, p)$, we have*

$$h([u]) + h([v]) = h([u * v]).$$

Proof. Let u and v be loops in X based at p. Define a continuous map $\sigma \colon \Delta^2 \to X$ as indicated by Figure 9.1. Specifically, we define σ to be u', v', and $(u * v)'$ on the boundary of $\partial\Delta^2$ as the picture suggests:

$$\sigma(1-s, s, 0) := u(s), \qquad \sigma(0, 1-s, s) := v(s), \qquad \sigma(1-s, 0, s) := (u * v)(s).$$

For fixed s, we then define σ on the interior of Δ^2 to be constant on the line segments from $a = a(s)$ to $b = b(s)$ and from $c = c(s)$ to $d = d(s)$. Here $a(s) = (1-s, s, 0)$, $b(s) = \left(\frac{1}{2}(2-s), 0, \frac{s}{2}\right)$, $c(s) = (0, 1-s, s)$ and $d(s) = \left(\frac{1}{2}(1-s), 0, \frac{1}{2}(1+s)\right)$. The gluing lemma shows that σ is continuous, and hence is a singular 2-simplex in X. An explicit formula for σ using (9.1) is

$$\sigma(s_0, s_1, s_2) := (u * v)' \left(s_0 + \frac{s_1}{2}, \frac{s_1}{2} + s_2 \right).$$

Now observe that

$$\partial\sigma = \sigma \circ \varepsilon_0 - \sigma \circ \varepsilon_1 + \sigma \circ \varepsilon_2 = v' - (u * v)' + u'.$$

Thus in $H_1(X)$, we have

$$\langle u' \rangle + \langle v' \rangle = \langle (u * v)' \rangle,$$

or equivalently

$$h([u]) + h([v]) = h([u * v]),$$

which shows that h is a homomorphism as desired. ■

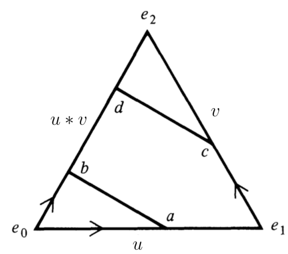


Figure 9.1: Proving h is a homomorphism.

We now present the following simple corollary of Proposition 2.15, which we will need when examining the kernel of h.

PROPOSITION 9.4. *Let $\sigma \colon \Delta^2 \to X$ be a singular 2-simplex. Abbreviate $\sigma_i := \sigma \circ \varepsilon_i$ for $i = 0, 1, 2$, so that σ_i is a singular 1-simplex in X. See Figure 9.2. Then the path $\sigma_0' * \bar{\sigma}_1' * \sigma_2'$ is nullhomotopic rel ∂I.*

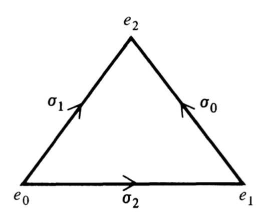


Figure 9.2: $\sigma \colon \Delta^2 \to X$

Proof. By Problem D.2, the map σ induces a map $g : B^2 \to X$, given by $g(x) = \sigma(h^{-1}(x))$, where $h : \Delta^2 \to B^2$ is the homeomorphism mapping $(\Delta^2, \partial\Delta^2) \to (B^2, S^1)$. Let $f := g|_{S^1}$, then f is a reparametrisation of the loop $\sigma_0' * \bar{\sigma}_1' * \sigma_2'$. By Proposition 2.15, f is nullhomotopic rel 1 in S^1. Thus $\sigma_0' * \bar{\sigma}_1' * \sigma_2'$ is nullhomotopic rel ∂I. ■

We will also need the following (trivial) piece of algebra. The lemma is basically a fancy way of saying "if something looks like it should cancel, then it does". The important thing in the following lemma is that the second group A is abelian.

Lemma 9.5 (Substitution Principle). *Let F be a free abelian group with basis B. Let $b_0, b_1, \dots, b_k$ be a list (possibly with repetitions) of elements of B, and assume $m_0, m_1, \dots, m_k$ are integers such that*

$$m_0\, b_0 = \sum_{i=1}^{k} m_i\, b_i.$$

Let A be an abelian group, and suppose $a_0, a_1, \dots, a_k$ are elements of A such that

$$b_i = b_j \qquad \Rightarrow \qquad a_i = a_j.$$

Then in A, one also has

$$m_0\, a_0 = \sum_{i=1}^{k} m_i\, a_i.$$

Proof. Define a function $\varphi : B \to A$ by setting $\varphi(b_i) = a_i$ for $i = 0, 1, \dots, k$ and $\varphi(b) = 0$ for all other elements of B. This is well defined by assumption. Then by Lemma 7.2, there exists a unique group homomorphism $\tilde{\varphi} : F \to A$ extending φ. Then

$$0 = \tilde{\varphi}\left(m_0\, b_0 - \sum_{i=1}^{k} m_i\, b_i \right) = m_0\, a_0 - \sum_{i=1}^{k} m_i\, a_i,$$

■

Now let us recall a standard piece of group theory.

Definition 9.6. Let G be any group (not necessarily abelian). Given $g, h \in G$, we define their **commutator** to be the element

$$[g, h] := ghg^{-1}h^{-1}.$$

We define the **commutator subgroup** of G to be the subgroup $[G, G]$ of G generated by all the commutators. This is a normal abelian subgroup of G, and $[G, G] = \{1\}$ (where 1 is the identity element of G) if and only if G is abelian. If N is a normal subgroup of G, then G/N is abelian if and only if $[G, G] \le N$. We define the **abelianisation** of G to be the quotient group $G^{\mathrm{ab}} = G\big/[G, G]$. As you will see on Problem E.1 on Problem Sheet E, the abelianisation G^{ab} together with the group homomorphism (the projection) $p\colon G \to G^{\mathrm{ab}}$ can be characterised by a universal property: namely that if A is any abelian group and $\varphi\colon G \to A$ is any group homomorphism, there exists a *unique* homomorphism $\tilde{\varphi}$ such that the following commutes:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & A \\ {\scriptstyle p}\downarrow & \nearrow_{\tilde{\varphi}} & \\ G^{\mathrm{ab}} & & \end{array}$$

We are now ready to state and prove the main result of today's lecture. As stated, this result is actually due to Poincaré, not Hurewicz, but there is a more general theorem[1] that extends this which is due to Hurewicz, and hence this result is commonly called the "Hurewicz Theorem".

[1] We will discuss the more general version at the end of Algebraic Topology II (Theorem 46.1.

THEOREM 9.7 (Hurewicz Theorem). *Let X be a path connected topological space. Then the induced homomorphism $\tilde{h} : \pi_1(X,p)^{\text{ab}} \to H_1(X)$ is a group isomorphism.*

Proof. Since X is path connected, choose a path $w_x : I \to X$ such that $w_x(0) = p$ and $w_x(1) = x$ for each $x \in X$. Let us insist that w_p is the constant path e_p.

We will first show that h is surjective. Suppose

$$c = \sum_{i=1}^{k} m_i \, \sigma_i \in Z_1(X).$$

Since c is a cycle, we have

$$0 = \partial c = \sum_{i=1}^{k} m_i\big(\sigma_i(e_1) - \sigma_i(e_0)\big), \tag{9.2}$$

which we view as an equation among the basis elements (i.e. points in X) of the free abelian group $C_0(X)$. Set

$$y_i := \sigma_i(e_1), \qquad z_i := \sigma_i(e_0).$$

See Figure 9.3. We can apply the substitution principle to the list

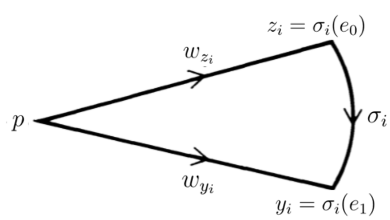


Figure 9.3: Proving h is surjective.

$$y_1, z_1, \ldots, y_k, z_k,$$

in the free abelian group $C_0(X)$ and compare it to the list

$$w'_{y_1}, w'_{z_1}, \ldots, w'_{y_k}, w'_{z_k},$$

in $C_1(X)$. Then (9.2) tells us[2] that

$$\sum_{i=1}^{k} m_i\big(w'_{y_i} - w'_{z_i}\big) = 0$$

[2]Less formally, this is simply the observation that since c is a cycle, the sum of the paths w_{y_i} and $\bar{w}_{z_i}$ cancel.

in $C_1(X)$. Thus

$$c = c - 0 = c - \left(\sum_{i=1}^{k} m_i \big(w'_{y_i} - w'_{z_i} \big) \right) = \sum_{i=1}^{k} m_i \big(w'_{z_i} + \sigma_i - w'_{y_i} \big). \tag{9.3}$$

But now by assumption $w_{z_i} * \sigma'_i * \bar{w}_{y_i}$ is a *loop* in X based at p, and hence we can feed it to h:

$$\begin{aligned} h\left(\prod_{i=1}^{k} \Big[w_{z_i} * \sigma'_i * \bar{w}_{y_i} \Big]^{m_i} \right) &\overset{(*)}{=} \sum_{i=1}^{k} m_i h \Big[w_{z_i} * \sigma'_i * \bar{w}_{y_i} \Big] \\ &\overset{(\dagger)}{=} \sum_{i=1}^{k} m_i \langle w'_{z_i} + \sigma_i + \bar{w}'_{y_i} \rangle \\ &\overset{(\ddagger)}{=} \sum_{i=1}^{k} m_i \langle w'_{z_i} + \sigma_i - w'_{y_i} \rangle \\ &\overset{(\heartsuit)}{=} \langle c \rangle. \end{aligned}$$

where $(*)$ used Proposition 9.3, $(\dagger)$ used Problem E.4, $(\ddagger)$ used Problem E.3 and finally $(\heartsuit)$ used (9.3). This shows that h is surjective.

We now build an inverse $\tilde{\eta} \colon H_1(X) \to \pi_1(X,p)^{\text{ab}}$ to the induced map $\tilde{h}$, which will prove that $\tilde{h}$ is an isomorphism. Suppose $\sigma \colon \Delta^1 \to X$ is a 1-simplex. We associate to σ the class in $\pi_1(X,p)^{\text{ab}}$ represented by the loop $w_{\sigma(e_0)} * \sigma' * \bar{w}_{\sigma(e_1)}$. Extend this by linearity to define a map

$$\eta \colon C_1(X) \to \pi_1(X,p)^{\text{ab}}.$$

We claim that η vanishes on $B_1(X)$. For this let $\tau : \Delta^2 \to X$ be a singular 2-simplex. We compute $\eta(\partial\tau)$. As in Proposition 9.4, let $\tau_i := \tau \circ \varepsilon_i$ for $i = 0, 1, 2$. See Figure 9.4. Then

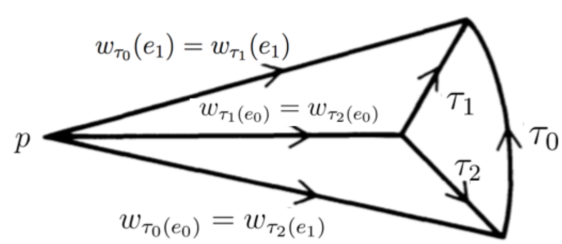


Figure 9.4: Showing $\eta(\partial\tau) = 1$.

$$
\begin{aligned}
\eta(\partial\tau) &= \eta(\tau_0) * \eta(\tau_1)^{-1} * \eta(\tau_2) \\
&= \left[w_{\tau_0(e_0)} * \tau_0' * \bar{w}_{\tau_0(e_1)}\right] * \left[w_{\tau_1(e_0)} * \tau_1' * \bar{w}_{\tau_1(e_1)}\right]^{-1} * \left[w_{\tau_2(e_0)} * \tau_2' * \bar{w}_{\tau_2(e_1)}\right] \\
&= \left[w_{\tau_0(e_0)} * \tau_0' * \bar{\tau}_1' * \tau_2' * \bar{w}_{\tau_2(e_1)}\right] \\
&\overset{(*)}{=} \left[w_{\tau_0(e_0)} * \bar{w}_{\tau_2(e_1)}\right] \\
&= 1,
\end{aligned}
$$

where $(*)$ used Proposition 9.4. Thus $\eta|_{Z_1(X)}$ factors to define a map $\tilde{\eta} : H_1(X) \to \pi_1(X, p)^{\mathrm{ab}}$. By construction $\tilde{\eta} \circ \tilde{h} = \mathrm{id}$. This completes the proof. ∎

LECTURE 10

Chain complexes

In this lecture we introduce a new category, Comp, and a new subject: **homological algebra**.

DEFINITION 10.1. A **chain complex** is a sequence of abelian groups and homomorphisms

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \dots$$

for $n \in \mathbb{Z}$ which satisfies

$$\partial^2 = 0, \qquad \forall\, n \in \mathbb{Z}.$$

We refer[1] to the entire complex as $(C_\bullet, \partial)$ or sometimes just $C_\bullet$. The maps ∂ are called the **boundary operators** of the chain complex.

Of course, we have already met one key example:

EXAMPLE 10.2. Let X be a topological space. Then the singular chains $(C_\bullet(X), \partial)$ is a chain complex. Note that in this example the abelian groups are all zero for negative subscripts; this however is not part of the definition in general.

DEFINITION 10.3. The fact that $\partial^2 = 0$ means that if we define

$$Z_n = Z_n(C_\bullet) = \ker \partial \colon C_n \to C_{n-1}$$

and

$$B_n = B_n(C_\bullet) = \operatorname{im} \partial \colon C_{n+1} \to C_n$$

then

$$B_n \subseteq Z_n.$$

By analogy with the singular chain complex, we call elements of Z_n n**-cycles** and elements of B_n n**-boundaries**. We define the n**th homology group** of the chain complex $C_\bullet$ to be the quotient group

$$H_n = H_n(C_\bullet) := Z_n(C_\bullet) \big/ B_n(C_\bullet).$$

We will continue to the use the notation $\langle c \rangle$ to denote the class of an element $c \in Z_n$ in H_n.

Now let us introduce another key notion.

[1]Yes, I know it is somewhat illogical to omit the subscript on the ∂ and not on the C, but in practice it makes things more convenient.

Definition 10.4. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of two homomorphisms of abelian groups is said to be **exact at** B if

$$\operatorname{im} f = \ker g.$$

More generally, a sequence

$$\dots A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \dots, \qquad n \in \mathbb{Z}$$

is said to be **exact** if it is exact at every A_n.

Example 10.5. Using the notion of exactness we can rephrase other definitions from basic algebra. Suppose $f\colon A \to B$ is a homomorphism of abelian groups.

- f is injective if and only if $0 \to A \xrightarrow{f} B$ is exact.
- f is surjective if and only if $A \xrightarrow{f} B \to 0$ is exact.
- f is an isomorphism if and only if $0 \to A \xrightarrow{f} B \to 0$ is exact.
- Slightly less obviously, if $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ is exact then f is surjective if and only if h is injective.

Definition 10.6. A **short exact sequence** of abelian groups is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

In this case, $A \cong \operatorname{im} f$ and $\operatorname{coker} f := B/\operatorname{im} f \cong C$ via $b + \operatorname{im} f \mapsto g(b)$.

In contrast, a **long exact sequence** is one that has (potentially) infinitely many terms.

Definition 10.7. A chain complex C is said to be **acyclic** if $C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1}$ is exact at C_n for all n.

We then have trivially:

Proposition 10.8. *A chain complex C is acyclic if and only if $H_n(C) = 0$ for all $n \in \mathbb{Z}$.*

Let us now make the chain complexes into a category.

Definition 10.9. Suppose $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ are two chain complexes. A **chain map** $f\colon C_\bullet \to C'_\bullet$ is a sequence of group homomorphisms $f_n\colon C_n \to C'_n$ such that the following diagram commutes for all $n \in \mathbb{Z}$:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \longrightarrow & \dots \\
 & & \big\downarrow{\scriptstyle f_{n+1}} & & \big\downarrow{\scriptstyle f_n} & & \big\downarrow{\scriptstyle f_{n-1}} & & \\
\dots & \longrightarrow & C'_{n+1} & \xrightarrow[\partial']{} & C'_n & \xrightarrow[\partial']{} & C'_{n-1} & \longrightarrow & \dots
\end{array}$$

Usually we will just write f for the maps f_n (to minimise the number of subscripts involved), which simplifies the formula to

$$\partial' \circ f = f \circ \partial.$$

Composition of two chain maps is defined as one would guess: if $f\colon C_\bullet \to C'_\bullet$ and $g\colon C'_\bullet \to C''_\bullet$ are two chain maps then $g \circ f\colon C_\bullet \to C''_\bullet$ is the chain map given by $(g \circ f)_n = g_n f_n$. This is indeed a valid chain map, i.e. $\partial''(g \circ f)_n = (g \circ f)_{n-1} \circ \partial$, as the following diagram commutes:

$$\begin{array}{ccccc} C_n & \xrightarrow{f_n} & C'_n & \xrightarrow{g_n} & C''_n \\ \partial \downarrow & & \downarrow \partial' & & \downarrow \partial'' \\ C_{n-1} & \xrightarrow[f_{n-1}]{} & C'_{n-1} & \xrightarrow[g_{n-1}]{} & C''_{n-1} \end{array}$$

Example 10.10. If $f\colon X \to Y$ is a continuous map between two topological spaces then $f_\#\colon C_\bullet(X) \to C_\bullet(Y)$ is a chain map.

Definition 10.11. The category Comp has objects the chain complexes $(C_\bullet, \partial)$, morphisms the chain maps $f\colon C_\bullet \to C'_\bullet$, and composition as specified above.

The singular chain complex can now be interpreted as a functor.

Proposition 10.12. *There is a functor* $\mathsf{Top} \to \mathsf{Comp}$ *that associates to a topological space* X *its singular chain complex* $C_\bullet(X)$ *and to a continuous map* $f\colon X \to Y$ *the associated map* $f_\#\colon C_\bullet(X) \to C_\bullet(Y)$.

Proof. This follows from the results of Lecture 7. ■

The reason for insisting that $\partial' f = f\partial$ in the definition of a chain map is that it means a chain map induces a map on the respective homologies:

$$H_n(f)\colon H_n(C_\bullet) \to H_n(C'_\bullet)$$

given by

$$H_n(f)\langle c\rangle := \langle f_n c\rangle.$$

This is well defined as by assumption

$$f_n(Z_n) \subseteq Z'_n, \qquad f_n(B_n) \subseteq B'_n.$$

This means that we can interpret H_n as a functor.

Proposition 10.13. *For each* $n \in \mathbb{Z}$*, there exists a functor* $H_n\colon \mathsf{Comp} \to \mathsf{Ab}$ *called the* n**th homology functor** *that sends* $C_\bullet$ *to* $H_n(C_\bullet)$ *and to a chain map* $f\colon C_\bullet \to C'_\bullet$ *the associated map* $H_n(f)\colon H_n(C_\bullet) \to H_n(C'_\bullet)$.

The proof is immediate. This means we can now see the construction of the singular homology as a two-stage process. The first is *topological*: this is assignment $X \mapsto C_\bullet(X)$. The second is *purely algebraic*: this is the assignment $C_\bullet(X) \mapsto H_n(C_\bullet(X)) = H_n(X)$.

Proposition 10.14. *The functor* $H_n\colon \mathsf{Comp} \to \mathsf{Ab}$ *is an* **additive functor**[2], *that is* $H_n(f + g) = H_n(f) + H_n(g)$.

[2] I will not give the precise definition of an additive functor, since we do not need it. It's more complicated then you think, since in order to define additive functors one first needs to define additive categories ...

Now let me bombard you with more definitions. They are all straightforward.

DEFINITION 10.15. A **subcomplex** $(C_\bullet, \partial)$ of chain complex $(C'_\bullet, \partial')$ is a chain complex such that $C_n \subseteq C'_n$ for all $n \in \mathbb{Z}$ and such that $\partial_n = \partial'_n|_{C_n}$. Denoting by $i_n \colon C_n \to C'_n$ the inclusion, the second condition is equivalent to saying that $i \colon C_\bullet \to C'_\bullet$ is a chain map.

DEFINITION 10.16. If $(C_\bullet, \partial)$ is a subcomplex of $(C'_\bullet, \partial')$, one can form the **quotient complex** $(\bar{C}_\bullet, \bar{\partial})$ where

$$\bar{C}_n = C'_n / C_n$$

and $\bar{\partial}$ is the induced map.

DEFINITION 10.17. Suppose $f \colon (C_\bullet, \partial) \to (C'_\bullet, \partial')$ is a chain map between two complexes. Then $(\ker f)_\bullet$ is a subcomplex of $C_\bullet$ and $(\operatorname{im} f)_\bullet$ is a subcomplex of $C'_\bullet$; the boundary operator of $(\ker f)_\bullet$ is simply

$$\partial_n|_{\ker f_n} \colon \ker f_n \to \ker f_{n-1},$$

and similarly the boundary operator of $(\operatorname{im} f)_\bullet$ is the restriction

$$\partial'_n|_{\operatorname{im} f_n} \colon \operatorname{im} f_n \to \operatorname{im} f_{n-1}.$$

Note these only form subcomplexes because f is a chain map! The **cokernel** of f is the chain complex $(\operatorname{coker} f)_\bullet$ given by

$$\operatorname{coker} f_n = C'_n / \operatorname{im} f_n,$$

which is itself a quotient complex of $C'_\bullet$ since $(\operatorname{im} f)_\bullet$ is a subcomplex of $C'_\bullet$.

This allows us to talk about a sequence of complexes being exact.

DEFINITION 10.18. Suppose we are given complexes $(C^m_\bullet, \partial^m)$ for $m \in \mathbb{Z}$ and chain maps $f^m \colon C^m_\bullet \to C^{m-1}_\bullet$. Pictorially, this means we have the following commutating mess:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C^{m+1}_{n+1} & \xrightarrow{f^{m+1}_{n+1}} & C^m_{n+1} & \xrightarrow{f^m_{n+1}} & C^{m-1}_{n+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \partial^{m+1}} & & \downarrow{\scriptstyle \partial^m} & & \downarrow{\scriptstyle \partial^{m-1}} & & \\
\cdots & \longrightarrow & C^{m+1}_{n} & \xrightarrow{f^{m+1}_{n}} & C^m_{n} & \xrightarrow{f^m_{n}} & C^{m-1}_{n} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \partial^{m+1}} & & \downarrow{\scriptstyle \partial^m} & & \downarrow{\scriptstyle \partial^{m-1}} & & \\
\cdots & \longrightarrow & C^{m+1}_{n-1} & \xrightarrow{f^{m+1}_{n-1}} & C^m_{n-1} & \xrightarrow{f^m_{n-1}} & C^{m-1}_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We say that sequence $(C^m_\bullet, f^m)$ is **exact** if (as complexes) $(\ker f^m)_\bullet = (\operatorname{im} f^{m+1})_\bullet$ for every m.

Definition 10.19. A **short exact sequence** of chain complexes is an exact sequence of chain complexes of the form

$$0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet \to 0,$$

where 0 denotes the chain complex all of whose entries are zero. By Problem E.5 on Problem Sheet E, the rows of short exact sequences of chain complexes are short exact sequences of abelian groups.

Definition 10.20. Let $C_\bullet$ and $C'_\bullet$ be two subcomplexes of $C''_\bullet$. The **intersection** of $C_\bullet$ and $C'_\bullet$ is the subcomplex $C_\bullet \cap C'_\bullet$ whose nth term is $C_n \cap C'_n$, and similarly the **sum** of $C_\bullet$ and $C'_\bullet$ is the subcomplex $C_\bullet + C'_\bullet$ whose nth term is $C_n + C'_n$.

One can also form direct sums: if $\{(C^\lambda_\bullet, \partial^\lambda) \mid \lambda \in \Lambda\}$ is a family of complexes indexed by a set Λ, then their **direct sum** is the complex $\bigoplus_\lambda C^\lambda_\bullet$ equipped with the boundary operator $\sum_\lambda \partial^\lambda$.

Now we define the abstract analogue of Lemma 8.7.

Definition 10.21. Let $f, g\colon (C_\bullet, \partial) \to (C'_\bullet, \partial')$ be two chain maps. We say that f and g are **chain homotopic**, written $f \simeq g$ if there exists a sequence of homomorphisms

$$P\colon C_n \to C'_{n+1}, \qquad n \in \mathbb{Z},$$

such that

$$\partial' P + P\partial = f_n - g_n, \qquad \forall\, n \in \mathbb{Z}.$$

The sequence $P = (P_n)$ is called a **chain homotopy**, and we write $P\colon f \simeq g$.

Definition 10.22. We say that $f\colon C_\bullet \to C'_\bullet$ is a **chain equivalence** if there exists $g\colon C'_\bullet \to C_\bullet$ such that $g \circ f \simeq \mathrm{id}_{C_\bullet}$ and $f \circ g \simeq \mathrm{id}_{C'_\bullet}$.

Remark 10.23. The relation of being chain homotopic is a congruence on Comp, and this can be used to define a category hComp.

The next result shows that the homology functor descends to hComp.

Proposition 10.24. *Let $f, g\colon (C_\bullet, \partial) \to (C'_\bullet, \partial')$ be two chain maps with $f \simeq g$. Then for all n,*

$$H_n(f) = H_n(g)\colon H_n(C_\bullet) \to H_n(C'_\bullet).$$

In particular, if f is a chain equivalence then $H_n(f)$ is an isomorphism for each n.

The proof is identical to Lemma 8.7, but let us repeat it anyway.

Proof. Take $c \in Z_n$. Then

$$(f_n - g_n)c = (\partial' P + P\partial)c = \partial' P c \in B'_n.$$

Thus $H_n(f)\langle c\rangle = H_n(g)\langle c\rangle$. ■

A special case of a chain homotopy is where one map is the identity and the other is the zero map. This gets its own name.

Definition 10.25. A **contracting homotopy** Q of a chain complex $C_\bullet$ is a sequence of maps $Q_n \colon C_n \to C_{n+1}$ such that

$$\partial Q + Q\partial = \mathrm{id}_{C_n}, \qquad \forall\, n \in \mathbb{Z}.$$

Corollary 10.26. *If a chain complex $C_\bullet$ has a contracting homotopy then it is acyclic.*

Proof. By Proposition 10.24, we see that for all n, $H_n(\mathrm{id}_{C_\bullet}) = H_n(0) = 0$. Since H_n is a functor, this implies that $H_n(C_\bullet) = 0$. ■

Remark 10.27. The converse to Corollary 10.26 is false. An example is given by taking

$$C_n := \begin{cases} \mathbb{Z}_2, & n = 0, \\ \mathbb{Z}, & n = 1, 2, \\ 0, & n \neq 0, 1, 2, \end{cases}$$

and defining $\partial \colon C_2 \to C_1$ to be $k \mapsto 2k$ and $\partial \colon C_1 \to C_0$ by $k \mapsto k \bmod 2$. This complex is acyclic but there does not exist a contracting homotopy. Indeed, if such a Q existed, then $Q_0 \colon C_0 \to C_1$ would define a right inverse to $\partial \colon C_1 \to C_0$. However any group homomorphism $\mathbb{Z}_2 \to \mathbb{Z}$ is trivial.

In fact, if $C_\bullet$ is a complex all of whose groups C_n are *free* abelian groups (such a chain complex is called a **free** chain complex), then the converse to Corollary 10.26 *does* hold: a free chain complex is acyclic if and only if it has a contracting homotopy. This is proved in Proposition 27.1 below. Similarly there is a partial converse to Proposition 10.24: if $f \colon C_\bullet \to C'_\bullet$ is a chain map between two free chain complexes such that $H_n(f)$ is an isomorphism for all n, then f is a chain equivalence. This is proved in Proposition 27.5 below.

LECTURE 11

The Snake Lemma and its friends

In this lecture we start by proving the so-called **Snake Lemma**. The reason for the name is the fact that the wiggly arrow in (11.1) below looks like a snake[1].

Proposition 11.1 (The Snake Lemma). *Suppose we are given a commutative diagram of abelian groups where the rows are exact:*

$$\begin{array}{ccccccc} & & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow[i']{} & B' & \xrightarrow[j']{} & C' & & \end{array}$$

Then there is a well defined homomorphism

$$\delta \colon \ker h \to \operatorname{coker} f$$

such that there is an exact sequence

$$\ker f \to \ker g \to \ker h \xrightarrow{\delta} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h.$$

Explicitly,

$$\delta(c) = (i')^{-1} g j^{-1}(c) + \operatorname{im} f,$$

where $(i')^{-1}(\cdot)$ and $j^{-1}(\cdot)$ denote any choice of preimage (the composition is independent of the choices).

Proof. The proof is very easy, but I will be kind and go through it in great detail. We will prove the result in three stages.

Will J. Merry and Berit Singer, Algebraic Topology I. Last modified: Sept 01, 2018.

[1]Right ... Evidently most mathematicians have never seen snakes.

1. We first enlarge our given diagram to include the kernels and cokernels:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \ker f & \xrightarrow{\ k\ } & \ker g & \xrightarrow{\ l\ } & \ker h & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A & \xrightarrow{\ i\ } & B & \xrightarrow{\ j\ } & C & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A' & \xrightarrow[\ i'\ ]{} & B' & \xrightarrow[\ j'\ ]{} & C' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker} f & \xrightarrow[\ p\ ]{} & \operatorname{coker} g & \xrightarrow[\ q\ ]{} & \operatorname{coker} h & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The maps k, l on the top row and the maps p, q are just the induced maps. That is,

$$k := i|_{\ker f}.$$

and

$$p(a' + \operatorname{im} f) := i'(a') + \operatorname{im} g,$$

as cosets, and similarly for l and q. We prove that both the top row and the bottom row are exact. Firstly, if $a \in \ker f$ then $gi(a) = i'f(a) = 0$, so that $k(a) := i(a)$ belongs to $\ker g$. Moreover $lk(a) = ji(a) = 0$ and hence $\operatorname{im} k \subseteq \ker l$. Conversely if $l(b) = 0$ then $j(b) = 0$ and hence $b = i(a)$ for some a as $A \xrightarrow{i} B \xrightarrow{j} C$ is exact. Moreover $i'f(a) = gi(a) = g(b) = 0$ as $b \in \ker g$. Since i' is injective, it follows $f(a) = 0$ and thus $a \in \ker(f)$ with so that $k(a) = b$. Thus $\ker l \subseteq \operatorname{im} k$ and we have exactness at $\ker g$.

Let us now prove exactness at $\operatorname{coker} g$. The composition qp is obviously zero since $j'i' = 0$ by exactness at B':

$$qp(a') = j'i'(a') + \operatorname{im} h = 0 + \operatorname{im} h = 0 \in \operatorname{coker} h.$$

Thus $\operatorname{im} p \subseteq \ker q$. Conversely suppose $q(b' + \operatorname{im} g) = 0$. This means that $j'(b') \in \operatorname{im} h$, so there exists $c \in C$ such that $h(c) = j'(b')$. Since j is surjective, there exists $b \in B$ such that $j(b) = c$. Now observe $j'(b' - g(b)) = j'(b') - j'g(b) = h(c) - hj(b) = h(c) - h(c) = 0$. Thus $b' - g(b) \in \operatorname{im} i'$ by exactness at B'. If a' is such that $i'(a') = b' - g(b)$ then

$$p(a' + \operatorname{im} f) = i'(a') + \operatorname{im} g = b' - g(b) + \operatorname{im} g = b' + \operatorname{im} g.$$

Thus $\ker q \subseteq \operatorname{im} p$, which proves exactness at $\operatorname{coker} g$.

2. Our aim now is to define a map δ such that the following sequence is exact:

$$\begin{array}{l}
\ker f \xrightarrow{\ k\ } \ker g \xrightarrow{\ l\ } \ker h \xrightarrow{\ \delta\ } \\
\qquad \operatorname{coker} f \xrightarrow[\ p\ ]{} \operatorname{coker} g \xrightarrow[\ q\ ]{} \operatorname{coker} h
\end{array} \tag{11.1}$$

Let's start with an element $c \in \ker h$. Since j is surjective, choose b such that $j(b) = c$. Observe $g(b) \in \ker j'$ since $j'g(b) = hj(b) = h(c) = 0$. By exactness at B', this implies that $g(b) \in \operatorname{im} i'$. Thus there exists $a' \in A'$ such that $i'(a') = g(b)$. In fact, since i' is injective, a' is unique. We define $\delta(c) \in \operatorname{coker} f$ as the coset $a' + \operatorname{im} f$.

We made a choice here: j is surjective, not necessarily an isomorphism, and hence we could have chosen a different element, say b_1, such that $j(b_1) = c$. This would give rise to a different element $a'_1 \in A'$ such that $i'(a'_1) = g(b_1)$. Nevertheless, we claim that the cosets $a' + \operatorname{im} f$ and $a'_1 + \operatorname{im} f$ coincide. This means we need to find $a \in A$ such that $f(a) = a' - a'_1$. But this is easy: since $j(b) = j(b_1)$ we have $b - b_1 \in \ker j$, and hence by exactness at B there exists $a \in A$ such that $i(a) = b - b_1$. Then by commutativity, $i'f(a) = gi(a) = g(b - b_1) = g(b) - g(b_1)$. Since i' is injective, it follows that $f(a) = a' - a'_1$ as required.

Thus $\delta(c) = (i')^{-1} g j^{-1}(c) + \operatorname{im} f$ is well defined. It is clear that δ is a homomorphism, i.e. that $\delta(c + c_1) = \delta(c) + \delta(c_1)$, since i', g and j are all homomorphisms.

3. Finally, let us check exactness at the two new places: $\ker h$ and $\operatorname{coker} f$. It is clear that $\operatorname{im} l \subseteq \ker \delta$. Indeed, if $c = l(b)$ for some b then we can choose this b in the definition of δ. Then $g(b) = 0$ and hence the unique preimage under i' is $a' = 0$. Then $\delta(c) = 0 + \operatorname{im} f = 0 \in \operatorname{coker} f$.

Now suppose that $\delta(c) = 0$. This means that the element a' we found belongs to the image of f, say $a' = f(a)$. The $gi(a) = i'(a') = g(b)$. Thus $b - i(a) \in \ker g$. Moreover $j(b - i(a)) = j(b) - ji(a) = c$ by exactness. This means that $l(b - i(a)) = c$ and thus $c \in \operatorname{im} l$. This proves exactness at $\ker h$.

Now we check exactness at $\operatorname{coker} f$. Again, one direction is immediate: $p\delta(c)$ is just the coset $i'(a') + \operatorname{im} g$ in $\operatorname{coker} g$. But since $i'(a') = g(b)$, this coset is zero, and hence $\operatorname{im} \delta \subseteq \ker p$. Conversely, suppose $p(a' + \operatorname{im} f) = 0$. This means that $i'(a') \in \operatorname{im} g$, so there exists $b \in B$ such that $g(b) = i'(a')$. Set $c = j(b)$. Then $h(c) = hj(b) = j'g(b) = j'i'(a') = 0$ by exactness at B'. Then by construction, $\delta(c) = a' + \operatorname{im} f$. Thus $\ker p \subseteq \operatorname{im} \delta$. This finally completes the proof. ■

Remark 11.2. This proof may look complicated, but in fact there was nothing to it: at every stage we just "did the only thing possible". This type of proof is rather relaxing, and it is usually referred to as **diagram chasing**. The best way to get used to the "yoga" of diagram chasing is to try some examples yourself. Thus you will no doubt be thrilled to discover that the next two results are left as exercises for you to solve on Problem Sheet F.

This first one is slightly less imaginatively named than the Snake Lemma.

Proposition 11.3 (The Five Lemma). *Suppose we have a commutative diagram of abelian groups, where the two rows are exact:*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Then:

1. *If g and k are injective and f is surjective, h is injective.*
2. *If g and k are surjective and l is injective, h is surjective.*
3. *If f, g, k, l are all isomorphisms then so is h.*

This is Problem F.1. The next one has no imagination whatsoever in its name ...

Proposition 11.4 (The Barratt-Whitehead Lemma). *Suppose we have the following commutative diagram of abelian groups, where the two rows are exact:*

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{j_n} & C_n & \xrightarrow{k_n} & A_{n-1} & \longrightarrow & \dots \\
 & & {\scriptstyle f_n}\big\downarrow & & {\scriptstyle g_n}\big\downarrow & & \big\downarrow{\scriptstyle h_n} & & \big\downarrow{\scriptstyle f_{n-1}} & & \\
\dots & \longrightarrow & A'_n & \xrightarrow[i'_n]{} & B'_n & \xrightarrow[j'_n]{} & C'_n & \xrightarrow[k'_n]{} & A'_{n-1} & \longrightarrow & \dots
\end{array}$$

Assume each map $h_n\colon C_n \to C'_n$ is an isomorphism. Then there is a long exact sequence:

$$\dots \to A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - i'_n} B'_n \xrightarrow{k_n h_n^{-1} j'_n} A_{n-1} \to \dots,$$

where $(i_n, f_n)\colon A_n \to B_n \oplus A'_n$ is given by $a \mapsto (i_n(a), f_n(a))$ and $g_n - i'_n\colon B_n \oplus A'_n \to B'_n$ is given by $(b, a') \mapsto g_n(b) - i'_n(a')$.

This is Problem F.2. Let us now use the Snake Lemma to prove the following foundational result in homological algebra.

Theorem 11.5 (The long exact sequence in homology). *Let*

$$0 \to C_\bullet \xrightarrow{f} C'_\bullet \xrightarrow{g} C''_\bullet \to 0 \tag{11.2}$$

be a short exact sequence of chain complexes. Then there is a sequence $\delta_n\colon H_n(C''_\bullet) \to H_{n-1}(C_\bullet)$ of homomorphisms such that there is a long exact sequence:

$$\dots \to H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(C'_\bullet) \xrightarrow{H_n(g)} H_n(C''_\bullet) \xrightarrow{\delta} H_{n-1}(C_\bullet) \to \dots$$

We call $\delta = (\delta_n)$ the **connecting homomorphism** *of the short exact sequence (11.2). Explicitly,*

$$\delta_n\langle c\rangle = \langle f_{n-1}^{-1}\partial' g_n^{-1} c\rangle, \qquad \forall c \in Z_n(C''_\bullet), \tag{11.3}$$

where ∂' is the boundary operator of $C'_\bullet$.

Proof. Write $Z_n = \ker\big(\partial\colon C_n \to C_{n-1}\big)$, $B_n = \operatorname{im}\big(\partial\colon C_{n+1} \to C_n\big)$ and $H_n = Z_n/B_n$, and similarly for the other complexes. Then the following diagram satisfies the requirements of the Snake Lemma, where the written maps are the induced ones:

$$\begin{array}{ccccccccc}
 & & C_n/B_n & \xrightarrow{f_n} & C'_n/B'_n & \xrightarrow{g_n} & C''_n/B''_n & \longrightarrow & 0 \\
 & & {\scriptstyle \partial}\big\downarrow & & \big\downarrow{\scriptstyle \partial'} & & \big\downarrow{\scriptstyle \partial''} & & \\
0 & \longrightarrow & Z_{n-1} & \xrightarrow[f_{n-1}]{} & Z'_{n-1} & \xrightarrow[g_{n-1}]{} & Z''_{n-1} & &
\end{array}$$

Adding in the kernels and cokernels of the vertical maps, we obtain (where now all the maps are omitted for clarity):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_n & \longrightarrow & H'_n & \longrightarrow & H''_n & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & C_n/B_n & \longrightarrow & C'_n/B'_n & \longrightarrow & C''_n/B''_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & Z'_{n-1} & \longrightarrow & Z''_{n-1} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_{n-1} & \longrightarrow & H'_{n-1} & \longrightarrow & H''_{n-1} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The Snake Lemma thus provides us with a map $\delta_n \colon H''_n \to H_{n-1}$, which is the map we are looking for. ■

We now prove that the long exact sequence is *natural*. It won't be until the end of the course that I explain the precise definition of the word "natural". For now, just think of "natural" meaning that whenever you draw a diagram that "ought" to commute, then it does.

PROPOSITION 11.6 (Naturality of the connecting homomorphism). *Suppose we are given a commutative diagram of chain complexes with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_\bullet & \xrightarrow{f} & B_\bullet & \xrightarrow{g} & C_\bullet & \longrightarrow & 0 \\
 & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & A'_\bullet & \xrightarrow[f']{} & B'_\bullet & \xrightarrow[g']{} & C'_\bullet & \longrightarrow & 0
\end{array}$$

Then there is a commutative diagram of abelian groups with exact rows:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & H_n(A_\bullet) & \xrightarrow{H_n(f)} & H_n(B_\bullet) & \xrightarrow{H_n(g)} & H_n(C_\bullet) & \xrightarrow{\delta_n} & H_{n-1}(A_\bullet) & \longrightarrow & \dots \\
 & & {\scriptstyle H_n(i)}\downarrow & & \downarrow{\scriptstyle H_n(j)} & & \downarrow{\scriptstyle H_n(k)} & & \downarrow{\scriptstyle H_{n-1}(i)} & & \\
\dots & \longrightarrow & H_n(A'_\bullet) & \xrightarrow[H_n(f')]{} & H_n(B'_\bullet) & \xrightarrow[H_n(g')]{} & H_n(C'_\bullet) & \xrightarrow[\delta'_n]{} & H_{n-1}(A'_\bullet) & \longrightarrow & \dots
\end{array}$$

Proof. We've already proved most things. That the rows are exact is the content of Theorem 11.5. The first two squares commute because H_n is a functor. Thus we need only check that the right-hand square commutes. So for this, suppose $c \in Z_n(C)$ is a cycle representing a homology class $\langle c \rangle \in H_n(C_\bullet)$. Since g is surjective (as a chain map), the map $g_n\colon B_n \to C_n$ is surjective (cf. Problem E.5). Thus there exists $b \in B_n$ with $g_n b = c$. Then

$$H_{n-1}(i)\delta_n\langle c\rangle = H_{n-1}(i)\delta_n\langle g_n b\rangle.$$

Let ∂ denote the boundary operator of $B_\bullet$ and ∂' the boundary operator of $B'_\bullet$ (these are the only two boundary operators we will need in the following). Now we note that from (11.3),

$$H_{n-1}(i)\delta_n\langle g_n b\rangle = H_{n-1}(i)\langle f_{n-1}^{-1}\partial b\rangle = \langle i_{n-1}f_{n-1}^{-1}\partial b\rangle.$$

Now using $j_{n-1}\circ f_{n-1} = f'_{n-1}\circ i_{n-1}$ we have

$$\langle i_{n-1}f_{n-1}^{-1}\partial b\rangle = \langle (f'_{n-1})^{-1} j_{n-1}\partial b\rangle.$$

Since j is a chain map, $\langle (f'_{n-1})^{-1} j_{n-1}\partial b\rangle = \langle (f'_{n-1})^{-1}\partial' j_n b\rangle$. Now using (11.3) for δ'_n, we see that:

$$\langle (f'_{n-1})^{-1}\partial' j_n b\rangle = \delta'_n\langle g'_n j_n b\rangle.$$

Now use $g'_n\circ j_n = k_n\circ g_n$ to obtain

$$\delta'_n\langle g'_n j_n b\rangle = \delta'_n\langle k_n g_n b\rangle.$$

Then as $\langle k_n g_n b\rangle = H_n(k)\langle g_n b\rangle = H_n(k)\langle c\rangle$ we finally have

$$H_{n-1}(i)\delta_n\langle c\rangle = \delta'_n H_n(k)\langle c\rangle,$$

which proves the last square commutes. ∎

LECTURE 12

Relative homology and reduced homology

In this lecture we extend H_n to a functor $\mathsf{Top}^2 \to \mathsf{Ab}$. To begin with, we need the following piece of pedantry.

LEMMA 12.1. *Let X' be a subspace of X with inclusion $\imath\colon X' \hookrightarrow X$. Then for every $n \geq 0$, the map $\imath_\#\colon C_n(X') \to C_n(X)$ is an injection.*

Proof. Let $c = \sum m_i \sigma_i \in C_n(X')$. We may assume all the σ_i are distinct. By definition, $\imath_\# c = \sum m_i \imath \circ \sigma_i$. Since $\imath \circ \sigma_i$ differs only from σ_i only by having its target enlarged, it follows that the $\imath \circ \sigma_i$ are all distinct. Now if $c \in \ker \imath_\#$ then we have

$$0 = \sum m_i \imath \circ \sigma_i.$$

Since $C_n(X)$ is free abelian with basis all the singular n-simplices in X, it follows that all the m_i are zero, and hence $c = 0$. ∎

This means that we can unambiguously think of $C_\bullet(X')$ as a *subcomplex* of $C_\bullet(X)$ (i.e. by identifying $C_\bullet(X')$ with $(\operatorname{im} \imath_\#)_\bullet$.) We shall do this without further comment. Thus we have a short exact sequence of complexes:

$$0 \to C_\bullet(X') \to C_\bullet(X) \to C_\bullet(X) \big/ C_\bullet(X') \to 0.$$

DEFINITION 12.2. Let $X' \subseteq X$ be a subspace. We define the **relative homology groups** $H_n(X, X')$ of the pair (X, X') to be the homology of the complex $C_\bullet(X) \big/ C_\bullet(X')$.

The next result is immediate from Theorem 11.5 and Proposition 11.6. Like the dimension axiom (Proposition 8.1) and the homotopy axiom (Theorem 8.9), we call this result an "axiom" since it will turn out to be one of the four axioms of a homology theory.

PROPOSITION 12.3 (The exact sequence axiom). *Let X' be a subspace of X. Then there is a long exact sequence*

$$\dots H_n(X') \to H_n(X) \to H_n(X, X') \xrightarrow{\delta} H_{n-1}(X') \to \dots$$

Moreover if $f\colon (X,X') \to (Y,Y')$ is a map of pairs then there is a commutative diagram:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H_n(X') & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,X') & \longrightarrow & H_{n-1}(X') & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H_n(Y') & \longrightarrow & H_n(Y) & \longrightarrow & H_n(Y,Y') & \longrightarrow & H_{n-1}(Y') & \longrightarrow & \dots \end{array}$$

where the vertical maps are all induced by f.

DEFINITION 12.4. This construction also allows us to see homology as a functor $H_n\colon \mathsf{Top}^2 \to \mathsf{Ab}$. Firstly we define the chain complex functor $\mathsf{Top}^2 \to \mathsf{Comp}$ that sends (X,X') to $C_\bullet(X)\big/C_\bullet(X')$ and sends a map $f\colon (X,X') \to (Y,Y')$ to the induced map

$$f_\#\colon C_\bullet(X)\big/C_\bullet(X') \to C_\bullet(Y)\big/C_\bullet(Y')$$

(this works as $f_\#(C_n(X')) \subseteq C_n(Y')$.) Then we apply the usual homology functor $H_n\colon \mathsf{Comp} \to \mathsf{Ab}$.

REMARK 12.5. Taking $X' = \emptyset$ recovers our original groups:

$$H_n(X,\emptyset) = H_n(X), \qquad \forall\, n \geq 0.$$

Let us now give a slightly more useful way of defining $H_n(X,X')$.

DEFINITION 12.6. Define the group of **relative n-cycles mod X'** to be

$$Z_n(X,X') := \left\{c \in C_n(X) \mid \partial c \in C_{n-1}(X')\right\},$$

and the group of **relative n-boundaries mod X'** to be

$$\begin{aligned} B_n(X,X') &:= \left\{c \in C_n(X) \mid c - c' \in B_n(X) \text{ for some } c' \in C_n(X')\right\} \\ &= B_n(X) + C_n(X'). \end{aligned}$$

Then $B_n(X,X') \subseteq Z_n(X,X')$.

We have:

PROPOSITION 12.7. *For all $n \geq 0$,*

$$H_n(X,X') \cong Z_n(X,X')\big/B_n(X,X').$$

Proof. By definition, the boundary operator $\bar\partial$ of the quotient complex $C_\bullet(X)\big/C_\bullet(X')$ is given by

$$\bar\partial\colon c + C_n(X') \mapsto \partial c + C_{n-1}(X'), \qquad c \in C_n(X), \qquad n \geq 0.$$

Thus

$$\ker\bar\partial = \left\{c + C_n(X') \mid \partial c \in C_{n-1}(X')\right\} = Z_n(X,X')\big/C_n(X')$$

and

$$\operatorname{im}\bar{\partial} = \{c + C_n(X') \mid c \in B_n(X)\} = B_n(X,X') \big/ C_n(X').$$

The claim thus follows from the third isomorphism for groups[1]. ∎

The following result shows that the relative homology groups often vanish in dimension zero, unlike the "absolute" ones which never do.

Proposition 12.8. *Suppose X is path connected and X' is a non-empty subspace. Then $H_0(X,X') = 0$.*

Proof. Fix $p \in X'$. Suppose $c = \sum m_x x \in Z_0(X,X')$. Choose a path[2] $\sigma_x \colon \Delta^1 \to X$ such that $\sigma_x(e_0) = p$ and $\sigma_x(e_1) = x$. Then $a = \sum m_x \sigma_x \in C_1(X)$ and

$$\partial a = c - \left(\sum m_x\right) p.$$

Since $c' := (\sum m_x)\, p \in C_0(X')$, we thus have $c - c' \in B_0(X)$, so that $c \in B_0(X,X')$. Thus $H_0(X,X') = 0$. ∎

Next, we have the following result.

Proposition 12.9. *Let $\{X_\lambda \mid \lambda \in \Lambda\}$ denote the path components of X, and let $X' \subseteq X$ denote a subspace. For each $n \geq 0$, one has*

$$H_n(X,X') \cong \bigoplus_{\lambda \in \Lambda} H_n(X_\lambda, X_\lambda \cap X').$$

Proof. Immediate from Problem D.4 and Problem E.6. ∎

Let us record a special case of this statement, since it will be useful in Lecture 22 when we discuss the axioms.

Corollary 12.10 (The additivity axiom). *Let (X_λ, X'_λ), $\lambda \in \Lambda$ be a family of pairs of spaces. Denote by*

$$\imath_\lambda \colon (X_\lambda, X'_\lambda) \hookrightarrow \left(\bigsqcup_{\lambda \in \Lambda} X_\lambda, \bigsqcup_{\lambda \in \Lambda} X'_\lambda \right)$$

the inclusion. Then for all $n \geq 0$, the map

$$\sum_{\lambda \in \Lambda} H_n(\imath_\lambda) \colon \bigoplus_{\lambda \in \Lambda} H_n(X_\lambda, X'_\lambda) \to H_n \left(\bigsqcup_{\lambda \in \Lambda} X_\lambda, \bigsqcup_{\lambda \in \Lambda} X'_\lambda \right).$$

is an isomorphism.

[1] Which states that if $N \leq K \leq G$ are normal subgroups then $(G/N)/(K/N) \cong G/K$, i.e. you can "cancel" the N.

[2] We won't bother distinguishing paths and 1-simplices in this lecture.

Corollary 12.11. *Let* $\{X_\lambda \mid \lambda \in \Lambda\}$ *denote the path components of* X*, and let* $X' \subseteq X$ *denote a subspace. The group* $H_0(X, X')$ *is free abelian, with*

$$\operatorname{rank} H_0(X, X') = \#\left\{\lambda \in \Lambda \mid X_\lambda \cap X' = \emptyset\right\}.$$

Proof. If $X_\lambda \cap X' \neq \emptyset$ then $H_0(X_\lambda, X_\lambda \cap X') = 0$ by Proposition 12.8, since X_λ is path connected for each λ by definition. If $X_\lambda \cap X' = \emptyset$ then $H_0(X_\lambda, X_\lambda \cap X') = H_0(X_\lambda) \cong \mathbb{Z}$ by Proposition 8.3. ∎

Now let us specialise to the case where X' is a single point $\{p\}$ for some $p \in X$.

Corollary 12.12. *If* (X, p) *is a pointed space then* $H_0(X, p)$ *is a free abelian group of (possibly infinite) rank* r*, where* X *has* $r + 1$ *path components.*

Meanwhile for $n \geq 1$, taking a single point doesn't change the homology:

Proposition 12.13. *Let* (X, p) *be a pointed space. Then for all* $n \geq 1$*,*

$$H_n(X, p) \cong H_n(X).$$

Proof. By Proposition 12.3 there is an exact sequence

$$\dots H_n(p) \to H_n(X) \to H_n(X, p) \xrightarrow{\delta} H_{n-1}(p) \to \dots$$

If $n \geq 2$ then $H_n(p) = 0$ and $H_{n-1}(p) = 0$ by the dimension axiom, and thus we immediately see $H_n(X, p) \cong H_n(X)$. The case $n = 1$ is slightly more tricky; it can be deduced directly from the long exact sequence by studying what the actual maps do at the tail end:

$$0 \to H_1(X) \to H_1(X, p) \to H_0(p) \to H_0(X) \to H_0(X, p) \to 0.$$

Namely, by the fourth item of Example 10.5, the map $H_1(X) \to H_1(X, p)$ is surjective if and only if the map $H_0(p) \to H_0(X)$ is injective. Since $H_0(p) = \mathbb{Z}$ and $H_0(X)$ is free abelian, either $H_0(p) \to H_0(X)$ is the zero map or it is injective. By exactness, if it was the zero map then $H_0(X) \to H_0(X, p)$ would have to be injective. Thus to complete the proof we need only exhibit an element in the kernel of the map $H_0(X) \to H_0(X, p)$. Such an element is provided by $\langle p \rangle$ (cf. the proof of Proposition 8.3.) ∎

This means that for $n \geq 1$ we can regard H_n as a functor on Top_*. Let us now introduce another algebraic concept.

Definition 12.14. Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of abelian groups. We say that the sequence **splits** if there exists a map $h \colon C \to B$ such that $gh = \mathrm{id}_C$. We call h a **splitting map**.

The splitting map h is *not* unique. On Problem Sheet F you will prove that an equivalent definition is asking that f admits a left inverse:

Proposition 12.15. *Suppose* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is a short exact sequence of abelian groups. Then the sequence splits if and only if there exists a map* $k\colon B \to A$ *such that* $kf = \mathrm{id}_A$.

Here we show that if a sequence splits the middle term is a direct sum.

Proposition 12.16. *Suppose* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is a split short exact sequence of abelian group. Then* $B \cong A \oplus C$.

Proof. Let $h\colon C \to B$ be such that $gh = \mathrm{id}_C$. We show that $B = \operatorname{im} f \oplus \operatorname{im} h$. If $b \in B$ then $g(b) \in C$ and $b - hg(b) \in \ker g$ because $g(b) - gh(gb) = 0$ as $gh = \mathrm{id}_C$. Thus by exactness, there exists $a \in A$ with $f(a) = b - hg(b)$. Thus $B = \operatorname{im} f + \operatorname{im} h$. It remains to show that $\operatorname{im} f \cap \operatorname{im} h = \{0\}$. If $f(a) = x = h(c)$ then $g(x) = gf(a) = 0$ and also $g(x) = gh(c) = c$, thus $x = h(c) = 0$. ■

Remark 12.17. It is important to realise that the isomorphism $B \cong A \oplus C$ depends on the choice of the splitting map h. More formally[3], this means that the splitting is *not* natural. Moreover the converse to Proposition 12.16 is *not* true, as you will show on Problem Sheet F.

The next idea is more important than it looks at first glance.

Definition 12.18. Let X be a non-empty topological space and let $\{*\}$ be a topological space with one point. Let $j\colon X \to \{*\}$ be the unique continuous map that sends every point in X to $*$. For any map $i\colon \{*\} \to X$ we have $j \circ i = \mathrm{id}_{\{*\}}$. Thus the induced map $H_n(j)\colon H_n(X) \to H_n(*)$ is always surjective. We define $\tilde{H}_0(X) := \ker H_0(j)$ and call it the **zeroth reduced homology group**. This gives us a short exact sequence:

$$0 \to \tilde{H}_0(X) \to H_0(X) \xrightarrow{H_0(j)} H_0(*) \to 0.$$

Since this sequence splits (via $H_0(i)$), we have

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z},$$

but this splitting is *not* natural, since it depends on the choice of map i.

Remark 12.19. Under the identification $H_0(*) \cong \mathbb{Z}$ given by

$$m\langle * \rangle \mapsto m, \qquad m \in \mathbb{Z},$$

the map $H_0(j)\colon H_0(X) \to H_0(*)$ can be identified with the map ϕ from (8.3). Indeed, if $c = \sum m_x x \in Z_0(X)$ represents a homology class $\langle c \rangle \in H_0(X)$, then

$$H_0(j)\langle c \rangle = \sum m_x j(x) = \left(\sum m_x\right) \langle * \rangle \in H_0(*),$$

and hence the map $H_0(j)$ sends $\sum m_x x \mapsto \sum m_x$, which is exactly how ϕ was defined in (8.3).

[3] We will define this properly in Lecture 22.

We now extend $\tilde{H}_n$ to all n by simply setting $\tilde{H}_n(X) := H_n(X)$ for $n \geq 1$, and call $\tilde{H}_\bullet(X)$ the **reduced homology groups of** X. We then have:

Corollary 12.20. *If X is a non-empty contractible space then $\tilde{H}_n(X) = 0$ for all $n \geq 0$.*

On Problem Sheet F you will show that the long exact sequence for pairs works for reduced homology too:

Proposition 12.21. *Let $\emptyset \neq X' \subseteq X$. There is an exact sequence*

$$\dots \to \tilde{H}_n(X') \to \tilde{H}_n(X) \to H_n(X, X') \to \tilde{H}_{n-1}(X') \to \dots$$

which ends with $\tilde{H}_0(X') \to \tilde{H}_0(X) \to H_0(X, X') \to 0$.

Corollary 12.22. *If $p \in X$ then $\tilde{H}_n(X) \cong H_n(X, p)$ for all $n \geq 0$.*

Remark 12.23. In Corollary 12.22 we can be slightly more explicit. Let us take our one point space $\{*\}$ to be $\{p\}$ itself. Then there is an "obvious" choice of map $p \to X$, namely the inclusion. This will be important later.

Remark 12.24. Remark 12.19 allows us to see the reduced homology groups as the homology groups of a chain complex. Let X be a non-empty topological space and define a chain complex $(\tilde{C}_\bullet(X), \tilde{\partial})$ by setting:

$$\tilde{C}_n(X) := \begin{cases} C_n(X), & n \geq 0, \\ \mathbb{Z}, & n = -1. \\ 0, & n \leq -2, \end{cases}$$

and for $n \geq 1$, define $\tilde{\partial}\colon \tilde{C}_n(X) \to C_{n-1}(X)$ to be the normal boundary operator, and for $n = 0$, set

$$\tilde{\partial}\colon \tilde{C}_0(X) \to \mathbb{Z} = \tilde{C}_{-1}(X), \qquad \tilde{\partial}\left(\sum_x m_x\, x\right) \mapsto \sum_x m_x.$$

Then by Remark 12.19, one has

$$H_n\big(\tilde{C}_\bullet(X), \tilde{\partial}\big) \cong \tilde{H}_n(X), \qquad \forall\, n \geq 0.$$

This will be important next lecture in Corollary 13.3.

Our first "real" use of the reduced homology groups will come in Lecture 15 when we finally compute the homology of S^n (it will turn out it is more convenient to compute $\tilde{H}_\bullet(S^n)$ using induction.) In Lecture 19, we will prove that if a pair (X, X') is sufficiently "nice" then

$$H_n(X, X') \cong \tilde{H}_n(X/X'), \tag{12.1}$$

where X/X' is the quotient space obtained by collapsing X' to a point (Corollary 12.22 is a special case of (12.1).)

We conclude this lecture by defining the homotopy version of Top^2.

Definition 12.25. If $f, g\colon (X, X') \to (Y, Y')$ are maps of pairs then we say $f \simeq g$ mod X' if there exists a continuous map $F\colon (X \times I, X' \times I) \to (Y, Y')$ with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$.

This notion is **not** the same as saying $f \simeq g$ rel X', since the definition does **not** require $f|_{X'} = g|_{X'}$ and that $F(x', t)$ is independent of t for all $x' \in X'$. This relation defines a congruence on Top^2 and thus yields a new category, hTop^2. Moreover H_n induces a functor $H_n\colon \mathsf{hTop}^2 \to \mathsf{Ab}$ thanks to the following result.

Theorem 12.26 (The homotopy axiom for pairs). *If $f, g\colon (X, X') \to (Y, Y')$ are maps of pairs such that $f \simeq g$ mod X' then for all $n \geq 0$,*

$$H_n(f) = H_n(g)\colon H_n(X, X') \to H_n(Y, Y').$$

The proof is analogous to the proof of the homotopy axiom (Theorem 8.9) and I won't bore you with it again.

LECTURE 13

Barycentric subdivision

In this lecture we give a systematic way of chopping up a singular simplex into a bunch of smaller ones. This process, known as *barycentric subdivision*, is interesting in its own right (we will use it to give another proof that a convex space has zero reduced homology), but for us the main application will be in the proof of *excision* that we will carry out next lecture.

DEFINITION 13.1. Let D be a bounded convex subset of some Euclidean space. Fix a point $p \in D$, and suppose $\sigma \colon \Delta^n \to D$ is a singular n-simplex. We define the **cone over σ with vertex** p to be the singular $(n+1)$-simplex $Q(p,\sigma) \colon \Delta^{n+1} \to D$ defined by

$$Q(p,\sigma)(s_0, s_1, \dots, s_{n+1}) := \begin{cases} p, & s_0 = 1, \\ s_0 p + (1-s_0)\sigma\left(\frac{s_1}{1-s_0}, \dots, \frac{s_{n+1}}{1-s_0}\right), & s_0 \neq 1. \end{cases}$$

This is well defined because if $s_0 \neq 1$ then $\frac{1}{1-s_0}\sum_{i=1}^{n+1} s_i = 1$, and it is easy to check that $Q(p,\sigma)$ is continuous and takes values in D by convexity.

We extend this by linearity to a map $C_n(D) \to C_{n+1}(D)$, which we write as $c \mapsto Q(p,c)$.

PROPOSITION 13.2. *If $c = \sum m_i \sigma_i \in C_n(D)$ then*

$$\partial Q(p,c) = \begin{cases} c - Q(p, \partial c), & \text{if } n > 0, \\ c - \left(\sum m_i\right) p, & \text{if } n = 0. \end{cases}$$

Proof. If $n \geq 1$ and $i = 0$ then

$$Q(p,\sigma) \circ \varepsilon_0^{n+1}(s_0, \dots, s_n) = Q(p,\sigma)(0, s_0, \dots, s_n) = \sigma(s_0, \dots, s_n),$$

and if $1 \leq i \leq n+1$ then

$$Q(p,\sigma) \circ \varepsilon_i^{n+1}(s_0, \dots, s_n) = Q(p,\sigma)(s_0, \dots, s_{i-1}, 0, s_i, \dots, s_n).$$

If $s_0 = 1$ then this reduces to

$$Q(p,\sigma)(1, 0, \dots, 0) = p,$$

meanwhile if $s_0 \neq 1$ then we have

$$\begin{aligned} Q(p,\sigma) \circ \varepsilon_i^{n+1}(s_0, \dots, s_n) &= s_0 p + (1-s_0)\sigma\left(\frac{s_1}{1-s_0}, \dots, \frac{s_{i-1}}{1-s_0}, 0, \frac{s_i}{1-s_0}, \dots, \frac{s_n}{1-s_0}\right) \\ &= s_0 p + (1-s_0)\sigma \circ \varepsilon_{i-1}^n\left(\frac{s_1}{1-s_0}, \dots, \frac{s_n}{1-s_0}\right) \\ &= Q(p, \sigma \circ \varepsilon_{i-1}^n)(s_0, \dots, s_n). \end{aligned}$$

Thus we see that for $n \geq 1$,

$$Q(p,\sigma) \circ \varepsilon_0^{n+1} = \sigma \tag{13.1}$$

and for $i \geq 1$,

$$Q(p,\sigma) \circ \varepsilon_i^{n+1} = Q(p, \sigma \circ \varepsilon_{i-1}^n). \tag{13.2}$$

Now taking alternating sums, we see that

$$\begin{aligned}
\partial Q(p,\sigma) &= \sum_{i=0}^{n+1} (-1)^i Q(p,\sigma) \circ \varepsilon_i^{n+1} \\
&\overset{(*)}{=} \sigma + \sum_{i=1}^{n+1} (-1)^i Q(p, \sigma \circ \varepsilon_{i-1}^n) \\
&= \sigma - \sum_{j=0}^{n} (-1)^j Q(p, \sigma \circ \varepsilon_j^n) \\
&= \sigma - Q\left(p, \sum_{j=0}^{n} (-1)^j \sigma \circ \varepsilon_j^n\right) \\
&= \sigma - Q(p, \partial\sigma),
\end{aligned}$$

where $(*)$ used (13.1) and (13.2). This proves the result in the case $n \geq 1$. For $n = 0$, it suffices to observe that if x is a point in X then $Q(p,x)$ is a 1-simplex σ with $\sigma(e_0) = p$ and $\sigma(e_1) = x$. Thus $\partial\sigma = \sigma(e_1) - \sigma(e_0) = x - p$. ■

This gives us another direct proof of the fact that the reduced homology groups of D vanish. (Of course, since D is contractible this already follows from Corollary 12.22.)

Corollary 13.3. *Let D be a bounded convex subset of some Euclidean space. Then the reduced homology groups vanish: $\tilde{H}_n(D) = 0$ for all $n \geq 0$.*

Proof. The operator $c \mapsto Q(p,c)$ is a contracting homotopy for the chain complex $\tilde{C}_\bullet(D)$ defined in Remark 12.24. Thus by Corollary 10.26 we see that $\tilde{H}_n(D) = 0$ for all $n \geq 0$. ■

Remark 13.4. Note that this proof did *not* use the homotopy axiom. We will use this fact in Lecture 23 when giving an alternative proof of the homotopy axiom using the Acyclic Models Theorem (if Corollary 13.3 used the homotopy axiom then our argument would be circular).

We now introduce the concept of an *affine* simplex.

Definition 13.5. Let D be convex. A singular n-simplex $\sigma\colon \Delta^n \to D$ is said to be **affine** if

$$\sigma\left(\sum_{i=0}^{n} s_i\, e_i\right) = \sum_{i=0}^{n} s_i\, \sigma(e_i), \qquad \forall\, (s_0, s_1, \ldots, s_n) \in \Delta^n.$$

If σ is affine then $\partial\sigma$ is also affine, and thus the space of affine singular n-simplices defines a subcomplex of $C_\bullet(D)$. We write this as $C_\bullet^{\text{affine}}(D)$. Note also that if σ is affine then so is $Q(p,\sigma)$ for any $p \in D$.

Let us denote by $b_n := \frac{1}{n+1}(e_0 + e_1 + \cdots + e_n)$ the *barycentre* (cf. (7.1) of the standard simplex Δ^n. We now define the barycentric subdivision operator. We will first define it for affine simplices in convex subsets, and then extend it to arbitrary simplices and spaces.

DEFINITION 13.6. Let D be convex. Define the **convex barycentric subdivision**

$$\mathrm{Sd}_n^{\mathrm{cv}} \colon C_n^{\mathrm{affine}}(D) \to C_n^{\mathrm{affine}}(D)$$

inductively for an affine singular n-simplex $\sigma \colon \Delta^n \to D$ by

$$\mathrm{Sd}_n^{\mathrm{cv}}(\sigma) := \begin{cases} \sigma & n = 0, \\ Q\big(\sigma(b_n), \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma)\big), & n \geq 1, \end{cases}$$

and then extending by linearity. See Figure 13.1 for a picture in the case $n = 2$.

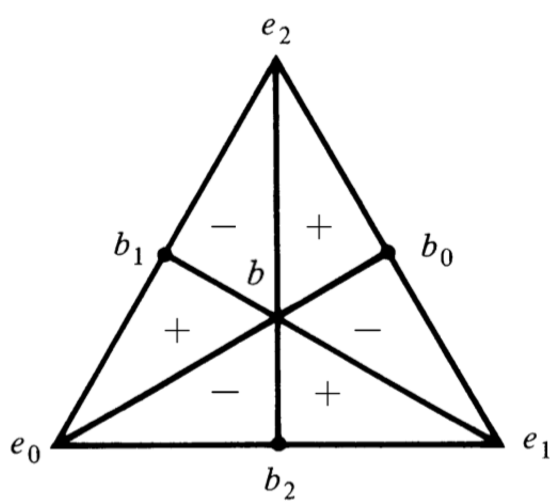


Figure 13.1: The barycentric subdivision of Δ^2.

If X is an arbitrary topological space then this definition doesn't make sense, since we cannot apply the cone construction. Nevertheless, there is an easy way to extend this. In the following, in order to minimise notational confusion, let us denote by $\ell_n \colon \Delta^n \to \Delta^n$ the identity map, *thought of as a singular n-simplex in Δ^n*.

DEFINITION 13.7. Let X be an arbitrary topological space. Define the **barycentric subdivision** $\mathrm{Sd}_n \colon C_n(X) \to C_n(X)$ by setting

$$\mathrm{Sd}_n(\sigma) := \sigma_\#\big(\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n)\big),$$

and then extending by linearity.

This makes sense. The simplex ℓ_n is certainly affine, and hence $\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n)$ is well-defined and belongs to $C_n^{\mathrm{affine}}(\Delta^n) \subset C_n(\Delta^n)$. Since $\sigma_\#$ is a map $C_n(\Delta^n) \to C_n(X)$, we see that $\sigma_\#\big(\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n)\big)$ does indeed belong to $C_n(X)$.

Lemma 13.8. *If D is a convex bounded subset of some Euclidean space then Definition 13.7 agrees with Definition 13.6 for all affine simplices:*

$$\mathrm{Sd}_n(\sigma) = \mathrm{Sd}_n^{\mathrm{cv}}(\sigma), \qquad \forall\, \sigma\colon \Delta^n \to D \text{ affine.}$$

The proof of Lemma 13.8 is on Problem Sheet G. Moreover on Problem Sheet G you are asked to write out explicit formulae for Sd_n for $n = 0, 1, 2$.

Proposition 13.9. *The barycentric subdivision is a chain map. Moreover if $f\colon X \to Y$ is continuous then the following diagram commutes for all $n \geq 0$:*

$$\begin{array}{ccc} C_n(X) & \xrightarrow{f_\#} & C_n(Y) \\ {\scriptstyle \mathrm{Sd}_n}\downarrow & & \downarrow{\scriptstyle \mathrm{Sd}_n} \\ C_n(X) & \xrightarrow[f_\#]{} & C_n(Y) \end{array} \tag{13.3}$$

Remark 13.10. The fact that the diagram (13.3) commutes means that Sd_n is a **natural** chain map, anticipating terminology will we introduce later on in the course.

Proof. We begin by showing that (13.3) commutes:

$$f_\#\mathrm{Sd}_n(\sigma) = f_\#\sigma_\#\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n) = (f\circ\sigma)_\#\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n) = \mathrm{Sd}_n(f\circ\sigma) = \mathrm{Sd}_n(f_\#\sigma).$$

Assume now that X is a bounded convex subset of some Euclidean space. Let us prove by induction on n that Sd_n is a chain map. By Lemma 13.8, it suffices to show that $\mathrm{Sd}_n^{\mathrm{cv}}$ is a chain map. The case $n = 0$ is obvious. For the inductive step we compute

$$\begin{aligned} \partial\mathrm{Sd}_n^{\mathrm{cv}}(\sigma) &= \partial Q\big(\sigma(b_n), \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma)\big) \\ &\overset{(*)}{=} \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma) - Q\big(\sigma(b_n), \partial\mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma)\big) \\ &\overset{(\dagger)}{=} \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma) - Q\big(\sigma(b_n), \mathrm{Sd}_{n-2}^{\mathrm{cv}}(\underbrace{\partial^2\sigma}_{=0})\big) \\ &= \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma), \end{aligned}$$

where $(*)$ used Proposition 13.2 and $(\dagger)$ used the inductive hypothesis[1]. We now prove the general case where X is not necessarily convex. If $\sigma\colon \Delta^n \to X$ we have

$$\begin{aligned} \partial\mathrm{Sd}_n(\sigma) &= \partial\sigma_\#\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n) \\ &\overset{(\ddagger)}{=} \sigma_\#\partial\mathrm{Sd}_n^{\mathrm{cv}}(\ell_n) \\ &\overset{(\spadesuit)}{=} \sigma_\#\mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\ell_n) \\ &\overset{(\clubsuit)}{=} \sigma_\#\mathrm{Sd}_{n-1}(\partial\ell_n) \\ &\overset{(\heartsuit)}{=} \mathrm{Sd}_{n-1}(\sigma_\#\partial(\ell_n)) \\ &\overset{(\ddagger)}{=} \mathrm{Sd}_{n-1}(\partial\sigma_\#(\ell_n)) \\ &= \mathrm{Sd}_{n-1}(\partial\sigma), \end{aligned}$$

where:

[1]To make the case $n = 1$ work, we can take $\mathrm{Sd}_{-1}^{\mathrm{cv}}$ to be the zero map.

1. (‡) used that $\sigma_\#$ is a chain map (both times),
2. (♠) used that we already know that convex barycentric subdivision is a chain map,
3. (♣) used the fact that for the affine simplex $\partial \ell_n$ in the convex set Δ^n one has $\mathrm{Sd}_{n-1}(\partial \ell_n) = \mathrm{Sd}^{\mathrm{cv}}_{n-1}(\partial \ell_n)$,
4. (♡) used naturality (13.3),
5. and the last line used the fact that

$$\sigma_\#(\ell_n) = \sigma. \tag{13.4}$$

This completes the proof. ■

Now that we know that barycentric subdivision is a chain map, we get an induced map on homology. What could this map be? The answer is about as boring as it can be (it shows we have accomplished nothing!), but this will be crucial in the proof of excision next lecture.

THEOREM 13.11. *For each $n \geq 0$, the induced map*

$$H_n(\mathrm{Sd})\colon H_n(X) \to H_n(X)$$

is the identity.

REMARK 13.12. Just as with Proposition 8.5, the fact that barycentric subdivision induces the identity on homology is also an immediate consequence of the **Acyclic Models Theorem** that we will prove later on in the course. However for completeness we will give an independent proof here.

Proof. It suffices by Proposition 10.24 to construct a chain homotopy between Sd and the identity. In other words, we need to build map $P_n\colon C_n(X) \to C_{n+1}(X)$ such that

$$\partial P_n + P_{n-1}\partial = \mathrm{id} - \mathrm{Sd}_n. \tag{13.5}$$

We prove the result in two steps.

1. We begin in the convex case with affine simplices. Assume D is a bounded convex subset of some Euclidean space. We define inductively:[2]

$$P^{\mathrm{cv}}_n(\sigma) := \begin{cases} 0 & n = 0, \\ Q\big(\sigma(b_n), \sigma - \mathrm{Sd}^{\mathrm{cv}}_n(\sigma) - P^{\mathrm{cv}}_{n-1}(\partial\sigma)\big), & n \geq 1, \end{cases}$$

Then $P^{\mathrm{cv}}_n(\sigma) \in C^{\mathrm{affine}}_{n+1}(D)$ by induction. Let us verify inductively that (13.5) holds. Since $\mathrm{Sd}^{\mathrm{cv}}_0(\sigma) = \sigma$ for an (affine) 0-simplex, both sides of (13.5) are zero for $n = 0$. Now suppose $n > 0$ and that (13.5) holds for $n - 1$, that is:

$$\partial P^{\mathrm{cv}}_{n-1} + P^{\mathrm{cv}}_{n-2}\partial = \mathrm{id} - \mathrm{Sd}^{\mathrm{cv}}_{n-1}. \tag{13.6}$$

[2] P^{cv}_{-1} is also the zero map; there is no choice about that!

Let $\sigma\colon \Delta^n \to D$ be an affine singular n-simplex. Then

$$\begin{aligned}\partial\big(\sigma - \mathrm{Sd}_n^{\mathrm{cv}}(\sigma) - P_{n-1}^{\mathrm{cv}}(\partial\sigma)\big) &\overset{(\dagger)}{=} \partial\sigma - \partial\mathrm{Sd}_n^{\mathrm{cv}}(\sigma) - (\mathrm{id} - \mathrm{Sd}_{n-1}^{\mathrm{cv}} - P_{n-2}^{\mathrm{cv}}\partial)\partial\sigma \\ &\overset{(\ddagger)}{=} -\partial\mathrm{Sd}_n^{\mathrm{cv}}(\sigma) + \mathrm{Sd}_{n-1}^{\mathrm{cv}}(\partial\sigma) \\ &= 0,\end{aligned}$$

where this time (†) used (13.6) and (‡) used that $\mathrm{Sd}^{\mathrm{cv}}$ is a chain map and $\partial^2 = 0$. Thus from Proposition 13.2 we have that

$$\partial P_n^{\mathrm{cv}}(\sigma) = \sigma - \mathrm{Sd}_n^{\mathrm{cv}}(\sigma) - P_{n-1}^{\mathrm{cv}}(\partial\sigma),$$

and hence (13.5) holds.

2. Now we prove the general case. The strategy is the same as the definition of Sd and the proof of Proposition 13.9. Using that ℓ_n is an affine n-simplex, given a singular n-simplex $\sigma\colon \Delta^n \to X$, we define

$$P_n\sigma := \sigma_\#(P_n^{\mathrm{cv}}(\ell_n)) \in C_{n+1}(X).$$

Just as with Lemma 13.8, if X was already convex and σ was affine, we have $P_n(\sigma) = P_n^{\mathrm{cv}}(\sigma)$. We claim that this map P_n is "natural", that is, if $f\colon X \to Y$ is any continuous map, then the following commutes:

$$\begin{array}{ccc} C_n(X) & \xrightarrow{f_\#} & C_n(Y) \\ {\scriptstyle P_n}\big\downarrow & & \big\downarrow{\scriptstyle P_n} \\ C_{n+1}(X) & \xrightarrow[f_\#]{} & C_{n+1}(Y) \end{array} \tag{13.7}$$

Indeed,

$$f_\# P_n\sigma = f_\#\sigma_\#(P_n^{\mathrm{cv}}(\ell_n)) = (f\circ\sigma)_\#(P_n^{\mathrm{cv}}(\ell_n)) = P_n(f\circ\sigma) = P_n(f_\#\sigma).$$

Then using (13.7) and arguing as in the proof of Proposition 13.9, we have

$$\partial P_n(\sigma) = \sigma_\#\big(\partial P_n^{\mathrm{cv}}(\ell_n)\big) \tag{13.8}$$

and

$$P_{n-1}(\partial\sigma) = \sigma_\#\big(P_{n-1}^{\mathrm{cv}}(\partial\ell_n)\big) \tag{13.9}$$

and thus adding (13.8) and (13.9) together and using the fact that we already know that (13.5) holds for P_n^{cv}, we see that

$$\partial P_n(\sigma) + P_{n-1}\partial\sigma = \sigma_\#\big(\partial P_n^{\mathrm{cv}}(\ell_n) + P_{n-1}^{\mathrm{cv}}(\partial\ell_n)\big) = \sigma_\#\big(\ell_n - \mathrm{Sd}_n^{\mathrm{cv}}(\ell_n)\big) = \sigma - \mathrm{Sd}_n(\sigma),$$

where we used (13.4) again in the last equality. ■

LECTURE 14

Excision and the homology of spheres

In this lecture we state and prove the excision axiom. This is the last of the four major axioms needed to define a homology theory. Thus by the end of this lecture we will have proved (using terminology we will introduce in Lecture 22) that *singular homology is a homology theory*. We use this to prove the *Mayer-Vietoris* sequence and to compute the homology of spheres.

Given a set $U \subseteq X$ of a topological space, we denote by U° the interior of U.

DEFINITION 14.1. Let X be a topological space and let $\mathfrak{U}$ be a family of subsets of X such that

$$X = \bigcup_{U \in \mathfrak{U}} U^\circ.$$

We say a singular n-simplex $\sigma\colon \Delta^n \to X$ is **$\mathfrak{U}$-small** if there exists $U \in \mathfrak{U}$ such that $\sigma(\Delta^n) \subseteq U$. We denote by $C_\bullet^{\mathfrak{U}}(X)$ the subcomplex of $C_\bullet(X)$ generated by $\mathfrak{U}$-small simplices (it is clear this is a subcomplex.) We denote by $H_\bullet^{\mathfrak{U}}(X)$ the homology of this chain complex.

There is an obvious chain map $i\colon C_\bullet^{\mathfrak{U}}(X) \to C_\bullet(X)$ given by inclusion. The main technical result we prove today is that this chain map induces an isomorphism on homology.

THEOREM 14.2. *The inclusion of chain complexes $i\colon C_\bullet^{\mathfrak{U}}(X) \to C_\bullet(X)$ induces an isomorphism $H_n^{\mathfrak{U}}(X) \to H_n(X)$ for all $n \geq 0$.*

We will need a few preliminary results. The next two pertain to genuine simplices (not *singular* simplices!).

PROPOSITION 14.3. *Let $S = [z_0, z_1, \dots, z_n]$ denote an n-simplex in some Euclidean space. Then if $x, y \in S$ one has*

$$|x - y| \leq \sup_i |z_i - y|, \tag{14.1}$$

and hence

$$\operatorname{diam} S = \max_{i,j} |z_i - z_j|. \tag{14.2}$$

Moreover if b is the barycentre of S then

$$|b - z_i| \leq \frac{n}{n+1} \operatorname{diam} S \tag{14.3}$$

Proof. Let $x, y \in S$, and write $x = \sum_i s_i\, z_i$ with $\sum_i s_i = 1$. Then

$$|x - y| = \left|\sum_i s_i\, z_i - y\right| \leq \sum_i s_i |z_i - y| \leq \max_i |z_i - y|$$

This proves (14.1), and (14.2) is an immediate consequence of this. Now since $b = \frac{1}{n+1}\sum_i z_i$ we have

$$\begin{aligned} |b - z_j| &= \left|\left(\sum_i \frac{1}{n+1} z_i\right) - z_j\right| \\ &\overset{(*)}{=} \left|\sum_i \frac{1}{n+1}(z_i - z_j)\right| \\ &\leq \frac{1}{n+1}\sum_i |z_i - z_j| \\ &\leq \frac{n}{n+1}\max_{i,j} |z_i - z_j| \\ &= \frac{n}{n+1}\operatorname{diam} S, \end{aligned}$$

where $(*)$ used the fact that $\sum_{i=0}^{n} \frac{1}{n+1} = 1$. This proves (14.3). ∎

We can regard any genuine simplex $[z_0, z_1, \ldots, z_n]$ as a singular n-simplex by choosing $\sigma \colon \Delta^n \to [z_0, z_1, \ldots, z_n]$ to be an affine map sending e_i to z_i (cf. Problem D.4). Thus if S_i are genuine n-simplices in some convex subset D and m_i are non-zero integers, we can regard $\sum_i m_i\, S_i$ as belonging to $C_n(D)$ (actually, to $C_n^{\text{affine}}(D)$.) We define the **mesh** of such a sum to the maximum diameter of the S_i.

In particular, if S is an n-simplex then $\operatorname{Sd}_n^{\text{cv}}(S)$ is an element of $C_n(D)$, and we have:

Corollary 14.4. *For any n-simplex S,*

$$\operatorname{mesh} \operatorname{Sd}_n^{\text{cv}}(S) \leq \frac{n}{n+1}\operatorname{diam} S.$$

This allows us to prove the following result, which tells us that we can make any singular simplex into a sum of $\mathfrak{U}$-small simplices by barycentrically subdividing enough times.

Proposition 14.5. *Let X be a topological space and let $\mathfrak{U}$ be a family of subsets of X whose interiors cover X. Let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. There exists $k \in \mathbb{N}$ such that every simplex in the n-chain $\operatorname{Sd}_n^k(\sigma)$ is $\mathfrak{U}$-small.*

Proof. Let $\delta > 0$ be a Lebesgue number (cf. Lemma 6.7) for the open covering $\{\sigma^{-1}(U^\circ) \mid U \in \mathfrak{U}\}$ of Δ^n. Choose $k \in \mathbb{N}$ large enough so that

$$\left(\frac{n}{n+1}\right)^k < \frac{\delta}{\sqrt{2}}.$$

The claim now follows from Corollary 14.4 and induction. ∎

With these preliminaries out of the way, we can now prove Theorem 14.2.

Proof of Theorem 14.2. Let $n \geq 0$. We first prove $H_n(i)\colon H_n^{\mathfrak{U}}(X) \to H_n(X)$ is injective. Suppose $c \in Z_n^{\mathfrak{U}}(X)$ belongs to the kernel of $H_n(i)$. This means there exists $a \in C_{n+1}(X)$ such that $\partial a = i_{\#}c = c$. By applying Proposition 14.5 to each of the finitely many simplices in a, we see there exists $k \in \mathbb{N}$ such that $\mathrm{Sd}_{n+1}^k(a) \in C_{n+1}^{\mathfrak{U}}(X)$. Recall Theorem 13.11 gives us a chain homotopy P from Sd to the identity: $\partial P + P\partial = \mathrm{Sd} - \mathrm{id}$. By induction, for any positive integer k we have

$$\partial P \mathrm{Sd}^{k-1} + P \mathrm{Sd}^{k-1} \partial = \mathrm{Sd}^k - \mathrm{Sd}^{k-1}.$$

Thus if we set

$$P^{(k)} := P \circ \left(\mathrm{id} + \mathrm{Sd} + \cdots + \mathrm{Sd}^{k-1}\right)$$

then we have

$$\partial P^{(k)} + P^{(k)}\partial = \mathrm{Sd}^k - \mathrm{id}.$$

Thus

$$\mathrm{Sd}_{n+1}^k(a) - a = \partial P_{n+1}^{(k)}(a) + P_n^{(k)}(\partial a) = \partial P_{n+1}^{(k)}(a) + P_n^{(k)}(c),$$

and hence as $\partial^2 = 0$ we have

$$c = \partial a = \partial\big(\mathrm{Sd}_{n+1}^k(a) - P_n^{(k)}(c)\big).$$

Since $c \in C_n^{\mathfrak{U}}(X)$ we have $P_n^{(k)}(c) \in C_{n+1}^{\mathfrak{U}}(X)$ as well; this can be seen from the naturality equation (13.7). Thus as $\mathrm{Sd}_{n+1}^k(a)$ also belongs to $C_{n+1}^{\mathfrak{U}}(X)$, we see that $c \in B_n^{\mathfrak{U}}(X)$. Hence $H_n(i)$ is injective.

Now we prove $H_n(i)$ is surjective. Suppose $d \in Z_n(X)$. Then using Proposition 14.5 for k large enough we have $\mathrm{Sd}_n^k(d) \in C_n^{\mathfrak{U}}(X)$. With $P^{(k)}$ as before we have

$$\mathrm{Sd}_n^k(d) - d = \partial P_n^{(k)}(d) + P_{n-1}^{(k)}(\partial d) = \partial P_n^{(k)}(d).$$

Since Sd^k is a chain map, $\mathrm{Sd}_n^k(d)$ is also a cycle. The previous equation thus shows that d is homologous to a cycle in $C_n^{\mathfrak{U}}(X)$. This shows that $H_n(i)$ is surjective, and hence completes the proof. ■

Now let (X, X') be a pair of spaces. Write

$$\mathfrak{U} \cap X' := \{U \cap X' \mid U \in \mathfrak{U}\},$$

and define the chain complex

$$C_\bullet^{\mathfrak{U}}(X, X') := C_\bullet^{\mathfrak{U}}(X) \Big/ C_\bullet^{\mathfrak{U}\cap X'}(X').$$

We denote its homology groups by $H_n^{\mathfrak{U}\cap X'}(X, X')$. This gives us a commutative diagram of chain complexes with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_\bullet^{\mathfrak{U}\cap X'}(X') & \longrightarrow & C_\bullet^{\mathfrak{U}}(X) & \longrightarrow & C_\bullet^{\mathfrak{U}}(X, X') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_\bullet(X') & \longrightarrow & C_\bullet(X) & \longrightarrow & C_\bullet(X, X') & \longrightarrow & 0 \end{array}$$

Each row has its own long exact sequence, giving the following commutating diagram:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & H_n^{\mathfrak{U}\cap X'}(X') & \longrightarrow & H_n^{\mathfrak{U}}(X) & \longrightarrow & H_n^{\mathfrak{U}\cap X'}(X,X') & \longrightarrow & H_{n-1}^{\mathfrak{U}\cap X'}(X') & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle j} & & \downarrow & & \\
\dots & \longrightarrow & H_n(X') & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,X') & \longrightarrow & H_{n-1}(X') & \longrightarrow & \dots
\end{array}$$

All the vertical maps apart from the one marked j are isomorphisms, thanks to Theorem 14.2. But now by the Five Lemma (Proposition 11.3), we see that j is also an isomorphism. This proves:

PROPOSITION 14.6. *The inclusion of chain complexes $C_\bullet^{\mathfrak{U}}(X,X') \to C_\bullet(X,X')$ induces an isomorphism in homology:*

$$H_n^{\mathfrak{U}\cap X'}(X,X') \cong H_n(X,X'), \qquad \forall\, n \geq 0.$$

We now state two forms of excision.

THEOREM 14.7 (The excision axiom). *Assume that $X'' \subset X' \subset X$ are subspaces with $\overline{X''} \subset (X')^\circ$. Then the inclusion $(X \setminus X'', X' \setminus X'') \hookrightarrow (X,X')$ induces an isomorphism in homology:*

$$H_n(X \setminus X'', X' \setminus X'') \cong H_n(X,X'), \qquad \forall\, n \geq 0.$$

THEOREM 14.8 (The excision axiom, second form). *Assume that X_1, X_2 are subspaces of X such that $X = X_1^\circ \cup X_2^\circ$. Then the inclusion $(X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ induces an isomorphism in homology:*

$$H_n(X_1, X_1 \cap X_2) \cong H_n(X, X_2), \qquad \forall\, n \geq 0.$$

On Problem Sheet G you will show that the two results Theorem 14.7 and Theorem 14.8 are equivalent. Here we will prove the second one.

Proof of Theorem 14.8. We take as our covering $\mathfrak{U} = \{X_1, X_2\}$. The hypotheses of Theorem 14.2 are satisfied. By definition we have

$$C_\bullet^{\mathfrak{U}}(X) = C_\bullet(X_1) + C_\bullet(X_2),$$

and hence if we look at long exact sequence in homology associated to the short exact sequence of chain complexes:

$$0 \to \Big(C_\bullet(X_1) + C_\bullet(X_2)\Big) \xrightarrow{f} C_\bullet(X) \to C_\bullet(X) \Big/ \Big(C_\bullet(X_1) + C_\bullet(X_2)\Big) \to 0,$$

every third map

$$H_n(f)\colon H_n\big(C_\bullet(X_1) + C_\bullet(X_2)\big) \to H_n\big(C_\bullet(X)\big)$$

is an isomorphism[1] by Theorem 14.2. Now we consider the short exact sequence of chain complexes:

$$0 \to \frac{C_\bullet(X_1)+C_\bullet(X_2)}{C_\bullet(X_2)} \xrightarrow{g} \frac{C_\bullet(X)}{C_\bullet(X_2)} \to \frac{C_\bullet(X)}{C_\bullet(X_1)+C_\bullet(X_2)} \to 0.$$

The corresponding long exact sequence has every third term zero, so that $H_n(g)$ is an isomorphism for every n. Next observe there is an isomorphism of chain complexes:

$$h\colon \frac{C_\bullet(X_1)}{C_\bullet(X_1\cap X_2)} \cong \frac{C_\bullet(X_1)+C_\bullet(X_2)}{C_\bullet(X_2)}.$$

This is just the fact that

$$C_\bullet(X_1\cap X_2) = C_\bullet(X_1)\cap C_\bullet(X_2),$$

together with the **second isomorphism theorem** for chain complexes, which is Problem G.6. We thus have a commuting triangle of chain complexes:

$$\begin{array}{ccc} \dfrac{C_\bullet(X_1)}{C_\bullet(X_1\cap X_2)} & \longrightarrow & \dfrac{C_\bullet(X)}{C_\bullet(X_2)} \\ & {\scriptstyle h}\searrow \quad \nearrow{\scriptstyle g} & \\ & \dfrac{C_\bullet(X_1)+C_\bullet(X_2)}{C_\bullet(X_2)} & \end{array}$$

The induced maps $H_n(h)$ and $H_n(g)$ are both isomorphisms, and hence the horizontal map also induces an isomorphism in homology:

$$H_n\left(\frac{C_\bullet(X_1)}{C_\bullet(X_1\cap X_2)}\right) \cong H_n\left(\frac{C_\bullet(X)}{C_\bullet(X_2)}\right), \qquad \forall\, n \geq 0.$$

This is exactly the statement of the theorem. ■

We now prove the "homology" version of the Seifert-van Kampen Theorem 6.5, which is a simple consequence of excision.

Theorem 14.9 (Mayer-Vietoris). *Let X_1 and X_2 be subspaces of X such that $X = X_1^\circ \cup X_2^\circ$. Set $X_0 := X_1 \cap X_2$ and let*

$$\imath_i\colon X_0 \hookrightarrow X_i, \qquad \jmath_i\colon X_i \hookrightarrow X$$

denote inclusions for $i = 1, 2$. Then there is a long exact sequence

$$\dots H_n(X_0) \xrightarrow{(H_n(\imath_1),H_n(\imath_2))} H_n(X_1)\oplus H_n(X_2) \xrightarrow{H_n(\jmath_1)-H_n(\jmath_2)} H_n(X) \xrightarrow{D} H_{n-1}(X_0) \to \dots$$

[1]If one has a long exact sequence

$$\dots \to A_n \xrightarrow{f_n} B_n \to C_n \to A_{n-1} \xrightarrow{f_{n-1}} B_{n-1} \to \dots$$

where every third map $f_n\colon A_n \to B_n$ is an isomorphism, then $C_n = 0$ for all n by exactness.

Proof. The following diagram of pairs of spaces commutes, where all the maps are inclusions[2]:

$$\begin{array}{ccccc}
(X_0, \emptyset) & \xrightarrow{\imath_1} & (X_1, \emptyset) & \xrightarrow{f} & (X_1, X_0) \\
\downarrow{\scriptstyle \imath_2} & & \downarrow{\scriptstyle \jmath_1} & & \downarrow{\scriptstyle h} \\
(X_2, \emptyset) & \xrightarrow[\jmath_2]{} & (X, \emptyset) & \xrightarrow[g]{} & (X, X_2)
\end{array}$$

By Problem F.4 we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & H_n(X_0) & \xrightarrow{H_n(\imath_1)} & H_n(X_1) & \xrightarrow{H_n(f)} & H_n(X_1, X_0) & \xrightarrow{\delta} & H_{n-1}(X_0) & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle H_n(\imath_2)} & & \downarrow{\scriptstyle H_n(\jmath_1)} & & \downarrow{\scriptstyle H_n(h)} & & \downarrow{\scriptstyle H_n(\imath_2)} & & \\
\dots & \longrightarrow & H_n(X_2) & \xrightarrow[H_n(\jmath_2)]{} & H_n(X) & \xrightarrow[H_n(g)]{} & H_n(X, X_2) & \xrightarrow[\delta]{} & H_{n-1}(X_2) & \longrightarrow & \dots
\end{array}$$

Now Theorem 14.8 tells us that the map $H_n(h)$ is an isomorphism for all n. This means that we can apply the Barratt-Whitehead Lemma (Proposition 11.4) to obtain the desired long exact sequence. ■

Provided X_0 is non-empty, the Mayer-Vietoris Theorem continuous to hold for reduced homology as well.

Corollary 14.10. *Let X_1 and X_2 be subspaces of X such that $X = X_1^\circ \cup X_2^\circ$. Set $X_0 := X_1 \cap X_2$ and assume $X_0 \neq \emptyset$. Then there is a long exact sequence*

$$\dots \tilde{H}_n(X_0) \to \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \to \tilde{H}_n(X) \to \tilde{H}_{n-1}(X_0) \to \dots$$

where the maps are the same as in Theorem 14.9. The sequence ends with

$$\dots \tilde{H}_0(X_1) \oplus \tilde{H}_0(X_2) \to \tilde{H}_0(X) \to 0.$$

Proof. Fix $p \in X_0$ and proceed as before, starting with the commutative diagram

$$\begin{array}{ccccc}
(X_0, p) & \longrightarrow & (X_1, p) & \longrightarrow & (X_1, X_0) \\
\downarrow & & \downarrow & & \downarrow \\
(X_2, p) & \longrightarrow & (X, p) & \longrightarrow & (X, X_2)
\end{array}$$

This gives the desired long exact sequence, albeit with $H_n(X, p)$ in place of $\tilde{H}_n(X)$, etc. However Corollary 12.22 then completes the proof. ■

We can now finally compute the homology of the sphere. We will state the result for reduced homology, because this is a neater statement and the proof is shorter.

Theorem 14.11. *For all $n \geq 0$, one has*

$$\tilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & k = n, \\ 0, & k \neq n. \end{cases}$$

[2] We identify eg. $\imath_1 \colon X_0 \hookrightarrow X_1$ with the map of pairs $(X_0, \emptyset) \hookrightarrow (X_1, \emptyset)$.

Proof. We induct on n. For $n = 0$ this follows from the definition of reduced homology, the dimension axiom (Proposition 8.1), and Proposition 8.3), since $S^0 = \{-1, 1\}$ is a space consisting of two points. For the inductive step, we apply Corollary 14.10. Suppose $n \geq 1$, and let p and q denote the "north pole" and "south pole" of S^n respectively. Set $X_1 = S^n \setminus \{p\}$ and $X_2 := S^n \setminus \{q\}$. Then X_1 and X_2 are contractible and $X_1 \cap X_2$ is homotopy equivalent to the "equator" S^{n-1}. Using Corollary 12.20, the Mayer-Vietoris sequence gives us for all $i \geq 0$ an exact sequence

$$0 \to \tilde{H}_k(S^n) \to \tilde{H}_{k-1}(S^{n-1}) \to 0.$$

The result follows by induction. ■

Just as with the Seifert-van Kampen Theorem 6.5, the Mayer-Vietoris exact sequence allows us to compute the homology of a number of standard spaces. We will return to this in Lecture 18, when we discuss the idea of *attaching cells*.

REMARK 14.12. In particular, $H_n(S^n) = \mathbb{Z} \neq 0$. Thus we have finally proved Lemma 1.3 from Lecture 1, and thus we have also finally proved the Brouwer Fixed Point Theorem 1.1.

LECTURE 15

The degree

In this lecture we define the *degree* of a continuous map from a sphere to itself. Recall in Lecture 5 we defined the degree of a *loop* $u\colon (I, \partial I) \to (S^1, 1)$. From Problem B.5, we have $\pi_1(S^1, 1) \cong [(S^1, 1), (S^1, 1)]$, and hence our earlier definition can be thought of as a map $[(S^1, 1), (S^1, 1)] \to \mathbb{Z}$. In this lecture we will work with homology, and thus it is convenient to ditch the basepoints. This means we need a stronger version of Problem B.5.

PROPOSITION 15.1. *Let X be path connected and let $\zeta\colon \pi_1(X, p) \to [S^1, X]$ be the function that sends a path class $[u]$ to the free homotopy class of the map $\hat{u}\colon S^1 \to X$ given by*

$$\hat{u}(e^{2\pi i s}) := u(s), \qquad s \in I.$$

This function is surjective. Moreover if $\zeta([u]) = \zeta([v])$ then there exists $[w] \in \pi_1(X, p)$ such that $[u] = [w] * [v] * [w]^{-1}$. In particular, if $\pi_1(X, p)$ is abelian then ζ is an isomorphism, and hence $\pi_1(X, p) \cong [S^1, X]$.

The proof of Proposition 15.1 is on Problem Sheet H as a test to see whether you've already forgotten all the homotopy theory we did...

Taking $X = S^1$, and using that $\pi_1(S^1) = \mathbb{Z}$ is abelian, this shows that our earlier definition of the degree (Definition 5.5) can be thought of as a map $[S^1, S^1] \to \mathbb{Z}$. We now use take advantage of the fact that $H_n(S^n) = \mathbb{Z}$ to extend this to higher-dimensional spheres using homology.

We will use the following simple observation: any group homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}$ is necessarily multiplication by an integer, that is, $\varphi(n) = mn$ for some $m \in \mathbb{Z}$. Indeed, if $m = \varphi(1)$ then

$$\varphi(n) = \varphi(\underbrace{1 + \cdots + 1}_{n \text{ times}}) = \underbrace{\varphi(1) + \cdots + \varphi(1)}_{n \text{ times}} = n \cdot \varphi(1) = nm.$$

DEFINITION 15.2. Let $n \geq 1$ and let $f\colon S^n \to S^n$ be continuous. Then $H_n(f)\colon H_n(S^n) \to H_n(S^n)$ is a group homomorphism, and hence is multiplication by an integer. This integer is called the **degree** of f and denoted by $\deg(f)$. Thus

$$H_n(f)\langle c\rangle = \deg(f)\langle c\rangle, \qquad \forall \, \langle c\rangle \in H_n(S^n).$$

The homotopy axiom (Theorem 8.9) implies that $\deg\colon [S^n, S^n] \to \mathbb{Z}$ is a well defined function, since if $f \simeq g$ are homotopic maps from S^n to itself then $H_n(f) = H_n(g)$ and thus in particular $\deg(f) = \deg(g)$.

Let us check that for S^1, this definition agrees with the old one. For clarity, since there are now (supposedly) three different definitions of "degree" in play, let us temporarily give them all different names. Firstly we have

$$\deg^{\text{loop}} \colon \pi_1(S^1, 1) \to \mathbb{Z},$$

the one from Theorem 5.6. Secondly, we have

$$\deg^{\text{old}} \colon [S^1, S^1] \to \mathbb{Z},$$

the one obtained from $\deg^{\text{loop}}$ using Proposition 15.1, and finally the function

$$\deg^{\text{new}} \colon [S^1, S^1] \to \mathbb{Z},$$

given from Definition 15.2. Of course $\deg^{\text{loop}}$ and $\deg^{\text{old}}$ are the same map, but in the proof of the next proposition it is convenient to keep the notation distinct.

Proposition 15.3. *Let $f \colon S^1 \to S^1$ be continuous. Then the homomorphism $\pi_1(f)$ is given by mulitplication by $\deg^{\text{old}}(f)$.*

Proof. Let $u_k(s) := e^{2\pi i k s}$ for $k \in \mathbb{N}$. Then from the proof of Theorem 5.6, $\deg^{\text{loop}}[u_k] = k$, and thus $[u_1]$ is a generator of $\pi_1(S^1, 1)$. Consider now first the special case where $f = g_m$ for $g_m(z) := z^m$. Since $g_m \circ u_1 = u_m$ we see that $\pi_1(g_m)[u_1] = [u_m]$. Thus $\pi_1(g_m)$ is given by multiplication by m under the identification $\pi_1(S^1, 1) \cong \mathbb{Z}$ given by $\deg^{\text{loop}}$.

For the general case, suppose $f \colon S^1 \to S^1$ has $\deg^{\text{old}}(f) = m$. Then $f \simeq g_m$ since $\deg^{\text{old}}$ is an isomorphism by Theorem 5.6 and Proposition 15.1. Since $\pi_1(S^1)$ is abelian, Corollary 4.14 shows that $\pi_1(f) = \pi_1(g_m)$. This completes the proof. ∎

We now prove that actually $\deg^{\text{old}} = \deg^{\text{new}}$.

Proposition 15.4. *If $f \colon S^1 \to S^1$ is continuous then $\deg^{\text{old}}(f) = \deg^{\text{new}}(f)$*

Proof. By Problem E.2, there is a commutative diagram

$$\begin{array}{ccc} \pi_1(S^1, 1) & \xrightarrow{\pi_1(f)} & \pi_1(S^1, 1) \\ \downarrow h & & \downarrow h \\ H_1(S^1) & \xrightarrow[H_1(f)]{} & H_1(S^1) \end{array}$$

where $h \colon \pi_1(S^1, 1) \to H_1(S^1)$ is the Hurewicz map. Since $\pi_1(S^1, 1) = \mathbb{Z}$ is abelian, the map h is an isomorphism, and $\deg^{\text{loop}}$ furnishes an explicit isomorphism $\pi_1(S^1, 1) \cong H_1(S^1) \cong \mathbb{Z}$. Now the result follows from Proposition 15.3. ∎

With this out the way, we will go back to just calling all three of the maps deg.

Proposition 15.5. *Let $n \geq 1$ and let $f, g \colon S^n \to S^n$ denote continuous maps. Then:*

1. $\deg(g \circ f) = \deg(g)\deg(f)$,
2. $\deg(\text{id}_{S^n}) = 1$,

3. *if f is a constant map then* $\deg(f) = 0$,
4. *if $f \simeq g$ then* $\deg(f) = \deg(g)$,
5. *if f is a homotopy equivalence then* $\deg(f) = \pm 1$.

Proof. All properties follow from the fact that H_n is a functor. Property (3) follows from the fact that a constant map f can be factored as a composition $S^n \to \{*\} \to S^n$ where $\{*\}$ is a one-point space. ■

We now prove a far less obvious result.

PROPOSITION 15.6. *Let $n \geq 1$ and let $A \in \mathrm{O}(n+1)$ denote an orthogonal linear transformation. Set $f := A|_{S^n}$. Then* $\deg(f) = \det(A)$.

Proof. The group $\mathrm{O}(n+1)$ has two connected components, distinguished by $\det\colon \mathrm{O}(n+1) \to \{+1, -1\}$. By homotopy invariance it suffices to check the result for one such A in each component. Since the identity matrix I_{n+1} induces $f = \mathrm{id}_{S^n}$, which has degree 1, it suffices to check the result for a single map A with $\det(A) = -1$. We take A to be reflection in a hyperplane $H \subset \mathbb{R}^{n+1}$. Divide S^n into two hemispheres that are *preserved* by A. Then the map f induces a reflection f' in the corresponding hyperplane H' in the equatorial S^{n-1}. Now applying the Mayer-Vietoris sequence and using naturality, we obtain the following commutative diagram:

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{D} & H_{n-1}(S^{n-1}) \\ {\scriptstyle H_n(f)}\downarrow & & \downarrow{\scriptstyle H_{n-1}(f')} \\ H_n(S^n) & \xrightarrow[D]{} & H_{n-1}(S^{n-1}) \end{array}$$

Here the maps D are the connecting maps from the Mayer-Vietoris sequence. These maps are isomorphisms, moreover they are the *same* isomorphism. Thus we see that

$$\deg(f) = \deg(f'),$$

and hence by induction it suffices to prove the result for $n = 1$. For this we write S^1 as the union of two open intervals A and B which contract onto the two given hemispheres preserved by our reflection. Then $A \cap B$ is homotopy equivalent to $S^0 = \{p, q\}$. Applying Mayer-Vietoris again, we see that $H_1(S^1)$ is isomorphic to the kernel of the map j:

$$0 \to H_1(S^1) \to H_0(S^0) \xrightarrow{j} H_0(A) \oplus H_0(B).$$

We take $\langle p \rangle$ and $\langle q \rangle$ as generators of $H_0(S^0)$. Both $\langle p \rangle$ and $\langle q \rangle$ generate $H_0(A)$ and $H_0(B)$ (cf. the last part of Proposition 8.3), and thus the map j here

$$0 \to H_1(S^1) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{j} \mathbb{Z} \oplus \mathbb{Z}$$

is given by $(u, v) \mapsto (u + v, u + v)$. In particular, the kernel of j is generated by $\langle p \rangle - \langle q \rangle$. Since the reflection f interchanges p and q, this shows that $\deg(f) = -1$ as claimed. ■

Corollary 15.7. *Let $n \geq 1$. The antipodal map $a\colon S^n \to S^n$ given by $a(x) = -x$ has degree $(-1)^{n+1}$.*

Proof. The antipodal map is the composition of the $n+1$ reflections in the coordinates axes of $\mathbb{R}^{n+1}$. These all have degree -1 by Proposition 15.6, and degree is multiplicative by part (1) of Proposition 15.5. ∎

This immediately gives a proof of the following famous result.

Theorem 15.8 (The Hairy Ball Theorem). *There exists a nowhere vanishing vector field on S^n if and only if n is odd.*

A vector field can be identified with a continuous map $v\colon S^n \to \mathbb{R}^{n+1}$ such that $\langle x, v(x)\rangle = 0$, where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^{n+1}$. Rather more visually, suppose we attach a "hair-vector" $v(x)$ at every point $x \in S^n$. If we could successfully "comb" the sphere so that every hair was tangential to S^n, we'd have successfully created a vector field on S^n. Thus the theorem tells us that if we try this on S^{2m}, either there will be a point x where the sphere is bald ($v(x) = 0$), or no matter how hard we try to comb, there will always be a tuft.

Proof. If $n = 2m-1$ then define $v\colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ by

$$v(x_1, y_1, \dots x_m, y_m) = (-y_1, x_1, \dots, -y_m, x_m).$$

The restriction of v to S^n is then a nowhere vanishing vector field. Conversely, suppose v is a nowhere vanishing vector field. Let $w\colon S^n \to S^n$ be defined by

$$w(x) = \frac{v(x)}{|v(x)|}.$$

Now define

$$W\colon S^n \times I \to S^n, \qquad W(x,t) := (\cos \pi t)x + (\sin \pi t)w(x),$$

(this does indeed take values in S^n since $\langle x, v(x)\rangle = 0$ for all x.) Then $W(x,0) = x$ and $W(x,1) = a(x)$, where a is the antipodal map. Thus the degree of the antipodal map is the same as the degree of the identity by part (4) of Proposition 15.5, and thus by Corollary 15.7, we see that n must be odd. ∎

Taking the earth to be S^2 and our vector field to be the wind, the theorem can also be interpreted as saying: *There is always somewhere on planet where there is no wind.*

Lemma 15.9. *Let $n \geq 1$. Suppose $f, g\colon S^n \to S^n$ are continuous maps such that $f(x) \neq g(x)$ for all $x \in S^n$. Then $f \simeq a \circ g$. In particular, if f has no fixed points then f is homotopic to the antipodal map.*

Proof. If $f(x) \neq g(x)$ for all x then

$$F(x,t) := \frac{(1-t)f(x) - tg(x)}{|(1-t)f(x) - tg(x)|}$$

is a well-defined homotopy from f to $a \circ g$ (since the denominator can never be zero.) ∎

We can use this lemma to obtain surprising information on what groups G can act freely on S^{2n}. First note that S^{2n-1} can be realised as the unit circle in $\mathbb{C}^n$, and thus carries a free action of S^1; namely $z \mapsto e^{i\theta}z$. Inside S^1 we then also have free actions of the mth roots of unity on S^{2n-1}, and thus $\mathbb{Z}_m$ acts on S^{2n-1} for each $m \in \mathbb{N}$. The same however is *not* true for S^{2n}.

Corollary 15.10. *If G acts freely on S^{2n} then G is either the trivial group or $G = \mathbb{Z}_2$.*

Proof. If G acts freely on S^{2n} then each non-trivial element $g \in G$ has no fixed points. Thus each g has degree -1 by Lemma 15.9 and Corollary 15.7. Thus the map $\deg \colon G \to \mathbb{Z}_2 = \{+1, -1\}$ is an injective group homomorphism. ∎

We conclude this lecture with a much deeper result.

Definition 15.11. A continuous map $f \colon S^n \to S^n$ is called an **odd** map if $f(-x) = -f(x)$ for all $x \in S^n$. Equivalently, $f \circ a = a \circ f$, where a is the antipodal map.

Theorem 15.12. *An odd map has odd degree.*

As you will see on Problem Sheet H, Theorem 15.12 implies two classical theorems, the **Borsuk-Ulam Theorem** and the **Lusternik-Schnirelmann Theorem**. There are several ways to prove Theorem 15.12. We will give a proof that (mostly) uses only material that we have covered so far. Alternative approaches use the *Smith Exact Sequence* and *homology with $\mathbb{Z}_2$-coefficients*, or the *ring structure* of the *cohomology $H^\bullet(\mathbb{R}P^n)$*. These are topics we will cover in Algebraic Topology II next semester.

Let us start with some notation. Let $B^n_\pm$ denote the upper and lower hemispheres of S^n, so that $B^n_+ \cap B^n_-$ is the equatorial S^{n-1}. This process can be iterated, so we can see S^i sitting inside S^n for all $0 \le i \le n$. We will need the following result in our proof of Theorem 15.12, which we can't properly prove just yet.

Proposition 15.13. *Let $f \colon S^n \to S^n$ be an odd map. Then there exists an odd map $f' \colon S^n \to S^n$ such that $f'(S^i) \subseteq S^i$ for each $i = 0, \dots, n$ and a homotopy $F \colon f \simeq f'$ with the property that $f_t := F(\cdot, t)$ is an odd map for each $t \in I$.*

Sketch proof. Let $p \colon S^n \to \mathbb{R}P^n$ denote the projection map (see Example 18.8 in Lecture 18). Since f is odd, there is an induced map $h \colon \mathbb{R}P^n \to \mathbb{R}P^n$ such that $h \circ p = p \circ f$. Using the **cellular structure** of $\mathbb{R}P^n$ and the **Cellular Approximation Theorem** (for the former, see Example 18.8 again, for the latter see Algebraic Topology II), there exists a homotopy $H \colon h \simeq h'$ such that $h'(\mathbb{R}P^i) \subseteq \mathbb{R}P^i$ for each $i = 0, 1, \dots, n$. Now using the **homotopy lifiting property** (also in Algebraic Topology II), we can **lift** H to a homotopy $F \colon S^n \times I \to S^n$. This means that

$$H(p(x), t)) = p(F(x, t)), \qquad \forall\, (x, t) \in S^n \times I \tag{15.1}$$

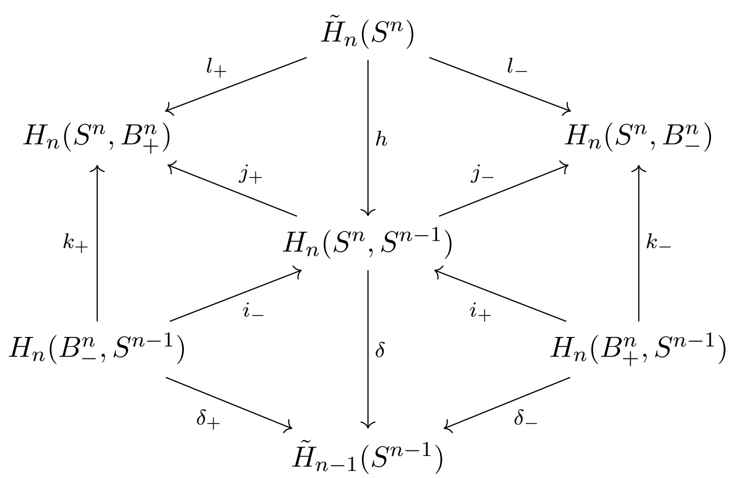
(compare this to Proposition 5.2), and such that $F(\cdot, 0)$ is our original map f. The map $f_t := F(\cdot, t)$ is odd for each t by (15.1), and $f' := f_1$ has the property that $f'(S^i) \subseteq S^i$ for each $i = 0, 1, \dots, n$. ∎

Proposition 15.13 implies that we can prove Theorem 15.12 via the following result. In the statement, it is convenient to formally define the degree of a map $f\colon S^0 \to S^0$ to be the the integer such that the induced map in *reduced homology* $\tilde{H}_0(S^0) \to \tilde{H}_0(S^0)$ is multiplication by this number.

Proposition 15.14. *Let $n \geq 1$ and let $f\colon S^n \to S^n$ denote an odd map such that $f(S^i) \subseteq S^i$ for each $i = 0, 1, \ldots, n$. Then $\deg(f)$ has the same parity as $\deg(f|_{S^{n-1}})$.*

Indeed, Theorem 15.12 immediately follows from this, since the only odd maps $S^0 \to S^0$ are the identity and the antipodal map itself, and these both have odd degree. Indeed, if $S^0 = \{p, q\}$ then $\langle p - q \rangle$ is a generator of $\tilde{H}_0(S^0)$. The identity map induces $\langle p - q \rangle \mapsto \langle p - q \rangle$, and the antipodal map induces $\langle p - q \rangle \mapsto -\langle p - q \rangle$. Hence the degree is $+1$ or -1, and thus in particular is odd. (This also shows that Corollary 15.7 also formally holds in the case $n = 0$.)

Proof of Proposition 15.14. Consider the following commuting hexagon (!):



We use reduced homology at the top and bottom only so the proof still works for $n = 1$. Here the maps $\delta_\pm$ and δ come from the long exact sequence for reduced homology (Proposition 12.21), and all the other maps are induced by inclusions. Moreover[1]:

1. All the groups apart from the middle one are isomorphic to $\mathbb{Z}$.
2. The maps $k_\pm$, $l_\pm$ and $\delta_\pm$ are all isomorphisms.
3. Exactness holds at $H_n(S^n, S^{n-1})$ for all three diagonals: $\operatorname{im} i_- = \ker j_-$ and $\operatorname{im} i_+ = \ker j_+$ and $\operatorname{im} h = \ker \delta$.

The notation in this proof is rather involved, so to simplify things, we will denote homology classes just by letters c etc. (i.e. no angle brackets). The map f induces maps

$$H_n(f)\colon \tilde{H}_n(S^n) \to \tilde{H}_n(S^n), \qquad H_{n-1}(f|_{S^{n-1}})\colon \tilde{H}_{n-1}(S^{n-1}) \to \tilde{H}_{n-1}(S^{n-1}),$$

[1] Consider it an exercise to verify all these claims!

and also (by assumption) a map $H_n(S^n, S^{n-1}) \to H_n(S^n, S^{n-1})$. We will denote them *all* by φ. Similarly we will denote by α all the maps on homology induced by the antipodal map. This should not cause confusion.

Fix a generator $c \in \tilde{H}_{n-1}(S^{n-1})$. This uniquely defines generators $c_\pm \in H_n(B^n_\mp, S^{n-1})$ via the equation $\delta_\pm(c_\pm) = c$. Moreover if $b_\pm := k_\pm(c_\pm)$ then $b_\pm$ are generators of $H_n(S^n, B^n_\pm)$, and there exist $a_\pm \in \tilde{H}_n(S^n)$ such that $l_\pm(a_\pm) = b_\pm$. Now set $u_\pm := i_\mp(c_\pm)$. Some diagram chasing tells us that $\{u_+, u_-\}$ is a basis of $H_n(S^n, S^{n-1})$ (see Problem H.2), and thus in particular

$$H_n(S^n, S^{n-1}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Let $d := \deg(f)$ and $d' := \deg(f|_{S^{n-1}})$. Then by definition,

$$\varphi a_+ = d a_+, \qquad \varphi c = d' c.$$

Since $\{u_+, u_-\}$ is a basis of $H_n(S^n, S^{n-1})$, there exist integers p, q such that $\varphi(u_+) = p u_+ + q u_-$. To complete the proof, we will show that

$$d = p - q, \qquad d' = p + q.$$

Thus $d' - d = 2q$ which is even. For this note that since $\delta(u_\pm) = c$ by commutativity,

$$\varphi(c) = \varphi\big(\delta(u_+)\big) = \delta\big(\varphi(u_+)\big) = \delta(p u_+ + q u_-) = (p+q)c,$$

where we used naturality (the commuting diagram part of Proposition 12.3) for $\varphi \circ \delta = \delta \circ \varphi$. Thus $d' = p + q$ as claimed.

Now consider α. By naturality $\delta_- \circ \alpha = \alpha \circ \delta_+$, and Corollary 15.7 we have $\alpha(c) = (-1)^n c$. Thus also $\alpha(c_+) = (-1)^n c_-$ and $\alpha(u_-) = (-1)^n u_+$. Next, since f is odd, $\varphi \circ \alpha = \alpha \circ \varphi$. Putting this together we see that

$$\varphi(u_-) = (-1)^n \varphi\big(\alpha(u_+)\big) = (-1)^n \alpha\big(\varphi(u_+)\big) = (-1)^n \alpha(p u_+ + q u_-) = p u_- + q u_+.$$

Next, since $\operatorname{im} h = \ker \delta$, the image of h is generated by $u_+ - u_-$. Thus $h(a_+) = r(u_+ - u_-)$ with $r = \pm 1$. In fact, we claim $r = +1$. For this we use that $j_+(u_+) = l_+(a_+)$ by definition, and hence

$$j_+(u_+) = l_+(a_+) = j_+\big(h(a_+)\big) = r j_+(u_+ - u_-) = r j_+(u_+),$$

since $u_- \in \operatorname{im} i_+ = \ker j_+$. Now we observe that by definition of d,

$$\begin{aligned} d(u_+ - u_-) &= d h(a_+) \\ &= \varphi\big(h(a_+)\big) \\ &= \varphi(u_+ - u_-) \\ &= (p u_+ + q u_-) - (p u_- + q u_+) \\ &= (p - q)(u_+ - u_-). \end{aligned}$$

Thus $d = p - q$. This completes the proof. ■

LECTURE 16

Colimits and filtered colimits

Let us now make good on the promise we made in Lecture 5 and formalise the notion of a pushout.

Definition 16.1. Let J be a small category[1]. Let C be another category. A **diagram of shape J in C** is simply a functor $T\colon \mathsf{J} \to \mathsf{C}$. We call J an **index category**.

This is easiest to parse with an example.

Example 16.2. Let J be a category with exactly three objects, $\{\heartsuit, \spadesuit, \diamondsuit\}$, and assume that there is unique morphism $\spadesuit \to \heartsuit$ and a unique morphism $\spadesuit \to \diamondsuit$, and that the only other morphisms are the identity morphisms (whose existence is forced). We write this pictorially as

$$\begin{array}{ccc} \spadesuit & \longrightarrow & \heartsuit \\ \downarrow & & \\ \diamondsuit & & \end{array}$$

A functor $T\colon \mathsf{J} \to \mathsf{C}$ is the same thing as a triple of objects (A, B_1, B_2) in $\mathrm{obj}(\mathsf{C})$ together with a choice of two morphisms $f_1\colon A \to B_1$ and $f_2\colon A \to B_2$.

$$\begin{array}{ccc} \spadesuit & \longrightarrow & \heartsuit \\ \downarrow & & \\ \diamondsuit & & \end{array} \quad \xrightarrow{\text{apply the functor } T} \quad \begin{array}{ccc} A & \xrightarrow{f_1} & B_1 \\ {\scriptstyle f_2}\downarrow & & \\ B_2 & & \end{array}$$

This thus recovers what we called a "diagram" in C in Definition 5.7.

We now generalise the notion of a "solution" to a diagram. To help keep the various objects distinct, we will usually use the letters α, β, γ to indicate objects of our indexing category J.

Definition 16.3. Let J be an index category and let $T\colon \mathsf{J} \to \mathsf{C}$ be a diagram in C. A **solution**[2] for T is an object C of C together with a family of morphisms $c_\alpha\colon T(\alpha) \to C$ in C for each object $\alpha \in \mathrm{obj}(\mathsf{J})$ such that if $i\colon \alpha \to \beta$ is any morphism in J then the following commutes:

$$\begin{array}{ccc} T(\alpha) & \xrightarrow{c_\alpha} & C \\ {\scriptstyle T(i)}\downarrow & \nearrow_{c_\beta} & \\ T(\beta) & & \end{array}$$

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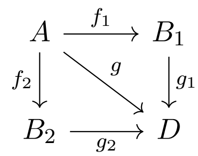
[1]This means that $\mathrm{obj}(\mathsf{J})$ is a set. It does *not* necessarily imply that J is actually "small" (since sets can be very large!) Nevertheless, in most our examples J is indeed rather small; for instance our running example (Example 16.2) has J having three objects and two non-identity morphisms.

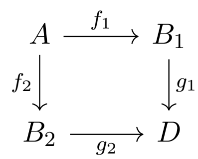
[2]This is usually called a "co-cone" but I prefer the name "solution".

We write $(C, \{c_\alpha\})$ to indicate the solution.

Example 16.4. Let us stick with the setup from Example 16.2. In this case a solution is simply an object D of C together with three morphisms $g = c_\spadesuit \colon A \to D$, $g_1 = c_\heartsuit \colon B_1 \to D$ and $g_2 = c_\diamondsuit \colon B_2 \to D$ such that the following commutes:



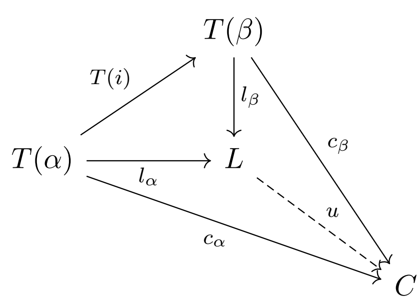
In fact, we don't need to specify the morphism g, since the commutativity requirement means that $g = g_1 \circ f_1 = g_2 \circ f_2$. So a solution is simply an object D of C together with two morphisms $g_1 \colon B_1 \to D$ and $g_2 \colon B_2 \to D$ such that the following commutes:



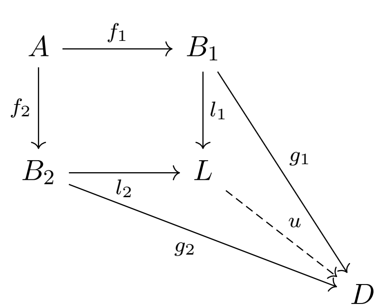
This recovers the notion of solution as given in Definition 5.7.

Now let us appropriately generalise the pushout construction.

Definition 16.5. Let J be an index category and let $T \colon \mathsf{J} \to \mathsf{C}$ be a diagram in C. A **colimit** is a solution $(L, \{l_\alpha\})$ that satisfies the following *universal property*: if $(C, \{c_\alpha\})$ is any other solution then there exists a *unique* morphism $u \colon L \to C$ such that the following diagram commutes for every morphism $i \colon \alpha \to \beta$ in J:



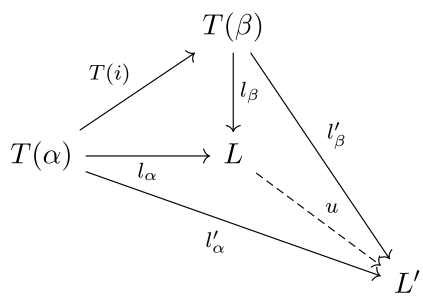
Example 16.6. Going back to Example 16.2, a colimit is simply a pushout in the sense of Definition 5.7: a solution (L, l_1, l_2) such that for any other solution (D, g_1, g_2) there is a unique map $u \colon L \to D$ such that the following commutes:

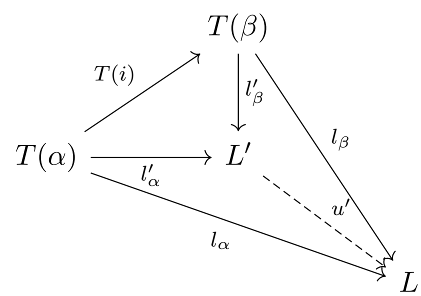


By now you should be completely happy with proving that colimits are unique if they exist. But since I'm exceedingly generous, I will do it for you.

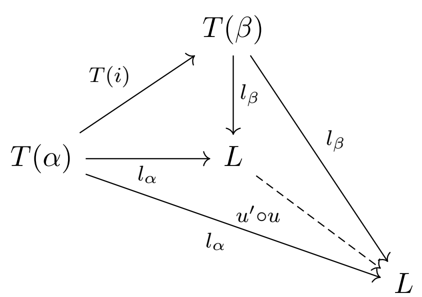
Lemma 16.7. *Let J be an index category and let $T\colon \mathsf{J} \to \mathsf{C}$ be a diagram in C. If a colimit exists then it is unique up to isomorphism.*

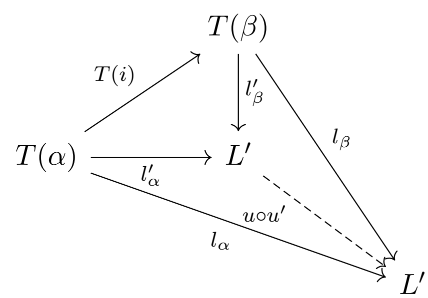
Proof. If $(L, \{l_\alpha\})$ and $(L', \{l'_\alpha\})$ are two limits then we get unique morphisms $u\colon L \to L'$ and $u'\colon L' \to L$ such that the following both commute for every morphism $i\colon \alpha \to \beta$ in J:





Then the composition $u' \circ u\colon L \to L$ and $u \circ u'\colon L \to L$ both make the following diagrams commute:





But since id_L and $\mathrm{id}_{L'}$ also make these two diagrams commute respectively, we see by uniqueness that $u' \circ u = \mathrm{id}_L$ and $u \circ u' = \mathrm{id}_{L'}$. This completes the proof. ∎

We usually write $\operatorname{colim} T$ to indicate the colimit. This notation is somewhat imprecise (as it should really include the maps l_α), but we do it anyway.

Here is another example:

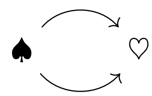
Example 16.8. Take J to have exactly two objects and no morphisms (apart from the identity morphisms).

$$\spadesuit \qquad\qquad \heartsuit$$

Let $T\colon \mathsf{J} \to \mathsf{Sets}$ or $T\colon \mathsf{J} \to \mathsf{Groups}$. This type of colimit is called a **coproduct**. In the category Sets, it is simply the disjoint union $T(\spadesuit) \sqcup T(\heartsuit)$. In the category of groups it is the free product $T(\spadesuit) * T(\heartsuit)$ (cf. Problem C.1.) In general, the coproduct (in an arbitrary category) C of two objects A and B is denoted by $A \sqcup B$ (if it exists).

One can also use colimits to get the category theory analogue of an equivalence relation.

Example 16.9. Let J have two objects and two morphisms:

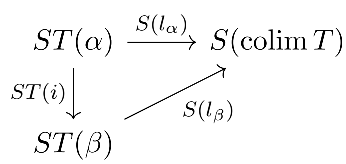


A colimit for this J is called a **coequaliser** in C. On Problem Sheet H you get to investigate what coequalisers are in Sets, Groups and Top.

Remark 16.10. Suppose J is an index category and $T\colon \mathsf{J} \to \mathsf{C}$ is a functor. Suppose $S\colon \mathsf{C} \to \mathsf{D}$ is another functor. Then $(S \circ T)\colon \mathsf{J} \to \mathsf{D}$ is a diagram in D. Assume that both the colimits $\operatorname{colim} T$ and $\operatorname{colim}(S \circ T)$ exist (as objects of C and D respectively). Then we claim there is a natural morphism

$$u\colon \operatorname{colim}(S \circ T) \to S(\operatorname{colim} T).$$

Indeed, if $l_\alpha\colon T(\alpha) \to \operatorname{colim} T$ are the maps from T being a colimit, then applying S we get maps $S(l_\alpha) : ST(\alpha) \to S(\operatorname{colim} T)$. Moreover if $i\colon \alpha \to \beta$ is a morphism in J then the following commutes:



This shows that $(S(\operatorname{colim} T), \{S(l_\alpha)\})$ form a solution to the diagram $S \circ T$. Thus by the universal property of the colimit, we get a unique morphism

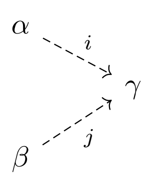
$$u\colon \operatorname{colim}(S \circ T) \to S(\operatorname{colim} T)$$

as claimed

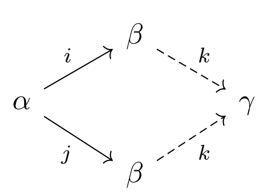
Let us now discuss a refinement of the idea of a colimit.

Definition 16.11. Let J be a small category with $\mathsf{obj}(\mathsf{J}) \neq \emptyset$. We say that J is **filtered** if the following two properties hold:

1. If $\alpha, \beta \in \operatorname{obj}(\mathsf{J})$ then there exists $\gamma \in \operatorname{obj}(\mathsf{J})$ such that $\operatorname{Hom}(\alpha, \gamma) \neq \emptyset$ and $\operatorname{Hom}(\beta, \gamma) \neq \emptyset$.



2. If $i, j\colon \alpha \to \beta$ are any two morphisms then there exists an object γ of J and a morphism $k\colon \beta \to \gamma$ such that $k \circ i = k \circ j$.



Example 16.12. Let $(\Lambda, \preceq)$ be a **directed set**. This means that $\preceq$ is a binary relation on Λ which is *reflexive* and *transitive*, which has the additional property that for all $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$. From $(\Lambda, \preceq)$ we can form a filtered category $\mathsf{J}(\Lambda, \preceq)$ as follows:

- Take $\operatorname{obj}(\mathsf{J}) = \Lambda$.
- Set $\operatorname{Hom}(\alpha, \beta) = \emptyset$ if $\alpha \not\preceq \beta$, and if $\alpha \preceq \beta$ let $\operatorname{Hom}(\alpha, \beta)$ consist of a single element $i_{\alpha,\beta}$.
- If $\alpha \preceq \beta \preceq \gamma$ then define $i_{\beta,\gamma} \circ i_{\alpha,\beta} := i_{\alpha,\gamma}$.

Definition 16.13. Let C be a category. A **filtered diagram** in C is a functor $T\colon \mathsf{J} \to \mathsf{C}$ where J is a filtered index category. A **filtered colimit** in C is a colimit of a filtered diagram $T\colon \mathsf{J} \to \mathsf{C}$. In this case we bung an arrow underneath and write $\underrightarrow{\mathrm{colim}}\, T$ instead of just $\operatorname{colim} T$.

In fact, in this course we will basically only ever need one type of filtered colimit (we will need the general case in Algebraic Topology II).

Example 16.14. A **sequential colimit** is a filtered colimit on the directed set $(\mathbb{N}, \leq)$. Let us spell out what this means explicitly, since this is the most important type of filtered colimit. Let C be a category, and assume we are given a sequence

$$f_n\colon C_n \to C_{n+1}, \qquad n \in \mathbb{N},$$

of morphisms in C. This data is equivalent to a filtered diagram $T\colon \mathsf{J}(\mathbb{N}, \leq) \to \mathsf{C}$: namely define $T(n) := C_n$ and for $n \leq m$ define

$$T(i_{n,m}) := f_{m-1,m} \circ f_{m-2,m-1} \circ \cdots \circ f_{n,n+1}\colon C_n \to C_m.$$

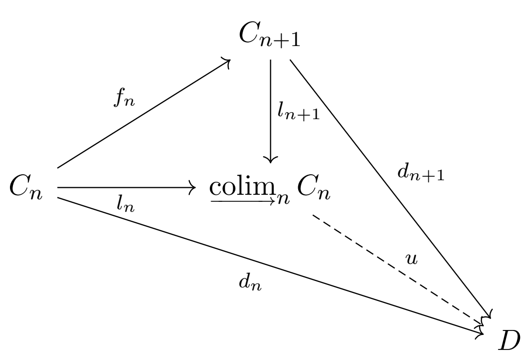
The filtered colimit of T is an object $\underrightarrow{\mathrm{colim}}\, T$ (which we will usually write as $\underrightarrow{\mathrm{colim}}_n\, C_n$ instead) together with a family of morphisms $l_n\colon C_n \to \underrightarrow{\mathrm{colim}}_i\, C_i$ for $n \in \mathbb{N}$ such that

$$l_{n+1} \circ f_n = l_n, \qquad \forall\, n \in \mathbb{N}.$$

This satisfies the universal property that if $(D, \{d_n\})$ is object of C and a family of morphisms $d_n\colon C_n \to D$ for $n \in \mathbb{N}$ such that

$$d_{n+1} \circ f_n = d_n, \qquad \forall\, n \in \mathbb{N},$$

then there exists a *unique* morphism $u\colon \underrightarrow{\mathrm{colim}}_n\, C_n \to D$ such that the following diagram commutes:



$\forall\, n \in \mathbb{N}$.

Remark 16.15. For the rest of this lecture I will work with a general filtered colimit. If you find the formalism confusing, I strongly urge you to rewrite everything out in the special case of a sequential colimit.

Let us now prove that filtered colimits exist in all our favourite categories. Actually this all works for arbitrary colimits (rather than filtered colimits) but the construction is rather messier. We begin with Sets.

Example 16.16. Let $T\colon \mathsf{J} \to \mathsf{Sets}$ be a filtered diagram. Let us construct the filtered colimit. First, form the disjoint union

$$Z := \bigsqcup_{\alpha \in \mathrm{obj}(\mathsf{J})} T(\alpha).$$

We now define an equivalence relation $\sim$ on Z by declaring that $a \in T(\alpha) \sim b \in T(\beta)$ if and only if there exists $i\colon \alpha \to \gamma$ and $j\colon \beta \to \gamma$ such that $T(i)a = T(j)b$:

$$a \sim b \qquad \Leftrightarrow \qquad T(i)a = T(j)b.$$

It is somewhat tedious to check that $\sim$ really is an equivalence relation on Z, and I will leave this you. Given this though, it is easy to see that $X := Z/\sim$ satisfies the universal property, where the maps $l_\alpha\colon T(\alpha) \to X$ are induced from the inclusions $T(\alpha) \to Z$.

Now let us check they exist in Top.

Example 16.17. Colimits always exist in Top. Suppose $T\colon \mathsf{J} \to \mathsf{Top}$ is a diagram. Let $F\colon \mathsf{Top} \to \mathsf{Sets}$ denote the "forgetful" functor (cf. Example 1.14.) Then $F \circ T$ is a diagram in Sets. We just constructed the filtered colimit in Set. Let us call this set X. In fact, X will also work for the filtered colimit in Top, once we give it a topology.

For this, note that the construction in the previous example provides us with functions $l_\alpha : F(T(\alpha)) \to X$ (here $F(T(\alpha))$ is just the underlying set of the topological space $T(\alpha)$.) Let us now endow X with a topology by declaring that the functions l_α are continuous. Explicitly, this means that a set $U \subseteq X$ is open if and only if $l_\alpha^{-1}(U)$ is open in $T(\alpha)$ for each $\alpha \in \mathrm{obj}(\mathsf{J})$. Then the functions $l_\alpha\colon T(\alpha) \to X$ are now well-defined morphisms in Top (i.e. continuous functions), and it is easy to see that $(X, \{l_\alpha\})$ verifies the universal property. Thus $X = \varinjlim T$. Note that this is an example where the natural map from Remark 16.10 is an isomorphism

$$F(\varinjlim T) = \varinjlim (F \circ T)$$

(as they are both the set X.)

Remark 16.18. Without additional hypotheses on the types of topological spaces we are working with, the colimit can be rather badly behaved. Indeed, sometimes the topology one gets on $\varinjlim T$ can be "wrong". We will discuss this more in later lectures. Next semester, we will introduce **homotopy colimits** which are much better behaved in Top.

Now we check that filtered colimits exist in Ab.

Example 16.19. Let J be a filtered index category and $T\colon \mathsf{J} \to \mathsf{Ab}$ be a filtered diagram. This time, we consider the forgetful functor $F\colon \mathsf{Ab} \to \mathsf{Sets}$. Our strategy is the same as before. We will endow the set $X = \varinjlim(F \circ T)$ from Example 16.16 with the structure of an abelian group. Given $a \in T(\alpha)$ and $b \in T(\beta)$, choose an object γ such that there exist morphisms $i\colon \alpha \to \gamma$ and $j\colon \beta \to \gamma$. Then define

$$[a] + [b] := [T(i)a + T(j)b].$$

It is again slightly tedious to check this operation is well-defined[3], and I will lazily leave this to you. Once this is done though, the functions $l_\alpha\colon T(\alpha) \to X$ are well-defined morphisms in Ab (i.e. group homomorphisms), and it is easy to see that $(X, \{l_\alpha\})$ verifies the universal property. Thus $X = \varinjlim T$, and once again we have

$$F(\varinjlim T) = \varinjlim(F \circ T).$$

Remark 16.20. Suppose $T\colon \mathsf{J} \to \mathsf{Ab}$ is a filtered diagram and $[a] \in \varinjlim T$ where $a \in T(\alpha)$. Then $[a] = 0 \in \varinjlim T$ *if and only if* there exists a morphism $i\colon \alpha \to \beta$ in J such that $T(i)a = 0 \in T(\beta)$. Here "if" is immediate from the definition, but "only if" requires a little bit of thought (which I invite you to do!)

In particular, if we restrict to sequential colimits, say $C = \varinjlim_n C_n$ where each C_n is an abelian group, then an element $c \in C_k$ represents the zero element in C if and only if there exists a *finite* $m \geq k$ such that c is mapped onto the zero element in C_m.

Filtered colimits always exist in Comp, too.

Example 16.21. The same thing works Comp. Let J be a filtered index category and let $T\colon \mathsf{J} \to \mathsf{Comp}$ be a filtered diagram. Thus for each $\alpha \in \mathrm{obj}(\mathsf{J})$ we get a chain complex $(T(\alpha)_\bullet, \partial^\alpha)$, and for each morphism $i\colon \alpha \to \beta$, we get a chain map $T(i)_\bullet\colon T(\alpha)_\bullet \to T(\beta)_\bullet$.

For each fixed $n \in \mathbb{Z}$, we get a functor $T_n\colon \mathsf{J} \to \mathsf{Ab}$ given by $T_n(\alpha) = T(\alpha)_n$. Since we already know filtered colimits exist in Ab, this gives us abelian groups $\varinjlim T_n$ for each n. Denote by $l_n^\alpha\colon T(\alpha)_n \to \varinjlim T_n$ the map associated to this colimit. The boundary operators ∂^α induce maps

$$\begin{array}{ccc} T(\alpha)_n & \xrightarrow{\partial^\alpha} & T(\alpha)_{n-1} \\ {\scriptstyle l_n^\alpha}\downarrow & & \downarrow{\scriptstyle l_{n-1}^\alpha} \\ \varinjlim T_n & \xrightarrow[\partial]{} & \varinjlim T_{n-1} \end{array}$$

These operators square to zero, and hence we get a chain complex $\left(\varinjlim T_\bullet, \partial\right)$. This is the filtered colimit of T.

We conclude this lecture by proving the following rather abstract result, which roughly speaking says that the homology functor *commutes* with taking filtered colimits. This result only works for filtered colimits (rather than arbitrary colimits.)

[3] That is, if $a \sim a'$ and $b \sim b'$ then with the obvious notation we need $T(i)a + T(j)b \sim T(i')a' + T(j')b'$.

If $T\colon \mathsf{J} \to \mathsf{Comp}$ is a filtered diagram, for any $n \in \mathbb{Z}$ the composition $H_n \circ T\colon \mathsf{J} \to \mathsf{Ab}$ is a filtered diagram of abelian groups. In order to make the notation a little bit less horrendous, let us abbreviate $C^\alpha_\bullet := T(\alpha)_\bullet$ for $\alpha \in \mathrm{obj}(\mathsf{J})$. Thus $H_n \circ T$ is the diagram where

$$\alpha \mapsto H_n(C^\alpha_\bullet), \qquad \alpha \in \mathrm{obj}(\mathsf{J})$$

and

$$i \in \mathrm{Hom}(\alpha, \beta) \qquad \mapsto \qquad H_n(T(i))\colon H_n(C^\alpha_\bullet) \to H_n(C^\beta_\bullet).$$

We denote by $\varinjlim(H_n \circ T)$ the associated filtered colimit.

THEOREM 16.22. *Let J be a filtered index category and $T\colon \mathsf{J} \to \mathsf{Comp}$ a filtered diagram. Then for any $n \in \mathbb{Z}$, one has*

$$\varinjlim(H_n \circ T) = H_n\left(\varinjlim T\right).$$

Proof. We will prove that the natural map

$$u\colon \varinjlim(H_n \circ T) \to H_n\left(\varinjlim T\right). \tag{16.1}$$

from Remark 16.10 is an isomorphism. Suppose $z \in \varinjlim T$ is a cycle. Choose a representative $a \in C^\alpha_n$ for some $\alpha \in \mathrm{obj}(\mathsf{J})$. We don't necessarily know that a is a cycle in C^α_n, but since z is a cycle, a must become a cycle "eventually". In other words, there exists $\beta \in \mathrm{obj}(\mathsf{J})$ and a morphism $i\colon \alpha \to \beta$ such that $\partial^\beta(T(i)a) = 0$ in C^β_{n-1}. Thus $T(i)a$ defines an element in $\langle T(i)a\rangle \in H_n(C^\beta_\bullet)$. This element represents an element $x \in \varinjlim(H_n \circ T)$, and by construction, $u(x) = \langle z\rangle$. This shows that (16.1) is surjective.

Now let us prove that the map (16.1) is injective. Suppose an element x in $\varinjlim(H_n \circ T)$ goes to zero under the map u. Choose a representative $\langle c\rangle \in H_n(C^\alpha_\bullet)$ of x. Let $c \in C^\alpha_n$ represent $\langle c\rangle$. Then c also represents an element z of $\varinjlim T$. By assumption z is a boundary ∂y; let us now choose a representative $b \in C^\beta_{n+1}$ of y. Now since J is filtered, we can choose an object $\gamma \in \mathrm{obj}(\mathsf{J})$ and morphisms $i\colon \alpha \to \gamma$ and $j\colon \beta \to \gamma$. Then $T(i)c \in C^\gamma_n$ and $T(j)b \in C^\gamma_{n+1}$, and $\partial^\gamma(T(j)b) = T(i)c$. This means that $H_n(T(i))\langle c\rangle = \langle T(i)c\rangle = 0 \in H_n(C^\gamma_\bullet)$, and thus $\langle c\rangle$ represents the zero element in $\varinjlim(H_n \circ T)$. ∎

LECTURE 17

The Jordan-Brouwer Separation Theorem

We begin this lecture with a bit more abstract nonsense. We then use this to prove the famouse *Jordan-Brouwer Separation Theorem*.

Our first result shows that under favourable assumptions, the singular chain complex functor also commutes with colimits. Before proving this, let us talk a little bit about the different *separation axioms* a topological space can have.

DEFINITION 17.1. Let X be a topological space. We say that:

- X is a T_1 **space** if the points are closed in X.
- X is a **weakly Hausdorff space** if for every continuous map $f\colon K \to X$ from a compact Hausdorff space, $f(K)$ is closed in X.

On Problem Sheet I you will show:

LEMMA 17.2. *One has*

$$\{\textit{Hausdorff spaces}\} \subsetneq \{\textit{weakly Hausdorff spaces}\} \subsetneq \{T_1 \textit{ spaces}\}.$$

REMARK 17.3. In algebraic topology, the weakly Hausdorff assumption is typically the most useful one to make. In fact, most of modern algebraic topology implicitly always works with the subcategory of Top of **compactly generated** spaces. A compactly generated space is (by definition) a weakly Hausdorff k-space (we will define k-spaces in Algebraic Topology II). This category is much more "convenient" than Top itself: it is large enough that only truly pathological topological spaces (that no self-respecting algebraic topologist would ever care about) fail to lie in it, and behaves nicely under various categorical operations. However most of this goes beyond the scope of the course, and so we will typically just use either the T_1 axiom or the Hausdorff axiom.

Let us consider sequential colimits of *embeddings* in Top. Thus suppose we are given a family

$$i_n\colon X_n \to X_{n+1}, \qquad n \in \mathbb{N},$$

of continuous maps such that each i_n is an *embedding* (this means that i_n is a homeomorphism onto its image.) Replacing X_n with the homeomorphic space $i_n(X_n)$, we may assume that $X_n \subseteq X_{n+1}$ and that i_n is the inclusion. Then the sequential colimit $X := \underrightarrow{\operatorname{colim}}_n X_n$ is simply the union:

$$X = \bigcup_{n\in\mathbb{N}} X_n,$$

topologised so that a set $C \subseteq X$ is closed if and only if $C \cap X_n$ is closed for each n. However the topology on X might be "wrong". Indeed, for each n the colimit gives us a map $l_n \colon X_n \to X$. One would hope that these maps would also be embeddings (i.e. homeomorphisms onto their images), but in general this is not true. In particular, X_n does *not* have to be closed in X.

If however we assume i_n is a *closed* embedding (i.e. $i_n(X_n)$ is a closed subspace of X_{n+1} for each $n \in \mathbb{N}$) then the map $l_n \colon X_n \to X$ is itself a closed inclusion[1], and hence also an embedding. In particular, X_n is closed in X. If each X_n is T_1 then so is X by definition of the colimit topology.

We now prove the following key result.

PROPOSITION 17.4. *Suppose we are given a family $i_n \colon X_n \to X_{n+1}$ for $n \in \mathbb{N}$ of closed inclusions. Assume in addition that for each n the space X_n is T_1. Then*

$$\varinjlim_n C_\bullet(X_n) = C_\bullet(\varinjlim_n X_n).$$

Proof. The main step in the proof is the following claim.

LEMMA 17.5. *Let $f \colon K \to X$ be a continuous map from a compact space K. Then $f(K)$ is contained in one of the X_n.*

Proof. Assume for contradiction the claim is false. Then for each $n \in \mathbb{N}$ we may select a point $x_n \in K$ such that $f(x_n) \notin X_n$. Let $S_m := \{f(x_n) \mid n \geq m\}$. Then $S_{m+1} \subset S_m$ for each $m \in \mathbb{N}$ and $\bigcap_m S_m = \emptyset$. Moreover S_m meets X_n in a finite set for each n, and thus since each X_n is T_1, it follows that $S_m \cap X_n$ is a closed set in X_n for each n. Thus by definition of the colimit topology on X, S_m is closed in X for each m. This means that if we set $Y_m := X \setminus S_m$ then Y_m is open in X for each m and $\bigcup_m Y_m = X$. In particular, the Y_m's form a cover of $f(K)$. But no finite subcover of them can cover $f(K)$, since any finite subcover is contained in the largest, and by construction $f(K)$ is not a subset of any of the Y_m. Thus $f(K)$ is not compact in X, which contradicts K being compact and f continuous. ∎

Going back to the proof of Proposition 17.4, it suffices now to observe that any singular simplex $\sigma \colon \Delta^m \to X$ is contained in some X_n by Lemma 17.5. The result now follows immediately from the definition of the colimit in Comp. ∎

The following corollary gives a topological version of Theorem 16.22. In the statement and proof, let us temporarily write H_k^{sing} for the singular homology functor, which is the composition $H_k^{\text{sing}} = H_k \circ C_\bullet$ where $C_\bullet$ is the singular chain complex functor and $H_k \colon \mathsf{Comp} \to \mathsf{Ab}$ is the usual homology functor.

COROLLARY 17.6. *Suppose we are given a family $i_n \colon X_n \to X_{n+1}$ for $n \in \mathbb{N}$ of closed inclusions. Assume in addition that for each n the space X_n is T_1. Then for each $k \geq 0$, the singular homology groups satisfy*

$$H_k^{\text{sing}}(\varinjlim_n X_n) = \varinjlim_n H_k^{\text{sing}}(X_n).$$

[1]Exercise: Why?

Proof. By Proposition 17.4, $C_\bullet(\varinjlim_n X_n) = \varinjlim_n C_\bullet(X_n)$ (as chain complexes). By Theorem 16.22 we have $H_k(\varinjlim_n C_\bullet(X_n)) = \varinjlim_n H_k(C_\bullet(X_n))$. The latter is by definition the sequential colimit $\varinjlim_n H_k^{\mathrm{sing}}(X_n)$. ■

Remark 17.7. The fundamental group of X is defined by looking at continuous maps $u\colon S^1 \to X$. Since S^1 is also a compact Hausdorff space, exactly the same argument as in the proof of Proposition 17.4 shows that under these hypotheses, one also has

$$\pi_1(X,p) = \varinjlim_n \pi_1(X_n,p), \qquad \forall\, p \in X_1$$

(note that the right-hand side is a colimit in the category Groups.)

Remark 17.8. In fact, in this lecture we will use an easier version of Proposition 17.4 and Corollary 17.6. Namely, both statements are still true if we assume instead that each $X_n \subset X$ is an *open* set. In fact, this argument is much easier (and does not require any separation axioms). Indeed, if each X_n is open and $f\colon K \to X$ is a continuous map from any compact space, then $\{X_n\}$ is an open cover of the compact set $f(K)$, and hence there is a finite subcover. Thus $f(K)$ is contained in a single X_n, and so the analogue of Lemma 17.5 holds.

Next lecture when we discuss cell complexes we will need the harder version we proved above.

We now move towards proving the Jordan-Brouwer Separation Theorem. The first step is the following two results, both of which concern the *reduced* homology groups of a sphere with a set removed.

Proposition 17.9. *Let $X \subset S^n$ be a subset homeomorphic to $I^m := [0,1]^m$ for $0 \le m \le n$. Then*

$$\tilde{H}_k(S^n \setminus X) = 0, \qquad \forall\, k \ge 0.$$

Proof. We argue by induction on m. If $m = 0$ then X is a point and $S^n \setminus X \cong \mathbb{R}^n$, which has zero reduced homology by Corollary 12.20. Now assume the result is true for $m-1$. Let $f\colon X \to I^m$ be our given homeomorphism. Split the m-cube I^m into its upper and lower halves:

$$I^+ = \left\{(x_1,\dots,x_m) \mid x_1 \ge \frac{1}{2}\right\}, \qquad I^- := \left\{(x_1,\dots,x_m) \mid x_1 \le \frac{1}{2}\right\}.$$

Then $I^+ \cap I^-$ is homeomorphic to I^{m-1}. Let $X^\pm := f^{-1}(I^\pm)$ and let $Y := X^+ \cap X^-$ so that $Y \cong I^{m-1}$. The set $S^n \setminus Y$ may be written as the union of two sets $(S^n \setminus X^+) \cup (S^n \setminus X^-)$ which satisfy the requirements of the Mayer-Vietoris sequence. Fix $k \ge 0$. We get an exact sequence

$$\dots \tilde{H}_{k+1}(S^n \setminus Y) \to \tilde{H}_k(S^n \setminus X) \to \left(\tilde{H}_k(S^n \setminus X^+) \oplus \tilde{H}_k(S^n \setminus X^-)\right) \to \tilde{H}_k(S^n \setminus Y) \to \dots$$

By the inductive hypotheses the end terms are both zero. Thus we have an isomorphism

$$\tilde{H}_k(S^n \setminus X) \xrightarrow{(H_k(\imath^+),H_k(\imath^-))} \left(\tilde{H}_k(S^n \setminus X^+) \oplus \tilde{H}_k(S^n \setminus X^-)\right)$$

Suppose now we have a non-zero homology class $\langle c\rangle \in \tilde{H}_k(S^n \setminus X)$. Then at least one of $H_k(\imath^+)\langle c\rangle$ and $H_k(\imath^-)\langle c\rangle$ are non-zero, where $\imath^\pm \colon S^n \setminus X \hookrightarrow S^n \setminus X^\pm$ are inclusions. Assume without loss of generality that $H_k(\imath^+)\langle c\rangle \neq 0$. Now we repeat the process, splitting X^+ into two pieces whose intersection is homeomorphic to I^{m-1}. In this manner a sequence of closed subsets of S^n may be constructed:

$$X = X_1 \supseteq X_2 \supseteq X_3 \supseteq \cdots$$

having that the property that the inclusion $S^n \setminus X \hookrightarrow S^n \setminus X_j$ induces a homomorphism in homology that takes our non-zero homology class to a non-zero element $\langle c_j\rangle \in \tilde{H}_k(S^n \setminus X_j)$. Set $Z := \bigcap_j X_j$. Since $S^n \setminus X_j \subset S^n \setminus Z$ is open for each j, the hypotheses of Remark 17.8 are satisfied[2], and thus

$$\tilde{H}_k(S^n \setminus Z) = \underrightarrow{\operatorname{colim}}_j \tilde{H}_k(S^n \setminus X_j). \tag{17.1}$$

Moreover since the map $\tilde{H}_k(S^n \setminus X_j) \to \tilde{H}_k(S^n \setminus X_{j+1})$ sends $\langle c_j\rangle$ to $\langle c_{j+1}\rangle$, by definition of the filtered colimit we end up with a non-zero element in $\langle c_\infty\rangle \in \tilde{H}_k(S^n \setminus Z)$ (cf. Remark 16.20.)

Now we play the joker: being an infinite intersection of shrinking m-cubes, Z is homeomorphic to I^{m-1}! Thus by the inductive hypothesis, $\tilde{H}_k(S^n \setminus Z) = 0$. This contradicts the existence of a non-zero class $\langle c\rangle \in \tilde{H}_k(S^n \setminus X)$, and thus we see that $\tilde{H}_k(S^n \setminus X) = 0$ as required. Since $k \geq 0$ was arbitrary, we are done. ∎

We now use this to prove the following corollary.

Corollary 17.10. *Let $S \subset S^n$ be a subset which is homeomorphic to S^m for some $0 \leq m \leq n-1$. Then*

$$\tilde{H}_k(S^n \setminus S) = \begin{cases} \mathbb{Z}, & k = n-m-1, \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Once again we induct on m. For $m = 0$, S is two points, and $S^n \setminus S \simeq S^{n-1}$. Since S^{n-1} has the desired homology, the case $m = 0$ follows. Now assume the result is true for $m-1$. Write $S = B^+ \cup B^-$ where $B^\pm$ are homeomorphic to closed hemispheres in S^m and $R := B^+ \cap B^-$ is homeomorphic to S^{m-1}. Now the Mayer-Vietoris sequence for reduced homology (Corollary 14.10) applied to $S^n \setminus R = (S^n \setminus B^+) \cup (S^n \setminus B^-)$ has the form

$$\Big(\tilde{H}_{k+1}(S^n \setminus B^+) \oplus \tilde{H}_{k+1}(S^n \setminus B^-)\Big) \to \tilde{H}_{k+1}(S^n \setminus R)$$
$$\to \tilde{H}_k(S^n \setminus S) \to \Big(\tilde{H}_k(S^n \setminus B^+) \oplus \tilde{H}_k(S^n \setminus B^-)\Big).$$

The end terms are zero by Proposition 17.9. Thus we obtain an isomorphism $\tilde{H}_{k+1}(S^n \setminus R) \to \tilde{H}_k(S^n \setminus S)$, which allows us to complete the inductive step. ∎

[2] Strictly speaking we are working with reduced homology here, but this makes no difference, as the reader is invited to check (alternatively, prove the case $k = 0$ directly!)

We can now prove the following famous result.

Theorem 17.11 (The Jordan-Brouwer Separation Theorem). *Suppose* $f\colon S^{n-1} \to S^n$ *is an embedding. Then* $S^n \setminus f(S^{n-1})$ *has two components, and* $f(S^{n-1})$ *is boundary of each component.*

Proof. Let $S = f(S^{n-1})$. Then by Corollary 17.10, $H_0(S^n \setminus S) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_k(S^n \setminus S) = 0$ for $k > 0$. Thus by Corollary 12.12, $S^n \setminus S$ has two path components. Since S is closed, $S^n \setminus S$ is open, and hence in particular locally pathwise connected. Thus the path components agree with the connected components.

Let X and Y be the two components of $S^n \setminus S$. Since $X \cup S$ is closed, the boundary of $\partial X := \overline{X} \setminus X^\circ$ is contained in S. We claim that also $S \subseteq \partial X$, whence $S = \partial X$. Let $p \in S$ and let U be a neighbourhood of p in S^n. Since S is an embedded copy of S^{n-1}, there is a subset C of $U \cap S$ with $p \in C$ and $S \setminus C$ homeomorphic to B^{n-1}. See Figure 17.1.

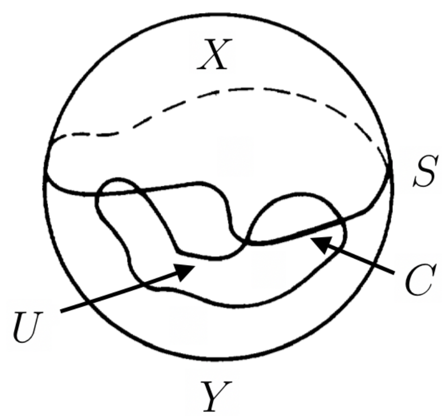


Figure 17.1: Proving the Jordan-Brouwer Separation Theorem.

Now by Proposition 17.9 and the dimension axiom, we see that $S^n \setminus (S \setminus C)$ has one path component. Suppose $x \in X$ and $y \in Y$. Then there is a path u from x to y with image in $S^n \setminus (S \setminus C)$. Since X and Y are distinct path components of $S^n \setminus S$, the path u must intersect C, and hence C contains points belonging to both $\overline{X}$ and $\overline{Y}$. Thus $p \in \partial X$. Since p was an arbitrary point of S, we have $S \subseteq \partial X$ and so also $S = \partial X$. Similarly $S = \partial Y$. The proof is complete. ■

On Problem Sheet I, you get to prove the following equally famous result, also due to Brouwer.

Theorem 17.12 (Invariance of Domain Theorem). *Suppose* U *and* U' *are two subsets of* S^n *and* $f\colon U \to U'$ *is a homeomorphism. If* U *is open then so is* U'.

The theorem is of course obvious if "open" is replaced by "closed". Likewise the theorem is also clear if $f = g|_U$, where $g\colon S^n \to S^n$ is a homeomorphism of the entire sphere S^n and $U' = g(U)$. However the theorem is not true for arbitrary spaces. For example, take $U = (1/2, 1]$ and $U' = (0, 1/2]$. Then $f\colon U \to U'$ given by $x - \frac{1}{2}$ is a homeomorphism and U is open in I. But U' is not. The proof of Theorem 17.12 is a simple application of the Jordan-Brouwer Separation Theorem 17.11, and the meaning of the name "invariance of domain" is explained by the following corollary (also on Problem Sheet I.)

COROLLARY 17.13. *If* $\mathbb{R}^n$ *contains a subspace homeomorphic to* $\mathbb{R}^m$ *then* $m \leq n$.

LECTURE 18

Attaching spaces and cell complexes

In this lecture we introduce a nice class of topological spaces called **cell complexes**[1]. We begin by investigating pushouts in Top.

DEFINITION 18.1. Let X and Y be topological spaces, and let $X' \subseteq X$ be a *closed* subspace. Let $f\colon X' \to Y$ be continuous. We define the **adjunction space** $X \cup_f Y$ to be obtained by taking the disjoint union $X \sqcup Y$ and then identifying x with $f(x)$ for all $x \in X'$. Slightly more formally, $X \cup_f Y$ is the space $(X \sqcup Y)/\sim$, where $\sim$ is the smallest equivalence relation (cf. the definition from the solution to Problem H.5) on $X \sqcup Y$ such that $x \sim f(x)$ for $x \in X'$.

The canonical inclusions $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$ induce maps $g\colon X \to X \cup_f Y$ and $j\colon Y \to X \cup_f Y$. The next piece of point-set topology is on Problem Sheet I.

LEMMA 18.2. *The diagram*

$$\begin{array}{ccc} X' & \xrightarrow{\;\;f\;\;} & Y \\ {\scriptstyle \imath}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow[\;\;g\;\;]{} & X \cup_f Y \end{array}$$

is a pushout in Top. *Moreover*

1. *The map j is a closed embedding,*
2. *The restriction of g to $X \setminus X'$ is an open embedding.*
3. *If X and Y are T_1 spaces then so is $X \cup_f Y$.*
4. *The quotient map $X \sqcup Y \to X \cup_f Y$ is closed if and only if f is closed.*
5. *If X and Y are Hausdorff and $X' \subseteq X$ is compact then $X \cup_f Y$ is Hausdorff.*
6. *If X is compact and $X \cup_f Y$ is Hausdorff then $X \mapsto g(X)$ is a quotient map.*

Informally, one should think of $X \cup_f Y$ as being obtained from Y by attaching $X \setminus X'$ to it. We call f the **attaching map** and g the **characteristic map** of the adjunction space.

EXAMPLE 18.3. Suppose $Y = \{*\}$ is a space with one point. Then there is only one map $f\colon X' \to \{*\}$ (for $X' \neq \emptyset$), and one has $X \cup_f \{*\} \cong X/X'$.

The following lemma is a partial converse to Lemma 18.2.

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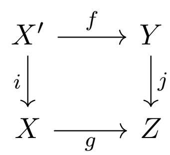
[1] They are often called *CW complexes* in the literature; the "C" stands for "closure finite" and the "W" stands for "weak topology". But I think the name "cell complex" is catchier.

Lemma 18.4. *Suppose we are given a commutative diagram of continuous maps:*

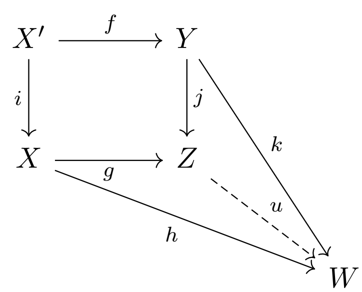


Assume that[2]

1. *i and j are closed embeddings,*
2. *g induces a bijection $X \setminus X' \to Z \setminus j(Y)$,*
3. *$g(X)$ is closed,*
4. *$X \to g(X)$ is a quotient map.*

Then (Z, j, g) is a pushout of $X \xleftarrow{i} X' \xrightarrow{f} Y$.

Proof. We verify the universal property. Suppose we are given a topological space W and continuous maps $h\colon X \to W$ and $k\colon Y \to W$ such that $h \circ i = k \circ f$. We need to build a unique continuous map $u\colon Z \to W$ such that the following commutes:



It is clear that Z is a *set-theoretic* pushout, so we get a unique map $u\colon Z \to W$ of *sets* that makes the diagram commute. It remains to show that u is continuous. Since j is a closed embedding, $u|_{j(Y)}$ is continuous. Since g is a quotient map, $u|_{g(X)}$ is continuous. Since $g(X)$ and $j(Y)$ are closed sets that cover Z, u is continuous. ∎

Definition 18.5. We denote $B^n \setminus S^{n-1}$ by E^n, so that E^n is the *open* unit ball[3]. We call E^n the **standard n-cell**. If X is a topological space, a set $E \subseteq X$ which is homeomorphic to E^n is called an **n-cell in X**. If $f\colon S^{n-1} \to Y$ is continuous, the space $B^n \cup_f Y$ is said to be obtained from Y by **attaching an n-cell.**

Proposition 18.6. *Let Z be a Hausdorff space and Y a closed subset. Suppose there exists a continuous map $g\colon B^n \to Z$ which induces a homeomorphism $E^n \to Z \setminus Y$. Then Z is obtained from Y by attaching an n-cell.*

[2]Condition (4) is automatic given the others if X is compact and Z is Hausdorff, as in the last part of Lemma 18.2.

[3]If $n = 0$, E^0 is just a point since S^0 is two points.

Proof. It suffices to show that $g(S^{n-1}) \subseteq Y$. Then if we set $f := g|_{S^{n-1}}$ the hypotheses of Lemma 18.4 are satisfied. Then Z is a pushout, and hence $Z \cong B^n \cup_f Y$ by uniqueness of the pushout. So suppose there exists $x \in S^{n-1}$ with $g(x) \in Z \setminus Y$. Since $g|_{E^n} : E^n \to Z \setminus Y$ is bijective, there exists a unique $y \in E^n$ with $g(x) = g(y)$. Let $U \subset B^n$ and $V \subset E^n$ be disjoint open neighbourhoods with $x \in U$ and $y \in V$. Then $g(V) \subset Z \setminus Y$ is open in Z, since $g|_{E^n} : E^n \to Z \setminus Y$ is a homeomorphism. But now using continuity of g, there exists an open neighbourhood $U' \subset U$ of x such that $g(U') \subset g(V)$. This contradicts $g|_{E^n}$ being injective. ∎

You will no doubt be surprised just how many of the "standard" spaces can be obtained by attaching cells. Let's see some examples, starting with a dumb one.

Example 18.7. For all $n \geq 1$, S^n is obtained by attaching an n-cell to a point. Indeed, this is simply Example 18.3 together with the observation that $B^n/S^{n-1} \cong S^n$, a fact that we have already used several times.

Example 18.8. Recall

$$\mathbb{R}P^n = \mathbb{R}^{n+1} \setminus \{0\}/\sim,$$

where $x \sim y$ if $x = ty$ for some $t \neq 0$. Restricting to vectors of length 1, we also see that

$$\mathbb{R}P^n = S^n/(x \sim -x),$$

that is, the sphere with antipodal points identified. But this is the same thing as taking the upper hemisphere and identifying antipodal points on the equator. Since the equator is just the sphere of dimension one less, we see that $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell. Explicitly, if $p\colon S^{n-1} \to \mathbb{R}P^{n-1}$ is the quotient map then

$$\mathbb{R}P^n = B^n \cup_p \mathbb{R}P^{n-1}.$$

This implies that we can write $\mathbb{R}P^n$ as a disjoint union

$$\mathbb{R}P^n = E_0 \cup E_1 \cup \cdots \cup E_n,$$

where each E_i denotes an i-cell.

Example 18.9. We can play a similar game with complex projective space $\mathbb{C}P^n$. This is the space of lines in $\mathbb{C}^{n+1}$ that go through the origin. Alternatively, $\mathbb{C}P^n = S^{2n+1}/\sim$, where two points $(z_1, \dots, z_{n+1}) \sim (w_1, \dots, w_{n+1})$ in $S^{2n+1} \subset \mathbb{C}^{n+1}$ are equivalent if and only if there exists $\lambda \in S^1$ such that $w_i = \lambda z_i$ for each $i = 1, \dots, n+1$. It follows that $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell, and hence we can write $\mathbb{C}P^n$ as a disjoint union

$$\mathbb{C}P^n = E_0 \cup E_2 \cup \cdots \cup E_{2n},$$

where each E_{2i} denotes a $2i$-cell.

Our next example requires a definition first.

Definition 18.10. Let X and Y be topological spaces, and let $p \in X$ and $q \in Y$. Define the **wedge** of the pointed spaces (X, p) and (Y, q) to be

$$X \vee Y := \big(X \times \{q\}\big) \cup \big(\{p\} \times Y\big) \subseteq X \times Y.$$

This is again a pointed space, where the basepoint is (p, q). This is actually the coproduct (cf. Example 16.8) in the category Top_*.

On Problem Sheet I you get to prove the following result.

Proposition 18.11. *For any $m, n \geq 0$, the space $S^m \times S^n$ can be obtained from $S^m \vee S^n$ by attaching a $(m+n)$-cell.*

Note that one cannot use Corollary 17.6 to compute the homology groups of $X \cup_f Y$, since a pushout is *not* a filtered colimit. Nevertheless, when we are attaching cells (i.e. X is a ball), we can use the Mayer-Vietoris sequence to compute the homology. Before stating the result, let us introduce a strengthening of the notion of a retract from Lecture 1.

Definition 18.12. Let $X' \subseteq X$ be a subspace. Let $\imath \colon X' \hookrightarrow X$ be the inclusion. We say that X' is a **deformation retract** of X if there exists a retract $r \colon X \to X'$ (as defined in Definition 1.2) such that $r \circ \imath = \mathrm{id}_{X'}$ and $\imath \circ r \simeq \mathrm{id}_X$. Equivalently, this means there exists a continuous function $H \colon X \times I \to X$ such that $H(x, 0) = x$ for all $x \in X$, $H(x, 1) \in X'$ for all $x \in X$, and $H(x', 1) = x'$ for all $x' \in X'$ (in this formulation, the retract r is given by $H(\cdot, 1)$.)

If X' is a deformation retract of X then the retract r is a homotopy equivalence, and hence X' and X have the same homotopy type. Next lecture we will introduce an even stronger version[4] called (rather unimaginatively) a *strong deformation retract*.

Proposition 18.13. *Let Y be a Hausdorff topological space. Let $n \geq 1$, and suppose $f \colon S^{n-1} \to Y$ is continuous. Then if $j \colon Y \to B^n \cup_f Y$ denotes the inclusion, there is an exact sequence*

$$\cdots \to H_k(S^{n-1}) \xrightarrow{H_k(f)} H_k(Y) \xrightarrow{H_k(j)} H_k\big(B^n \cup_f Y\big) \to H_{k-1}(S^{n-1}) \to \dots$$

which ends with

$$\dots H_0(S^{n-1}) \to \mathbb{Z} \oplus H_0(Y) \to H_0\big(B^n \cup_f Y\big) \to 0.$$

Proof. Write $B^n \cup_f Y$ as $U \cup V$, where U is the ball of radius $1/2$ inside B^n and V is $\big(B^n \cup_f Y\big) \setminus 0 \in B^n$. Then $U \cap V$ is homotopy equivalent to S^{n-1} and U is contractible. We claim that V has Y as a deformation retract. For this, let $g \colon B^n \to B^n \cup_f Y$ denote the characteristic map of the adjunction space, and define $H \colon V \times I \to V$ by

$$H(x, t) := \begin{cases} x, & \text{if } x \in Y, \\ g\big((1-t)z + tz/|z|\big), & \text{if } x = g(z) \in g(E^n \setminus 0). \end{cases} \tag{18.1}$$

[4]**Warning**: There is a slight discrepancy in the terminology here. Whilst most of the literature defines a deformation retract as we have in Definition 18.12, Hatcher's textbook instead defines a deformation retract to be what we will call a "strong deformation retract" next lecture.

H is well-defined and continuous[5]. Denote by $h\colon U \cap V \hookrightarrow V$ the inclusion. The Mayer-Vietoris sequence then gives us a long exact sequence

$$\cdots \to H_k(U \cap V) \xrightarrow{H_k(h)} H_k(U) \oplus H_k(V) \to H_k\big(B^n \cup_f Y\big) \to H_{k-1}(U \cap V) \to \dots$$

For $k > 0$ we have $H_k(U) = 0$, and for $k = 0$ we have $H_k(U) = \mathbb{Z}$ as U is path connected. After replacing $U \cap V$ with S^{n-1} and V with Y, we are almost done. It remains to see that for $k > 0$ we can identify $H_k(h)\colon H_k(U \cap V) \to H_k(V)$ with $H_k(f)\colon H_k(S^{n-1}) \to H_k(Y)$. For this we consider the commutative diagram:

$$\begin{array}{ccc}
U \setminus \{0\} & \xrightarrow{\;g\;} & U \cap V \\
\big\downarrow & & \big\downarrow h \\
B^n \setminus \{0\} & \longrightarrow & V \\
\big\uparrow & & \big\uparrow \\
S^{n-1} & \xrightarrow[f]{} & Y
\end{array}$$

The unlabelled vertical maps all induce isomorphisms on homology, since the respective subspaces are deformation retracts. The top horizontal map, which is the restriction of g, also induces an isomorphism, since $g|_{U \setminus \{0\}}$ is a homeomorphism. Thus for any $k \geq 0$ we get a commutative diagram in homology where the vertical maps are isomorphisms:

$$\begin{array}{ccc}
H_k(U \cap V) & \xrightarrow{H_k(h)} & H_k(V) \\
\big\downarrow & & \big\uparrow \\
H_k(S^{n-1}) & \xrightarrow[H_k(f)]{} & H_k(Y)
\end{array}$$

Thus the map $H_k(h)$ can be identified with the map $H_k(f)$. This completes the proof. ■

If $n \geq 2$ we can say a little more:

COROLLARY 18.14. *Let Y be a Hausdorff topological space. Let $n \geq 2$, and suppose $f\colon S^{n-1} \to Y$ is continuous. Then if $k \neq n-1, n$, one has*

$$H_k(Y) \cong H_k\big(B^n \cup_f Y\big),$$

and there is an exact sequence

$$0 \to H_n(Y) \xrightarrow{H_n(j)} H_n\big(B^n \cup_f Y\big) \to H_{n-1}(S^{n-1}) \xrightarrow{H_{n-1}(f)} H_{n-1}(Y) \to H_{n-1}\big(B^n \cup_f Y\big).$$

If $n \geq 3$ then the last map $H_{n-1}(Y) \to H_{n-1}\big(B^n \cup_f Y\big)$ is a surjection.

Proof. This is immediate from Proposition 18.13 and our computation of the homology of S^{n-1} in Theorem 14.11. ■

[5]Continuity of H is a little exercise using quotient maps.

We can also attach several cells at the same time.

Definition 18.15. Let Y be topological space. We say that a topological space Z is **obtained from Y by attaching n-cells** if there exists a pushout

$$\begin{array}{ccc} \bigsqcup_{\lambda\in\Lambda} S^{n-1}_\lambda & \xrightarrow{f} & Y \\ \big\downarrow & & \big\downarrow j \\ \bigsqcup_{\lambda\in\Lambda} B^n_\lambda & \xrightarrow[g]{} & Z \end{array}$$

Here the index $\lambda \in \Lambda$ just enumerates different copies of the same space. The map $j \colon Y \to Z$ is again a closed embedding, and g induces a homeomorphism $\bigsqcup_\lambda E^n_\lambda \to Z \setminus Y$. We set $f_\lambda := f|_{S^{n-1}_\lambda}$ and similarly $g_\lambda := g|_{B^n_\lambda}$. We call g_λ the **characteristic map** of the n-cell $g(E^n_\lambda)$ and we call f_λ its **attaching map**. Note that the definition still makes sense if our indexing set Λ is empty; then of course $Y = Z$.

We can now define the titular cell complexes.

Definition 18.16. Let $X' \subseteq X$ be topological spaces such that X' is closed in X. A **cellular decomposition** of the pair (X, X') consists of a sequence of subspaces:

$$X' = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X$$

such that:

1. X is the colimit of the (X^n) in Top. In other words, X carries the colimit topology: a set $C \subseteq X$ is closed if and only if $C \cap X^n$ is closed for each $n \geq -1$.
2. For each $n \geq 0$, X^n is obtained from X^{n-1} by attaching n-cells[6].

We call the pair (X, X') a **relative cell complex**. If $X' = \emptyset$ then X is called a **cell complex** and we just write X instead of $(X, \emptyset)$. The space X^n is called the **n-skeleton** of (X, X'), and we call the decomposition (X^n) for $n \geq -1$ the **skeleton filtration** of (X, X'). We say (X, X') is **finite** (resp. **countable**) if $X \setminus X'$ consists of a finite (resp. countable) number of cells. If $X = X^n$ for some n then the mininal such n is called the **dimension** of (X, X'). Note that if (X, X') is a relative cell complex then so is (X, X^n) and (X^n, X') for any $n \geq -1$.

A **subcomplex** of a cell complex X is a subspace $X' \subseteq X$ with the property that for any cell E of X, if $X' \cap E \neq \emptyset$ then $\overline{E} \subseteq X'$. In this case (X, X') is a relative cell complex, and X' itself is a cell complex whose cellular decomposition is inherited from X.

Proposition 18.17. *Let (X, X') be a relative cell complex.*

1. *The inclusion $X^n \subset X^{n+1}$ is a closed embedding for all $n \geq -1$.*
2. *If X' is a T_1 space then so is X, and a compact subset of X only meets finitely many cells of X.Thus if X' is T_1 then*

$$H_k(X) = \varinjlim_n H_k(X^n), \qquad \forall\, k \geq 0.$$

[6] If $X^{-1} = \emptyset$ then for $n = 0$ read this to mean: X^0 is a discrete set of points.

3. *If X' is Hausdorff*[7] *then so is X.*

4. *If X' is Hausdorff then X also carries the colimit topology with respect to the family which consists of X' and the closures of all the cells.*

In particular, taking $X' = \emptyset$ we see that these properties always hold for a cell complex X.

Proof. The first three points follow from Lemma 18.2, the proof of Lemma 17.5 and Corollary 17.6. Let us prove that (4). Suppose $C \subseteq X$ has the property that $C \cap X'$ is closed and $C \cap \overline{E}$ is closed for each cell E. We prove inductively that $C \cap X^n$ is closed for each n. This is true for $n = -1$ by assumption. The space X^n is a quotient of

$$Y^n := X^{n-1} \sqcup \left(\bigsqcup_{\Lambda_n} B^n_\lambda \right),$$

where Λ_n is the (possibly empty, possibly uncountable) set indexing the n-cells of X. Each characteristic map $g_\lambda \colon B^n_\lambda \to \overline{E}_\lambda$ is a quotient map since X is Hausdorff (cf. part (6) of Lemma 18.2.) By assumption $X^n \cap C$ has a closed preimage in Y^n. Then by part (4) of Lemma 18.2, we see that $X^n \cap C$ is closed in X^n. ■

We conclude this lecture by explaining why cell complexes are so important. Let X be a topological space and $p \in X$. In Algebraic Topology II, we will define the **higher homotopy groups** $\pi_n(X, p)$ for all $n \geq 0$ (we already did $n = 0$ in Lecture 3 and the fundamental group $n = 1$ in Lecture 4.) For $n \geq 2$, π_n is a functor $\mathsf{hTop}_* \to \mathsf{Ab}$.

Definition 18.18. A continuous map $f \colon X \to Y$ is called a **weak homotopy equivalence** if the induced map $\pi_n(f) \colon \pi_n(X, p) \to \pi_n(Y, f(p))$ is an isomorphism for all $n \geq 0$ and all $p \in X$.

For now this definition won't mean much to you (since we haven't defined π_n yet!) In general a weak homotopy equivalence is strictly weaker than an actual homotopy equivalence[8]. However a weak homotopy equivalence is still strong enough for all homology groups to coincide. Indeed, one of the axioms of a homology theory is that if $f \colon X \to Y$ is a weak homotopy equivalence then $H_n(f) \colon H_n(X) \to H_n(Y)$ is an isomorphism for each $n \geq 0$; more on this in Lecture 22. We will discuss the following theorem at the end of Algebraic Topology II (Theorem 46.15).

Theorem 18.19. *Let Y be any (!) topological space. Then there is a cell complex X and a weak homotopy equivalence $f \colon X \to Y$.*

Theorem 18.19 implies that as far as homology is concerned, *all* spaces are cell complexes. If this isn't sufficient motivation to study cell complexes, I don't know what is!

[7] In fact, if X' is *normal* then so is X.

[8] Although if $f \colon X \to Y$ is a weak homotopy equivalence between two connected cell complexes then f is automatically a homotopy equivalence. This result is called **Whitehead's Theorem** and will be one of the major results we prove next semester.

The Relative Homeomorphism Theorem

In this lecture we first complete the proof of a claim from Lecture 12: that if (X, X') is a sufficiently "nice" pair then

$$H_n(X, X') \cong \tilde{H}_n(X/X'), \qquad \forall n \geq 0.$$

We will then prove that if X is a cell complex and X' is a subcomplex then the pair (X, X') is always nice in this sense. Let us begin by specifying exactly what we mean by "nice".

DEFINITION 19.1. Let X be a topological space and $X' \subseteq X$ be a subspace. Denote by $\imath\colon X' \hookrightarrow X$ the inclusion. We say that X' is a **strong deformation retract** of X if there exists a continuous map $r\colon X \to X'$ such that $r \circ \imath = \mathrm{id}_{X'}$ and $\imath \circ r \simeq \mathrm{id}_X$ rel X'. Thus a strong deformation retract is a deformation retract where the homotopy from $\imath \circ r$ to id_X can be chosen to be a homotopy which is relative to X'. Equivalently, this means there exists a continuous function $H\colon X \times I \to X$ such that $H(x, 0) = x$ for all $x \in X$, $H(x, 1) \in X'$ for all $x \in X$, and $H(x', t) = x'$ for all $x' \in X'$ and $t \in I$.

This really is a stronger condition, as you will see on Problem Sheet J.

THEOREM 19.2. *Let $X' \subset X$ be a closed subspace with the property that there exists a neighbourhood U of X' in X such that X' is a strong deformation retract of U. Let $\rho\colon X \to X/X'$ denote the quotient map, and denote by $*$ the point in X/X' corresponding to X'/X'. Then for all $n \geq 0$, the map*

$$H_n(\rho)\colon H_n(X, X') \to H_n(X/X', *)$$

is an isomorphism.

Here is an example where the theorem is applicable.

EXAMPLE 19.3. Let Y be a Hausdorff space and $f\colon S^{n-1} \to Y$ a continuous map. Let $j\colon Y \to B^n \cup_f Y$ denote the map induced from the inclusion $Y \hookrightarrow B^n \sqcup Y$, so that j is a closed embedding (Lemma 18.2). Then $Y \cong j(Y) \subset B^n \cup_f Y$ satisfies the requirements of Theorem 19.2. Indeed, from the proof of Proposition 18.13, if $V := (B^n \cup_f Y) \setminus 0 \in B^n$, then V is an open neighbourhood of Y and (18.1) shows that Y is a strong deformation retract of V.

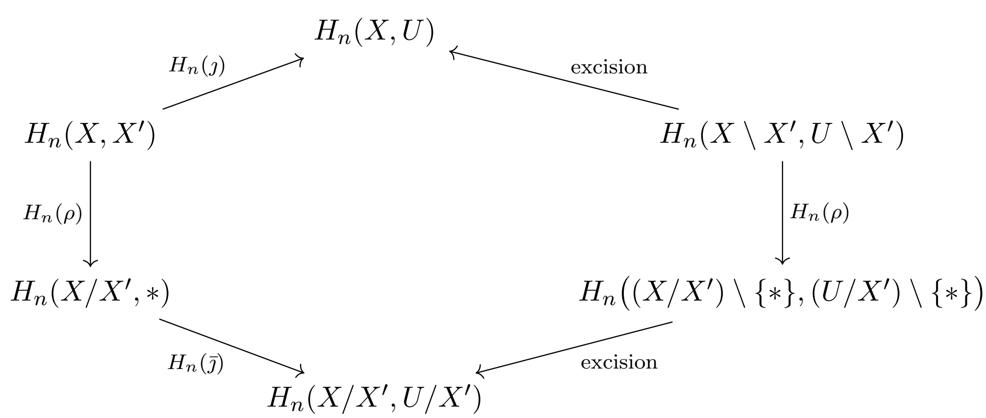
Proof of Theorem 19.2. Let U be as specified in the theorem, and denote by $\jmath\colon X' \hookrightarrow U$ the inclusion. Then we have a commutative diagram, where the horizontal maps come from the long exact sequence for pairs:

$$\begin{array}{ccccccccc}
H_n(X') & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,X') & \longrightarrow & H_{n-1}(X') & \longrightarrow & H_{n-1}(X) \\
\downarrow{\scriptstyle H_n(\jmath)} & & \downarrow{\scriptstyle H_n(\mathrm{id})} & & \downarrow{\scriptstyle H_n(\jmath)} & & \downarrow{\scriptstyle H_{n-1}(\jmath)} & & \downarrow{\scriptstyle H_{n-1}(\mathrm{id})} \\
H_n(U) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,U) & \longrightarrow & H_{n-1}(U) & \longrightarrow & H_{n-1}(X)
\end{array}$$

Since X' is a strong deformation retract of U, the left-hand $H_n(\jmath)$ and the right-hand $H_{n-1}(\jmath)$ are isomorphisms. Since $H_n(\mathrm{id})$ is certainly an isomorphism, the Five Lemma (Proposition 11.3) tells us that the middle $H_n(\jmath)$ is also an isomorphism. Next, since $\{*\}$ is a strong deformation retract of U/X' in X/X' (see Problem J.2), the same argument shows that the induced map $\bar{\jmath}\colon * \hookrightarrow U/X'$ induces an isomorphism

$$H_n(\bar{\jmath})\colon H_n(X/X', *) \to H_n(X/X', U/X')$$

for all $n \geq 0$. Now consider the following diagram:



The right-hand side $H_n(\rho)$ is an isomorphism since ρ is a homeomorphism away from X'. The two maps labelled "excision" are isomorphisms, and we just proved the two diagonal maps on the left-hand side are isomorphisms. Thus the left-hand $H_n(\rho)$ is also an isomorphism, which is what we wanted to prove. ∎

Using Corollary 12.22, we immediately obtain the claim (12.1) made in Lecture 12:

COROLLARY 19.4. *Let $X' \subset X$ be a closed subspace with the property that there exists a neighbourhood U of X' in X such that X' is a strong deformation retract of U. Then*

$$H_n(X, X') \cong \tilde{H}_n(X/X'), \qquad \forall\, n \geq 0.$$

Here is another application. An arbitrary wedge sum is defined again as a co-product[1].

[1]This means that the wedge sum is topologised as a quotient of the disjoint union.

Corollary 19.5. *Suppose* (X_λ, x_λ), $\lambda \in \Lambda$ *is a collection of pointed spaces. Assume that each* x_λ *is closed in* X_λ *and has a neighbourhood* $U_\lambda \subseteq X_\lambda$ *for which* x_λ *is a strong deformation retract of* U_λ*. Consider the wedge sum* $\bigvee_{\lambda\in\Lambda} X_\lambda$ *along the points* x_λ *(cf. Definition 18.10). Then the inclusions* $X_\lambda \hookrightarrow \bigvee_{\lambda\in\Lambda} X_\lambda$ *induce an isomorphism*

$$\bigoplus_{\lambda\in\Lambda} \tilde{H}_n(X_\lambda) \cong \tilde{H}_n\left(\bigvee_{\lambda\in\Lambda} X_\lambda\right).$$

Proof. By assumption $\left(\bigsqcup_{\lambda\in\Lambda} X_\lambda, \bigsqcup_{\lambda\in\Lambda}\{x_\lambda\}\right)$ satisfies the hypotheses of Theorem 19.2. The claim them following from the definition of the wedge product and Corollary 12.22. ■

Definition 19.6. Suppose $f\colon (X, X') \to (Y, Y')$ is a map of pairs. We say that f is a **relative homeomorphism** if f restricts to define a homeomorphism $f|_{X\setminus X'}\colon X \setminus X' \to Y \setminus Y'$.

With this terminology, the quotient map $\rho\colon X \to X/X'$ from Theorem 19.2 can also be thought of as a relative homeomorphism $(X, X') \to (X/X', *)$. We now prove a variant of Theorem 19.2.

Theorem 19.7 (The Relative Homeomorphism Theorem). *Let* $f\colon (X, X') \to (Y, Y')$ *be a relative homeomorphism. Assume that* X *is compact and that* Y *is compact Hausdorff, and that* X' *and* Y' *are closed in* X *and* Y *respectively. Assume further that there exists a neighbourhood* U *of* X' *in* X *such that* X' *is a strong deformation retract of* U*, and a neighbourhood* V *of* Y' *in* Y *such that* Y' *is a strong deformation retract of* V*. Then*

$$H_n(f)\colon H_n(X, X') \to H_n(Y, Y') \qquad \text{is an isomorphism for all } n \geq 0.$$

Proof. Denote by $\rho\colon X \to X/X'$ and $\rho'\colon Y \to Y/Y'$ the quotient maps. Then there is a well-defined continuous bijective map $f'\colon X/X' \to Y/Y'$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \rho\downarrow & & \downarrow\rho' \\ X/X' & \xrightarrow[f']{} & Y/Y' \end{array}$$

Since X/X' is compact and Y/Y' is Hausdorff[2], the map f' is a homeomorphism[3]. Passing to homology, for any $n \geq 0$ we obtain a commutative diagram:

$$\begin{array}{ccc} H_n(X, X') & \xrightarrow{H_n(f)} & H_n(Y, Y') \\ H_n(\rho)\downarrow & & \downarrow H_n(\rho)' \\ H_n(X/X', *) & \xrightarrow[H_n(f')]{} & H_n(Y/Y', *') \end{array}$$

[2]Y/Y' is Hausdorff as Y is compact and Y' is closed.

[3]A continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism.

where we denoted by $*$ the point in X/X' corresponding to X'/X' and by $*'$ the point in Y/Y' corresponding to Y'/Y'. By Theorem 19.2 the two maps $H_n(\rho)$ and $H_n(\rho')$ are isomorphisms, and $H_n(f')$ is an isomorphism as f' is a homeomorphism. Thus so is the map $H_n(f)\colon H_n(X, X') \to H_n(Y, Y')$. ■

We now prove that Theorem 19.2 is always applicable to a pair (X, X') where X is a cell complex and X' is a subcomplex. The first step is the following technical statement.

PROPOSITION 19.8. *Let X be a cell complex and X' be a subcomplex. For each cell E_λ in X which is not in X', choose a single point $x_\lambda \in E_\lambda$. Let*

$$Y^n := \{x_\lambda \mid E_\lambda \text{ is an } n\text{-cell in } X \text{ which is not in } X'\}.$$

Regard (X, X') as a relative cell complex with skeleton filtration $X' = X^{-1} \subset X^0 \subset X^1 \subseteq \cdots \subseteq X$. Then for every $n \geq 1$, X^{n-1} is a strong deformation retract of $X^n \setminus Y^n$.

Proof. We use the same idea as in (18.1) last lecture. Namely, without loss of generality we may assume that for each n-cell not in X', the corresponding characteristic map $g_\lambda\colon B^n_\lambda \to X^n$ satisfies $g_\lambda(0) = x_\lambda$. Then we define $H\colon X^n \setminus Y^n \times I \to X^n \setminus Y^n$ by

$$H(x,t) := \begin{cases} x, & \text{if } x \in X^{n-1}, \\ g_\lambda\big((1-t)z + tz/|z|\big), & \text{if } x = g_\lambda(z) \text{ for } z \in E^n \setminus \{0\}. \end{cases} \tag{19.1}$$

We need only check that H is continuous. Since X^n has the colimit topology from X' and the cells in X not belonging to X' (cf. the last part of Proposition 18.17), $X^n \setminus Y^n$ has the colimit topology determined by the cells in X' and the punctured cells $E_\lambda \setminus \{x_\lambda\}$ for the cells in X that are not in X'. It then follows that $X^n \setminus Y^n \times I$ has the colimit topology associated to the sets of the form $E' \times \{0\}$, $E' \times \{1\}$ and $E' \times (0,1)$, where E' is either a cell in X' or a punctured cell $E_\lambda \setminus \{x_\lambda\}$. The restriction of H to any of these subsets is continuous (this is proved in the same way that we proved the map H from (18.1) was continuous), and hence H is continuous by definition of the colimit topology. ■

We now prove the desired result.

THEOREM 19.9. *Let X be a cell complex and X' be a subcomplex. Then there exists an open set U in X containing X' such that X' is a strong deformation retract of U.*

Proof. Using the notation from the previous proposition, let $r_n\colon X^n \setminus Y^n \to X^{n-1}$ denote a strong deformation retract for $n \geq 1$. Set $U_0 = X'$ and set $U_n := r_n^{-1}(U_{n-1})$ for $n \geq 1$. Then each U_n is open in $X^n \setminus Y^n$ and thus $U := \bigcup_n U_n$ is an open set[4] in X containing X'. Since the composition of two strong deformation retracts is itself a strong deformation retract, we see that X' is a strong deformation retract of U_n

[4]Note Y^n is closed in X^n due to part (4) of Proposition 18.17.

for each $n \geq 1$. This means that there exist continuous maps $F_n \colon U_n \times I \to U_n$ such that

$$F_n(x, 0) = x, \qquad F_n(x, 1) = r_1 r_2 \cdots r_n(x) \in X', \qquad \forall\, x \in U_n,$$

and such that $F_k(x', t) = x'$ for all $x' \in X'$ and $t \in I$. Moreover by induction we may even require that $F_{n+1}|_{U_n \times I} = F_n$. This means that we can define $F \colon U \times I \to U$ by setting $F = F_n$ on U_n. Then F is continuous by definition of the colimit topology, and F exhibits X' as a strong deformation retract of X. ■

Theorem 19.9 implies that for cell complexes, we can use subcomplexes instead of open sets for excision and the Mayer-Vietoris sequence.

COROLLARY 19.10. *Suppose X is a cell complex and X', X'' are subcomplexes such that $X = X' \cup X''$. Then the inclusion $(X'', X' \cap X'') \hookrightarrow (X, X')$ induces isomorphisms on homology $H_n(X'', X' \cap X'') \to H_n(X, X')$ for all $n \geq 0$.*

Proof. We may assume $X' \cap X'' \neq \emptyset$, otherwise the result trivially follows from Proposition 8.2. Then the quotient spaces $X''/X' \cap X''$ and X/X' are homeomorphic (they both can be identified with the cells in X'' that are not in X'). Then by Theorem 19.2 and Theorem 19.9, we obtain for any $n \geq 0$

$$H_n(X'', X' \cap X'') \cong \tilde{H}_n(X''/X' \cap X'') \cong \tilde{H}_n(X/X') \cong H_n(X, X').$$

■

COROLLARY 19.11. *Suppose X is a cell complex and X', X'' are subcomplexes such that $X = X' \cup X''$. Then the there is an exact sequence*

$$\ldots H_n(X' \cap X'') \to H_n(X') \oplus H_n(X'') \to H_n(X) \to H_{n-1}(X' \cap X'') \to \ldots$$

Proof. The Mayer-Vietoris sequence is a formal consequence of excision and the Baratt-Whitehead Lemma (Proposition 11.4). In other words, the proof of Theorem 14.9 goes through without any changes. ■

LECTURE 20

Cellular homology

In this lecture we introduce a new homology theory which is tailored specifically for cell complexes. *Cellular homology* is much more efficient than singular homology for computational purposes, as the cellular chain complex is much smaller. Indeed, a basis for the nth cellular chain group is given by the n-cells of the cell complex. For many spaces, the chain groups are then finitely generated abelian group. The resulting homology is the same as singular homology.

The key result that gets the construction of the cellular chain complex going is the following proposition.

PROPOSITION 20.1. *Let X be a cell complex with skeleton filtration $\emptyset = X^{-1} \subset X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X$. Then for all $n \geq 0$,*

$$H_k(X^n, X^{n-1}) = 0, \qquad k \neq n.$$

Meanwhile $H_n(X^n, X^{n-1})$ is free abelian with a basis in one-to-one correspondence with the n-cells of X.

Proof. Suppose that n-cells of X are given by maps $g_\lambda \colon (B^n, S^{n-1}) \to (X^n, X^{n-1})$, where λ ranges over an index set Λ_n. Since X^{n-1} is a subcomplex of X^n for all $n \geq 1$, by Theorem 19.9 and Theorem 19.2 we see that

$$H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n/X^{n-1}), \qquad \forall\, k \geq 0.$$

But X^n/X^{n-1} is a wedge sum of spheres, one for each of the n-cells of X. The claim then follows from Corollary 19.5. ■

We now step away from cell complexes for a moment and abstract the conclusion of Proposition 20.1 into a definition.

DEFINITION 20.2. Let X be a topological space. A **cell-like filtration** $\mathcal{F}$ of X is an expanding sequence of T_1 closed subspaces $\emptyset = F^{-1} \subseteq F^0 \subseteq F^1 \subseteq F^2 \subseteq \cdots$ such that $X = \bigcup_{n \geq 0} F^n$ carries the colimit topology, and with the property that

$$H_k(F^n, F^{n-1}) = 0, \qquad \forall\, k \neq n.$$

Thus Proposition 20.1 tells us that the skeleton filtration associated to a cell complex is a cell-like filtration. However the notion of a cell-like filtration is much more general, since F^n does not have to be obtained from F^{n-1} by adjoining cells, and the topological space X does not have to be Hausdorff. In this course we will have no need for the extra level of generality afforded by cell-like filtrations; nevertheless it makes[1] the proof of Theorem 20.5 below more transparent to work with cell-like filtrations, since it shows exactly what properties we need.

Our goal is to construct a new homology theory $H_\bullet(X, \mathcal{F})$ associated to a topological space with a cell-like filtration. In the special case where the space is a cell complex and the filtration is the skeleton filtration, this will be called the *cellular homology* of the cell complex. Let us first prove the following statement.

PROPOSITION 20.3. *Let X be a topological space with cell-like filtration $\mathcal{F} = (F^n)$. Then $H_k(F^n) = 0$ for $k > n$ and the inclusion $F^n \hookrightarrow X$ induces an isomorphism $H_k(F^n) \to H_k(X)$ for $k < n$.*

Proof. We examine the long exact sequence of the pair (F^n, F^{n-1}). It contains the chain

$$H_{k+1}(F^n, F^{n-1}) \to H_k(F^{n-1}) \to H_k(F^n) \to H_k(F^n, F^{n-1}).$$

For $k \neq n, n-1$ the outer two groups are zero by assumption, and hence $H_k(F^{n-1}) \cong H_k(F^n)$ for $k \neq n, n-1$. In particular, if $k > n$ then

$$H_k(F^n) \cong H_k(F^{n-1}) \cong \cdots \cong H_k(F^0) = H_k(F^0, F^{-1}) = 0,$$

The same argument shows that the maps $H_k(F^{k+1}) \to H_k(F^{k+2}) \to H_k(F^{k+3}) \to \dots$ are all isomorphisms. Thus

$$\underset{n \geq k+1}{\underrightarrow{\operatorname{colim}}}\, H_k(F^n) = H_k(F^{k+1}).$$

But by Corollary 17.6, the colimit on the left-hand side is just the homology $H_k(X)$. This completes the proof. ■

We now define our new chain complex.

DEFINITION 20.4. Let X be a topological space with cell-like filtration $\mathcal{F}$. We define a chain complex $\big(C_\bullet(X, \mathcal{F}), \partial^{\mathcal{F}}\big)$ as follows. Firstly, set $C_n(X, \mathcal{F}) = 0$ for $n < 0$. Given $n \geq 0$, define $C_n(X, \mathcal{F})$ to be the abelian group

$$C_n(X, \mathcal{F}) := H_n(F^n, F^{n-1}).$$

Let

$$j_n \colon (F^n, \emptyset) \hookrightarrow (F^n, F^{n-1})$$

denote the inclusion, and abbreviate by

$$\eta_n = H_n(j_n) \colon H_n(F^n) \to H_n(F^n, F^{n-1}).$$

For $n \geq 1$ we define[2] the boundary operator $\partial^{\mathcal{F}} \colon C_n(X, \mathcal{F}) \to C_{n-1}(X, \mathcal{F})$ as the composition

$$H_n(F^n, F^{n-1}) \xrightarrow{\delta_n} H_{n-1}(F^{n-1}) \xrightarrow{\eta_{n-1}} H_{n-1}(F^{n-1}, F^{n-2}),$$

[1]To me at least...

[2]For $n = 0$ the boundary operator is of course the zero map.

where δ_n is the connecting homomorphism for the long exact sequence of the pair (F^n, F^{n-1}). This does indeed define a chain complex as for $n \geq 1$ the composition $\partial^{\mathcal{F}} \circ \partial^{\mathcal{F}} \colon C_{n+1}(X, \mathcal{F}) \to C_{n-1}(X, \mathcal{F})$ is given by

$$H_{n+1}(F^{n+1}, F^n) \to H_n(F^n) \xrightarrow{\eta_n} H_n(F^n, F^{n-1}) \xrightarrow{\delta_n} H_{n-1}(F^{n-1}) \to H_{n-1}(F^{n-1}, F^{n-2})$$

and this is zero since the composition in the middle

$$H_n(F^n) \xrightarrow{\eta_n} H_n(F^n, F^{n-1}) \xrightarrow{\delta_n} H_{n-1}(F^{n-1})$$

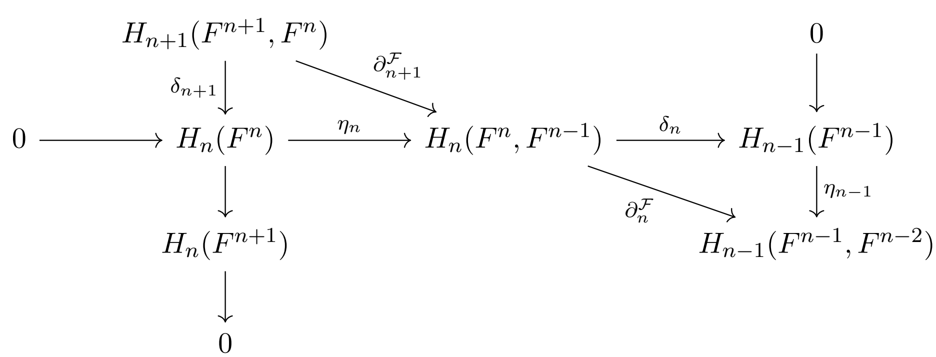
is zero as these are two adjacent maps in the long exact sequence of the pair (F^n, F^{n-1}). We denote the associated homology by $H_n(X, \mathcal{F}) := H_n\big(C_\bullet(X, \mathcal{F}), \partial^{\mathcal{F}}\big)$.

Here is main result of today's lecture.

Theorem 20.5. *Let X be a topological space with cell-like filtration $\mathcal{F} = (F^n)$. Then for every $n \geq 0$, one has*

$$H_n(X) \cong H_n(X, \mathcal{F}).$$

Proof. Let us break with out longstanding convention and temporarily give the boundary operator $\partial^{\mathcal{F}}$ a subscript (otherwise the chain of equations below makes no sense.) We have the following commuting diagram, where the row is part of the long exact sequence of the pair (F^n, F^{n-1}), the left-hand column is part of the long exact sequence of the pair (F^{n+1}, F^n), and the right-hand column is part of the long exact sequence of the pair (F^{n-1}, F^{n-2}). The left-most and the top-right zero entries follow from the Proposition 20.3, and the bottom zero entry is from the definition of a cell-like filtration.



Using this diagram, we now argue as follows:

$$\begin{aligned}
H_n(X) &\cong H_n(F^{n+1}) && \text{by Proposition 20.3,}\\
&\cong H_n(F^n)\big/ \operatorname{im}\delta_{n+1} && \text{by exactness of first column,}\\
&\cong \operatorname{im}\eta_n \big/ \operatorname{im}\eta_n\delta_{n+1} && \text{as } \eta_n \text{ is an injection,}\\
&\cong \ker\delta_n \big/ \operatorname{im}\eta_n\delta_{n+1} && \text{by exactness of the row,}\\
&\cong \ker\delta_n \big/ \operatorname{im}\partial^{\mathcal{F}}_{n+1} && \text{by commutativity of the top triangle,}\\
&\cong \ker\eta_{n-1}\delta_n \big/ \operatorname{im}\partial^{\mathcal{F}}_{n+1} && \text{as } \eta_{n-1} \text{ is an injection,}\\
&\cong \ker\partial^{\mathcal{F}}_n \big/ \operatorname{im}\partial^{\mathcal{F}}_{n+1} && \text{by commutativity of the bottom triangle,}\\
&= H_n(X, \mathcal{F}) && \text{by the definition of homology.}
\end{aligned}$$

This completes the proof. ∎

Remark 20.6. The isomorphism $H_n(X) \cong H_n(X, \mathcal{F})$ can be written down explicitly. This is the subject of Problem J.5 on Problem Sheet J.

Now, with this detour out of the way, let us go back to cell complexes. If X is a cell complex with skeleton filtration $\mathcal{F} = (X^n)$ then we will use the notation $\left(C_\bullet^{\text{cell}}(X), \partial^{\text{cell}}\right)$ instead of $\left(C_\bullet(X, \mathcal{F}), \partial^{\mathcal{F}}\right)$ for the chain complex associated to the skeleton filtration, and by $H_\bullet^{\text{cell}}(X)$ the homology. We call $C_\bullet^{\text{cell}}(X)$ the **cellular chain complex** and we call $H_\bullet^{\text{cell}}(X)$ the **cellular homology** of the cell complex X.

Using Proposition 20.1, we have

$$C_n^{\text{cell}}(X) = H_n(X^n, X^{n-1}),$$

and thus we see that the cellular chain group is free abelian with *generators in a one-to-one correspondence with the n-cells of X.*

Remark 20.7. The chain complex $C_\bullet^{\text{cell}}(X)$ depends on not only the space X but *also* the choice of cellular decomposition. The same topological space can have different cellular decompositions. For example, S^3 has a cellular decomposition with one 0-cell and one 3-cell (cf Example 18.7), but it also has a more complicated one (cf. Problem K.2.) However Theorem 20.5 implies that the *homology* $H_\bullet^{\text{cell}}(X)$ does *not* depend on the choice of cellular decomposition (since it agree with the singular homology.)

The following result applies to nearly all of the spaces we will ever meet in this course.

Corollary 20.8. *Let X be a compact cell complex of dimension n. Then*

1. *If X has N_k cells of dimension k then $H_k(X)$ has rank at most N_k. In particular, $H_k(X)$ is finitely generated for all k.*
2. *$H_k(X) = 0$ for all $k > n$.*
3. *$H_n(X)$ is free abelian.*

Proof. Since X is compact, it is necessarily a finite cell complex. The first two statements are clear for the cellular homology groups $H_k^{\text{cell}}(X)$, and hence also for the singular homology groups $H_k(X)$ by Theorem 20.5. The last statement follows as $H_n^{\text{cell}}(X) = \ker \partial^{\text{cell}} : C_n^{\text{cell}}(X) \to C_{n-1}^{\text{cell}}(X)$ is a subgroup of $C_n^{\text{cell}}(X)$, and a subgroup of a free abelian group is necessarily free abelian. ∎

Here is another immediate corollary.

Corollary 20.9. *Let X be a cell complex. Suppose X has N cells in dimension n, and no cells in dimension $n-1$ and $n+1$. Then $H_n(X) \cong \mathbb{Z}^N$.*

Proof. We necessarily have $C_{n+1}^{\text{cell}}(X) = C_{n-1}^{\text{cell}} = 0$, and thus $H_n^{\text{cell}}(X) \cong C_n^{\text{cell}}(X)$. ∎

Corollary 20.9 gives us a new (and much quicker) way to compute the homology of S^n for $n \geq 2$, of $S^m \times S^n$ for $m, n \geq 2$, and for $\mathbb{C}P^n$ for all $n \geq 0$. Note however we still *cannot* compute the homology of $\mathbb{R}P^n$.

To rectify this, we need to have a way of computing the boundary operator ∂^{cell}. We conclude this lecture by giving an explicit formula for ∂^{cell}.

So let X be a cell complex as before, and fix $n \geq 2$. To help keep the notation transparent, in the following we will use the letter λ to index the n-cells of X and the letter ν to index the $(n-1)$-cells. Given an n-cell E_λ, we denote by e_λ the corresponding generator in $C_n^{\text{cell}}(X)$, and similarly $e_\nu \in C_{n-1}^{\text{cell}}(X)$ is the generator corresponding to an $(n-1)$-cell.

The idea behind the cellular boundary formula is that we can get an $(n-1)$-sphere from both an n-cell and an $(n-1)$-cell. Indeed, if $g_\lambda \colon B^n \to X^n$ is an n-cell then $f_\lambda := g_\lambda \colon S_\lambda^{n-1} := \partial B_\lambda^n \to X^{n-1}$ is a map from an $(n-1)$-cell. Meanwhile if $g_\nu \colon B_\nu^{n-1} \to X^{n-1}$ is an $(n-1)$-cell, then if we collapse $f_\nu(\partial B_\nu^{n-1})$ to a point, the quotient space

$$S_\nu^{n-1} := g_\nu(B_\nu^{n-1}) / f_\nu(\partial B_\nu^{n-1})$$

is also an $(n-1)$-sphere[3].

Let us denote by

$$\rho \colon X^{n-1} \to X^{n-1}/X^{n-2}$$

the quotient map, and denote by

$$q_\nu \colon X^{n-1}/X^{n-2} \to S_\nu^{n-1}$$

the quotient map that collapses all the other $(n-1)$-spheres in X^{n-1}/X^{n-2} to a point.

Definition 20.10. Suppose e_λ is an n-cell and e_ν is an $(n-1)$-cell. We define a map

$$h_{\lambda,\nu} \colon S_\lambda^{n-1} \to S_\nu^{n-1}$$

to be the composition:

$$S_\lambda^{n-1} \xrightarrow{f_\lambda} X^{n-1} \xrightarrow{\rho} X^{n-1}/X^{n-2} \xrightarrow{q_\nu} S_\nu^{n-1}.$$

We then define the integer

$$[e_\lambda \colon e_\nu] := \deg(h_{\lambda,\nu}),$$

where we are using Definition 15.2. If $* \in X^{n-1}/X^{n-2}$ denotes the point corresponding to X^{n-2}, then since $g_\lambda(B_\lambda^n)$ intersects at most finitely many cells (part (2) of Proposition 18.17), for fixed e_λ, the map $h_{\lambda,\nu}$ is not the constant map $S_\lambda^{n-1} \to *$ for at most finitely many e_ν.

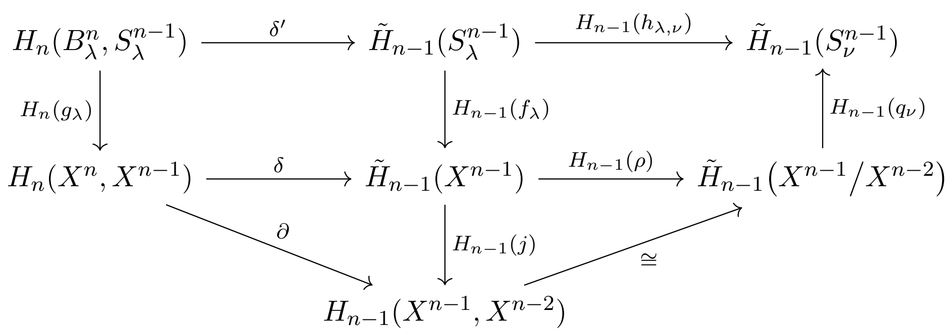
[3]For $n = 1$, the quotient space is a point, not S^0. To avoid introducing excessive notation, we will tacitly assume $n \geq 2$ in the following, and leave the case $n = 1$ as an exercise (the only difference is notation).

Theorem 20.11 (The cellular boundary formula). *The boundary operator $\partial^{\mathrm{cell}} \colon C_n^{\mathrm{cell}}(X) \to C_{n-1}^{\mathrm{cell}}(X)$ is given explicitly by the formula*

$$\partial^{\mathrm{cell}} e_\lambda = \sum_\nu [e_\lambda \colon e_\nu] \cdot e_\nu.$$

Note the right-hand side is a well-defined element of $C_{n-1}^{\mathrm{cell}}(X)$ since $h_{\lambda,\nu}$ is non-constant (and hence has non-zero degree) for at most finitely many e_ν.

Proof. Let $j \colon (X^{n-1}, \emptyset) \hookrightarrow (X^{n-1}, X^{n-2})$ denote the inclusion. Consider the following diagram, where the two maps labelled δ, δ' are the connecting homomorphisms coming from the long exact sequence in reduced homology:



The left-hand square commutes by naturality of the long-exact sequence in reduced homology, the right-hand square commutes by definition of $h_{\lambda,\nu}$ and the fact that $\tilde{H}_{n-1}$ is a functor. The left-hand triangle commutes by definition of ∂, and the right-hand triangle commutes from the proof of Theorem 19.2. The map labelled $\cong$ is an isomorphism from Theorem 19.2 as well. Let $\langle c \rangle$ be the generator of $H_n(B_\lambda^n, S_\lambda^{n-1}) \cong \mathbb{Z}$ such that $e_\lambda = H_n(g_\lambda)\langle c \rangle$. Then by commutativity,

$$\partial e_\lambda = H_{n-1}(j) \circ H_{n-1}(f_\lambda) \circ \delta' \langle c \rangle.$$

In terms of the basis of $H_{n-1}(X^{n-1}, X^{n-2})$ given by $(n-1)$-cells, the map $H_{n-1}(q_\nu)$ is the projection of $\tilde{H}_{n-1}\big(X^{n-1}/X^{n-2}\big)$ onto the $\mathbb{Z}$-summand corresponding to e_ν. Thus commutativity of the right-hand square tells us that if $\partial e_\lambda = \sum_\nu m_{\lambda,\nu}\, e_\nu$ for $m_{\lambda,\nu} \in \mathbb{Z}$ then

$$m_{\lambda,\nu} = \deg(h_{\lambda,\nu}).$$

This completes the proof. ■

We will shortly use this to calculate the homology of $\mathbb{R}P^n$. First, however, we need another formula for computing the degree of a map from the sphere to itself. Suppose

$$f \colon (X, p) \to (Z, z), \qquad g \colon (Y, q) \to (Z, z)$$

are two pointed continuous maps. Then there is a well defined pointed continuous map

$$f \vee g \colon X \vee Y \to Z \vee Z$$

defined by

$$(f \vee g)(x, y) := \begin{cases} (f(x), z), & y = q, \\ (z, g(y)), & x = p. \end{cases}$$

Now consider the continuous maps

$$\text{Pinch}\colon S^n \to S^n \vee S^n, \qquad \text{Fold}\colon S^n \vee S^n \to S^n$$

that "pinch" and "fold" the sphere as in Figure 20.1. Explicitly, Pinch collapses the equator S^{n-1} to a single point, and, denoting by p the basepoint in S^n, the map Fold does the following:

$$\text{Fold}(x, y) := \begin{cases} x, & y = p, \\ y, & x = p. \end{cases}$$

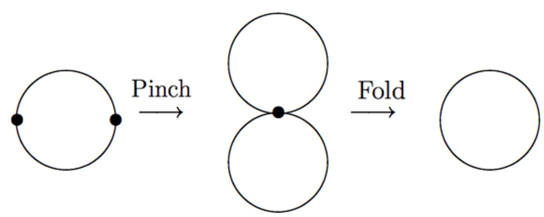


Figure 20.1: Pinching and folding.

LEMMA 20.12. *Let $f, g\colon S^n \to S^n$ be continuous maps. Let $h\colon S^n \to S^n$ denote the composition*

$$h = \text{Fold} \circ (f \vee g) \circ \text{Pinch}.$$

Then

$$\deg(h) = \deg(f) + \deg(g).$$

Proof. We use the isomorphism $H_n(S^n \vee S^n) \cong \tilde{H}_n(S^n) \oplus \tilde{H}_n(S^n)$ from Corollary 19.5. Let $\langle c \rangle, \langle c' \rangle$ denote elements of $H_n(S^n)$. Then one readily checks that

$$H_n(\text{Pinch})\langle c \rangle = (\langle c \rangle, \langle c \rangle),$$

and

$$H_n(f \vee g)\left(\langle c \rangle, \langle c' \rangle\right) = \left(H_n(f)\langle c \rangle, H_n(g)\langle c' \rangle\right),$$

and finally

$$H_n(\text{Fold})\left(\langle c \rangle, \langle c' \rangle\right) = \langle c \rangle + \langle c' \rangle.$$

The claim follows. ■

Here is the promised calculation of the homology of $\mathbb{R}P^n$.

COROLLARY 20.13. *The homology of the real projective space is given by*

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{if } k = 0 \text{ or } k = n \text{ and } n \text{ is odd}, \\ \mathbb{Z}_2, & \text{for odd } 0 < k < n, \\ 0, & \text{otherwise}. \end{cases}$$

Proof. The composition we need to look at is

$$S^{n-1} \xrightarrow{f} \mathbb{R}P^{n-1} \xrightarrow{q} \mathbb{R}P^{n-1}\big/\mathbb{R}P^{n-2} = S^{n-1}.$$

The map $q \circ f$ is a homeomorphism when restricted to each component of $S^{n-1} \setminus S^{n-2}$, and these two homeomorphisms are obtained from each other by precomposing with the antipodal map $a\colon S^{n-1} \to S^{n-1}$. This has degree $(-1)^n$ by Corollary 15.7. Thus by Lemma 20.12

$$\begin{aligned} \deg(q \circ f) &= \deg(\mathrm{Fold} \circ (\mathrm{id} \vee a) \circ \mathrm{Pinch}) \\ &= \deg(\mathrm{id}) + \deg(a) \\ &= 1 + (-1)^n. \end{aligned}$$

Thus the cellular chain complex for $\mathbb{R}P^n$ is given by

$$\begin{aligned} &0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0, \qquad \text{if } n \text{ is even}, \\ &0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0, \qquad \text{if } n \text{ is odd}. \end{aligned}$$

The result now follows from the fact that $H_\bullet^{\mathrm{cell}}(\mathbb{R}P^n) = H_\bullet(\mathbb{R}P^n)$. ■

A more involved application of the cellular boundary formula is on Problem Sheet K.

LECTURE 21

Natural transformations and the Eilenberg-Steenrod Axioms

We begin this lecture by finally define "naturality" properly.

Definition 21.1. Let C and D be two categories, and let $S, T\colon \mathsf{C} \to \mathsf{D}$ be two functors. A **natural transformation** $\Phi\colon S \to T$ is a family of morphisms $\Phi(C)\colon S(C) \to T(C)$ for each $C \in \operatorname{obj}(\mathsf{C})$ such that for any morphism $f\colon A \to B$ in C the following diagram commutes:

$$\begin{array}{ccc} S(A) & \xrightarrow{\Phi(A)} & T(A) \\ {\scriptstyle S(f)}\big\downarrow & & \big\downarrow{\scriptstyle T(f)} \\ S(B) & \xrightarrow[\Phi(B)]{} & T(B) \end{array}$$

If each morphism $\Phi(C)$ is an isomorphism then we say that Φ is a **natural isomorphism**.

If $\Psi\colon R \to S$ and $\Phi\colon S \to T$ are two natural transformations then there it is easy to check that there is a well-defined natural transformation

$$\Phi \circ \Psi\colon R \to T, \qquad (\Phi \circ \Psi)(C) := \Phi(C) \circ \Psi(C).$$

Given any functor T, there is a well-defined natural transformation $\mathrm{id}_T\colon T \to T$ given by $\mathrm{id}_T(C) = \mathrm{id}_{T(C)}$ for each object $C \in \mathsf{C}$. The following easy lemma is on Problem Sheet K

Lemma 21.2. *Let C and D be two categories and $S, T\colon \mathsf{C} \to \mathsf{D}$ two functors. Suppose $\Phi\colon S \to T$ is a natural transformation. Then Φ is a natural isomorphism if and only if there is a natural transformation $\Psi\colon T \to S$ such that $\Psi \circ \Phi = \mathrm{id}_S$ and $\Phi \circ \Psi = \mathrm{id}_T$.*

By now you can probably guess what's coming next. Natural transformations "look" like morphisms between functors, and that means time for a new category. Let us suggestively write

$$\mathrm{Nat}(S, T) := \{\text{natural transformations } \Phi\colon S \to T\}.$$

We would like to define a new category called the **functor category** whose objects are all the functors from one category to another, and whose morphisms are the natural transformations between the functors. Unfortunately we run into a set-theoretic bug! Recall that in the definition of a category in Lecture 1, we required the Hom-sets to be actual sets. However sadly $\mathrm{Nat}(S, T)$ need not be a set. Worse, it does not even have to be a class. The following result is not hard to prove, but isn't particularly relevant to our purposes, so I'll just state it.

PROPOSITION 21.3. *Let* C *be a small category (i.e.* $\text{obj}(C)$ *is a set). Then for any category* D *and any two functors* $S, T\colon \mathsf{C} \to \mathsf{D}$, $\text{Nat}(S,T)$ *is a set.*

This means that we can formally only define the functor category when the domain category is small.

DEFINITION 21.4. Let C be a small category, and let D be an arbitrary category. The **functor category** $\mathsf{Fun}(\mathsf{C},\mathsf{D})$ is the category with:

- $\text{obj}(\mathsf{Fun}(\mathsf{C},\mathsf{D}))$ the class of all functors $T\colon \mathsf{C} \to \mathsf{D}$.
- $\text{Hom}_{\mathsf{Fun}(\mathsf{C},\mathsf{D})}(S,T) = \text{Nat}(S,T)$,
- composition is given by composition of natural transformations.

This is well-defined due to the preceding proposition.

Let us give a concrete example of formulating a (non-trivial) result you probably all already know in the language of natural isomorphisms.

THEOREM 21.5. *A finite-dimensional vector space is naturally isomorphic to its double dual.*

What exactly does this mean? For simplicity, lets work with real vector spaces. If V is a real vector space then $V^* := \text{Hom}(V, \mathbb{R})$ denotes the set of all linear functionals $V \to \mathbb{R}$. More categorically, if Vect is the category of all vector spaces, then we can think of $V \mapsto V^* = \text{Hom}(V,\mathbb{R})$ as a functor $\text{Hom}(\Box, \mathbb{R})\colon \mathsf{Vect} \to \mathsf{Vect}$ (since $\mathbb{R}$ is itself a real vector space.) Actually this is not quite true: this functor *reverses* the direction of morphisms (compare Problem K.5.) Indeed, a morphism in Vect is a linear transformation $A\colon V \to W$ between two vector spaces. A induces a map A^* between the dual spaces, but it goes the "wrong" way round: $A^*\colon W^* \to V^*$. Explicitly, if $\lambda \in W^*$ (so $\lambda\colon W \to \mathbb{R}$ is a linear functional) then $A^*\lambda \in V^*$ is defined by

$$A^*\lambda(v) := \lambda(Av), \qquad \forall\, v \in V.$$

Next semester we will study the idea of functors going the "wrong way round" in detail when we study *cohomology*. For now though, let us side-step this issue by applying the functor twice. So let $T\colon \mathsf{Vect} \to \mathsf{Vect}$ denote the functor

$$T(V) := \text{Hom}(\mathsf{Hom}(V,\mathbb{R}),\mathbb{R}).$$

One usually denotes $T(V)$ by V^{**} and calls it the *double dual*. If $A\colon V \to W$ is a linear map then $T(A) : T(V) \to T(W)$ is the linear map usually written as $A^{**}\colon V^{**} \to W^{**}$ and defined by

$$A^{**}(\varphi)(\lambda) = \varphi(A^*\lambda) \qquad \varphi \in V^{**}, \lambda \in W^*.$$

As you probably remember from linear algebra, there is a map $\text{ev}_V\colon V \mapsto V^{**}$ called *evaluation* that simply evaluates a linear functional at a vector:

$$\text{ev}_V(v)(\mu) := \mu(v), \qquad \mu \in V^*.$$

We claim that ev is a natural transformation from the identity functor on Vect to T. This comes down to showing that the following diagram commutes for any pair of vector spaces V, W and any linear map $A\colon V \to W$:

$$\begin{array}{ccc} V & \xrightarrow{\mathrm{ev}_V} & V^{**} \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A^{**}} \\ W & \xrightarrow[\mathrm{ev}_W]{} & W^{**} \end{array}$$

This is trivial: take $\lambda \in W^*$ and observe:

$$A^{**}\mathrm{ev}_V(v)(\lambda) = \mathrm{ev}_V(v)(A^*\lambda) = A^*\lambda(v) = \lambda(Av) = \mathrm{ev}_{Av}(\lambda).$$

The map ev_V is easily seen to be an injection $V \to V^{**}$ but in general if V is infinite-dimensional then $\dim V^* > \dim V$ and hence ev_V is not an isomorphism. But if V is finite-dimensional then hopefully you all know how to prove that $\dim V = \dim V^*$ and thus in this case $\mathrm{ev}_V\colon V \to V^{**}$ is an isomorphism. Hence by definition ev is a *natural isomorphism* when restricted to the subcategory FiniteVect of finite-dimensional vector spaces, and this is precisely the statement of Theorem 21.5.

Here are a few more examples of things that we have already proved are natural transformations.

PROPOSITION 21.6. *Let $C_\bullet\colon$ Top $\to$ Comp denote the singular chain functor, and given $n \geq 0$ let $C_n\colon$ Top $\to$ Ab denote the singular chain functor restricted to a single C_n. Then:*

1. *The boundary operator in singular homology is a natural transformation $C_n \to C_{n-1}$.*
2. *The barycentric subdivision operator $\mathrm{Sd}\colon C_\bullet \to C_\bullet$ is a natural transformation.*

Proof. The first statement is just Proposition 7.20 in fancy language. The second is is just (13.9). ∎

Recall for any topological space X and any $p \in X$, one has $\tilde{H}_1(X) \cong H_1(X, p)$, cf. Corollary 12.22.

PROPOSITION 21.7. *Regard π_1 and $\tilde{H}_1$ as functors Top_* $\to$ Groups (i.e. forget that $\tilde{H}_1(X)$ is abelian). Then the Hurewicz map defines a natural transformation $\pi_1 \to \tilde{H}_1$.*

Proof. This is Problem E.2. ∎

PROPOSITION 21.8. *Define a functor $R\colon \mathsf{Top}^2 \to \mathsf{Top}^2$ by $R(X, X') = (X', \emptyset)$. Then the connecting homomorphism δ of the long exact sequence of a pair defines a natural transformation $\delta\colon H_n \to H_{n-1} \circ R$.*

Proof. This is just the fact that the right-hand square commutes in the diagram of Proposition 12.3. ∎

With the definition of natural transformations out of the way, we can finally introduce the famous *Eilenberg-Steenrod axioms* for a homology theory. Let $R\colon \mathsf{Top}^2 \to \mathsf{Top}^2$ be as in the previous proposition.

Definition 21.9 (The Eilenberg-Steenrod Axioms). A **homology theory** is a sequence $\mathcal{H}_n\colon \mathsf{Top}^2 \to \mathsf{Ab}$ of functors for $n \ge 0$ and a sequence $\delta = \delta_n\colon \mathcal{H}_n \to \mathcal{H}_{n-1} \circ R$ of natural transformations for $n \ge 1$ satisfying the following four axioms:

- **The homotopy axiom**: If $f, g\colon (X, X') \to (Y, Y')$ are homotopic mod X' (as in Definition 12.25) then $\mathcal{H}_n(f) = \mathcal{H}_n(g)$ for all $n \ge 0$. Thus $\mathcal{H}_n$ factors to define functors $\mathsf{hTop}^2 \to \mathsf{Ab}$ for each $n \ge 0$.
- **The exact sequence axiom**: For every pair (X, X') with inclusions $\imath\colon (X', \emptyset) \hookrightarrow (X, \emptyset)$ and $\jmath\colon (X, \emptyset) \hookrightarrow (X, X')$, there is an exact sequence
$$\cdots \to \mathcal{H}_n(X') \xrightarrow{\mathcal{H}_n(\imath)} \mathcal{H}_n(X) \xrightarrow{\mathcal{H}_n(\jmath)} \mathcal{H}_n(X, X') \xrightarrow{\delta} \mathcal{H}_{n-1}(X') \to \dots,$$
where we abbreviate $\mathcal{H}_n(X) = \mathcal{H}_n(X, \emptyset)$ etc.
- **The excision axiom**: For every pair (X, X') and every subset $U \subseteq X$ with $\overline{U} \subset (X')^\circ$, the inclusion $(X \setminus U, X' \setminus U) \hookrightarrow (X, X')$ induces an isomorphism
$$\mathcal{H}_n(X \setminus U, X' \setminus U) \cong \mathcal{H}_n(X, X'), \qquad \forall\, n \ge 0.$$
- **The dimension axiom**: If $\{*\}$ is a one-point space then $\mathcal{H}_n(*) = 0$ for all $n > 0$ and $\mathcal{H}_0(*) = \mathbb{Z}$.

There are additionally two “optional” axioms:

- **The additivity axiom**: Let (X_λ, X'_λ), $\lambda \in \Lambda$ be a family of pairs of spaces. Denote by
$$\imath_\lambda\colon (X_\lambda, X'_\lambda) \hookrightarrow \left(\coprod_{\lambda\in\Lambda} X_\lambda, \coprod_{\lambda\in\Lambda} X'_\lambda \right)$$
the inclusion. Then for all $n \ge 0$, the map
$$\sum_{\lambda\in\Lambda} \mathcal{H}_n(\imath_\lambda)\colon \bigoplus_{\lambda\in\Lambda} \mathcal{H}_n(X_\lambda, X'_\lambda) \to \mathcal{H}_n\left(\coprod_{\lambda\in\Lambda} X_\lambda, \coprod_{\lambda\in\Lambda} X'_\lambda \right).$$
is an isomorphism.
- **The weak equivalence axiom**: If $f\colon (X, X') \to (Y, Y')$ is a weak equivalence (cf. Definition 18.18) then $\mathcal{H}_n(f)\colon \mathcal{H}_n(X, X') \to \mathcal{H}_n(Y, Y')$ is an isomorphism for all $n \ge 0$.

Some remarks:

1. The last two axioms are less important, and are both sometimes omitted from the treatment of axiomatic homology theory. The additivity axiom is implied by the excision axiom whenever Λ is a finite set (see Problem K.6), and thus is only of interest for infinite disjoint unions.

2. The weak equivalence axiom is used only to ensure that a homology theory is uniquely determined by what it does to cell complexes, cf. Theorem 18.19. We will ignore this axiom for the time being, since we haven't defined the higher homotopy groups. In particular, we currently have no way of proving that singular homology is a homology theory (!) if we insist on this axiom, since we cannot verify that singular homology satisfies the weak equivalence axiom. We will discuss this in more detail at the end of Algebraic Topology II (Theorem 46.9.)

3. As currently defined, cellular homology is also *not* a homology theory, since we have only defined cellular homology for cell complexes. Nevertheless, it can be made into a homology theory by arguing as follows. Let us denote by Cell the category of cell complexes, and denote by $I\colon \mathsf{Cell} \to \mathsf{Top}$ the inclusion functor. A more precise version of Theorem 18.19 (this is stated as Theorem 46.15) tell us that there is a functor $\Gamma\colon \mathsf{Top} \to \mathsf{Cell}$ that assigns to any space X a cell complex $\Gamma(X)$, together with a natural transformation $\Phi\colon I \circ \Gamma \to \mathrm{id}_{\mathsf{Top}}$ such that $\Phi(X)\colon \Gamma(X) \to X$ is a weak equivalence. The functor Γ is called a **cellular approximation functor**. Extending this to pairs of spaces, one then defines $H_n^{\text{cell}}\colon \mathsf{Top}^2 \to \mathsf{Ab}$ by first applying Γ: $H_n^{\text{cell}} \circ \Gamma$. The resulting sequence of homology functors is a genuine homology theory.

4. Many of the properties of singular homology continue to hold for an arbitrary homology theory. For instance, if X is contractible then by the homotopy axiom and the dimension axiom one sees $\mathcal{H}_n(X) = 0$ for $n > 0$ and $\mathcal{H}_n(X) = \mathbb{Z}$ for $n = 0$. A more involved fact is that the Relative Homeomorphism Theorem 19.7 continues to hold; more on this later.

5. The exact sequence axiom implies the long exact sequence of a triple: if $X'' \subseteq X' \subseteq X$ are subspaces then there is a long exact sequence

$$\dots \mathcal{H}_n(X', X'') \to \mathcal{H}_n(X, X'') \to \mathcal{H}_n(X, X') \xrightarrow{\delta'} \mathcal{H}_{n-1}(X', X'') \to \dots,$$

where the undecorated maps are induced by inclusion and the map δ' is the composition

$$\mathcal{H}_n(X, X') \xrightarrow{\delta} \mathcal{H}_{n-1}(X') \to \mathcal{H}_{n-1}(X', X'')$$

Indeed, this follows from the solution of Problem F.4, since the solution given there (using the commutative braid of Problem F.3) used nothing other than the exact sequence axiom.

6. Both of the homology theories (singular and cellular) that we have constructed have arisen from first defining a functor $\mathsf{Top}^2 \to \mathsf{Comp}$ and then composing this with the functor $H_n\colon \mathsf{Comp} \to \mathsf{Ab}$ that takes the homology of the chain complex. However this is *not* part of the axioms, and any proof we gave for singular/cellular homology that used this are therefore *not* valid for an arbitrary homology theory. This is the reason we proved Problem F.4 using the commutative braid (as remarked at the end of the solution to Problem F.4, there is a much quicker proof that is valid only for homology theories that are the homology of a chain complex.)

7. The only axiom that guarantees non-triviality of $\mathcal{H}_n$ is the dimension axiom, which at least tells us that the zeroth homology of a point is non-zero. Without this axiom, a perfectly valid theory would be $\mathcal{H}_n \equiv 0$. Nevertheless, there are many examples of things that one would like to be a "homology theory" that do *not* satisfy the dimension axiom. Two examples are *topological K-theory* and *symplectic homology*[1]. Thus a **generalised homology theory** is a sequence $(\mathcal{H}_\bullet, \delta)$ that satisfies all the axioms apart from the dimension axiom.

Now let us define what it means for two homology theories to be the same.

Definition 21.10. Let $(\mathcal{H}_\bullet, \delta)$ and $\mathcal{K}_\bullet, \varepsilon)$ be two homology theories. A **natural transformation** $\Phi_\bullet \colon (\mathcal{H}_\bullet, \delta) \to (\mathcal{K}_\bullet, \varepsilon)$ is a sequence of natural transformations $\Phi_n \colon \mathcal{H}_n \to \mathcal{K}_n$ for $n \geq 0$ such that the following diagram commutes for all $n \geq 1$ and all pairs (X, X'):

$$\begin{array}{ccc} \mathcal{H}_n(X, X') & \xrightarrow{\delta} & \mathcal{H}_{n-1}(X') \\ \downarrow{\scriptstyle \Phi_n(X,X')} & & \downarrow{\scriptstyle \Phi_{n-1}(X')} \\ \mathcal{K}_n(X, X') & \xrightarrow[\varepsilon]{} & \mathcal{K}_{n-1}(X') \end{array}$$

If Φ_n is a natural isomorphism for each n then we say that the two homology theories $(\mathcal{H}_\bullet, \delta)$ and $\mathcal{K}_\bullet, \varepsilon)$ are **naturally isomorphic**.

It is now easy to formulate the big theorem.

Theorem 21.11 (Existence and uniqueness of a homology theory). *Singular homology is a homology theory. Moreover if $(\mathcal{H}_\bullet, \delta)$ is any homology theory then $(\mathcal{H}_\bullet, \delta)$ is naturally isomorphic to singular homology.*

We can't really come close to proving this as currently stated. Indeed, as remarked above, we cannot even show existence, since we don't know that singular homology satisfies the weak equivalence axiom. If we drop the weak equivalence axiom then we have already shown that singular homology satisfies the other axioms. However the main tool needed to construct a natural isomorphism between singular homology and an arbitrary homology theory is the higher dimensional analogue of the Hurewicz Theorem 9.7. We will discuss Theorem 21.11 right at the end of Algebraic Topology II (cf. Theorem 46.17.)

Nevertheless, the techniques we have developed thus far in the course allow us to prove the following weaker version.

Theorem 21.12 (Baby Uniqueness Theorem). *Suppose $(\mathcal{H}_\bullet, \delta)$ and $\mathcal{K}_\bullet, \varepsilon)$ satisfy the first four axioms (homotopy, exact sequence, excision and dimension) and suppose $\Phi_\bullet \colon (\mathcal{H}_\bullet, \delta) \to \mathcal{K}_\bullet, \varepsilon)$ is a sequence of natural transformations such that $\Phi_0(*) \colon \mathcal{H}_0(*) \to \mathcal{K}_0(*)$ is an isomorphism, where $*$ is a one-point space. Then*

$$\Phi_n(X, X') \colon \mathcal{H}_n(X, X') \to \mathcal{K}_n(X, X')$$

is an isomorphism for all pairs (X, X') consisting of a finite cell complex X and a subcomplex X'.

[1]The former is the content of next semester's student seminar entitled "Vector Bundles in Algebraic Topology". The latter is my favourite (generalised) homology theory.

In other words, if we *already* have a natural transformation between two homology theories, it suffices to check it's an isomorphism on a one-point space to conclude it's an isomorphism on any finite cell complex. Of course, by assumption one always has $\mathcal{H}_0(*) \cong \mathbb{Z} \cong \mathcal{K}_0(*)$, but the hypotheses of the theorem are asserting much more: that there exists a natural transformation between the two homology theories that realises this isomorphism. This theorem is not too hard to prove, and we will do so next lecture.

LECTURE 22

Free chain complexes

In this lecture we first prove Theorem 21.12. Our proof will use the Relative Homeomorphism Theorem 19.7, which is valid for an arbitrary homology theory. Let us begin by stating this precisely.

THEOREM 22.1 (The Relative Homeomorphism Theorem Reborn). *Let* $(\mathcal{H}_\bullet, \delta)$ *denote a homology theory satisfying the first four axioms. Let* $f\colon (X, X') \to (Y, Y')$ *be a relative homeomorphism. Assume that* X *is compact and that* Y *is compact Hausdorff, and that* X' *and* Y' *are closed in* X *and* Y *respectively. Assume further that there exists a neighbourhood* U *of* X' *in* X *such that* X' *is a strong deformation retract of* U*, and a neighbourhood* V *of* Y' *in* Y *such that* Y' *is a strong deformation retract of* V*. Then*

$$\mathcal{H}_n(f)\colon \mathcal{H}_n(X, X') \to \mathcal{H}_n(Y, Y') \qquad \text{is an isomorphism for all } n \geq 0.$$

Proof. Go through the proof of the Relative Homeomorphism Theorem 19.7 and check we only used the axioms. ■

We now prove Theorem 21.12.

Proof. We prove the theorem in three steps. In this proof, all vertical maps are induced by Φ, and we won't label them on the diagrams. The moral of the proof is: use the Five Lemma five times.

1. We first prove the result for $X = S^0$ and $X' = \emptyset$. We think of S^0 as the union of two points p and q, and consider the following diagram:

$$\begin{array}{ccc}
\mathcal{H}_n(q) & \longrightarrow & \mathcal{H}_n(p \cup q, p) \\
\downarrow & & \downarrow \\
\mathcal{K}_n(q) & \longrightarrow & \mathcal{K}_n(p \cup q, p)
\end{array}$$

The two horizontal maps are excision isomorphisms, and the left-hand vertical map is an isomorphism by hypothesis for $n = 0$ and by the dimension axiom for $n > 0$. Thus the right-hand vertical map is also an isomorphism. We now consider the diagram:

$$\begin{array}{ccccccccc}
\mathcal{H}_{n+1}(S^0, p) & \xrightarrow{\delta} & \mathcal{H}_n(p) & \longrightarrow & \mathcal{H}_n(S^0) & \longrightarrow & \mathcal{H}_n(S^0, p) & \longrightarrow & \mathcal{H}_{n-1}(p) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{K}_{n+1}(S^0, p) & \xrightarrow[\varepsilon]{} & \mathcal{K}_n(p) & \longrightarrow & \mathcal{K}_n(S^0) & \longrightarrow & \mathcal{K}_n(S^0, p) & \longrightarrow & \mathcal{K}_{n-1}(p)
\end{array}$$

The rows are exact by the exact sequence axiom. All the vertical maps apart from the middle one are isomorphisms. Thus the middle one is too by the Five Lemma (Proposition 11.3).

2. We now prove the result for an arbitrary sphere S^k. Let us inductively assume that $\Phi_n(S^{k-1})\colon \mathcal{H}_n(S^{k-1}) \to \mathcal{K}_n(S^{k-1})$ is an isomorphism for all $n \geq 0$. Since B^k is contractible, by the homotopy axiom, the dimension axiom and the hypotheses, the map $\Phi_n(B^k)\colon \mathcal{H}_n(B^k) \to \mathcal{K}_n(B^k)$ is an isomorphism. Then we apply the Five Lemma again to the following diagram:

$$\begin{array}{ccccccccc}
\mathcal{H}_n(S^{k-1}) & \longrightarrow & \mathcal{H}_n(B^k) & \longrightarrow & \mathcal{H}_n(B^k, S^{k-1}) & \xrightarrow{\delta} & \mathcal{H}_{n-1}(S^{k-1}) & \longrightarrow & \mathcal{H}_{n-1}(B^k) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{K}_n(S^{k-1}) & \longrightarrow & \mathcal{K}_n(B^k) & \longrightarrow & \mathcal{K}_n(B^k, S^{k-1}) & \xrightarrow[\varepsilon]{} & \mathcal{K}_{n-1}(S^{k-1}) & \longrightarrow & \mathcal{K}_{n-1}(B^k)
\end{array}$$

Again, the rows are exact and all vertical maps apart from the middle one are isomorphisms. Thus the middle one is too. Now consider the following diagram:

$$\begin{array}{ccc}
\mathcal{H}_n(B^k, S^{k-1}) & \xrightarrow{\mathcal{H}_n(f)} & \mathcal{H}_n(S^k, p) \\
\downarrow & & \downarrow \\
\mathcal{K}_n(B^k, S^{k-1}) & \xrightarrow{\mathcal{K}_n(f)} & \mathcal{K}_n(S^k, p)
\end{array}$$

The horizontal maps come from a relative homeomorphism $(B^k, S^{k-1}) \to (S^k, p)$, and is thus an isomorphism by the Relative Homeomorphism Theorem. We have just shown that the left-hand map is an isomorphism, and hence the right-hand vertical map is too. Now we apply the Five Lemma again to this diagram:

$$\begin{array}{ccccccccc}
\mathcal{H}_{n+1}(S^k, p) & \xrightarrow{\delta} & \mathcal{H}_n(p) & \longrightarrow & \mathcal{H}_n(S^k) & \longrightarrow & \mathcal{H}_n(S^k, p) & \longrightarrow & \mathcal{H}_{n-1}(p) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{K}_{n+1}(S^k, p) & \xrightarrow[\varepsilon]{} & \mathcal{K}_n(p) & \longrightarrow & \mathcal{K}_n(S^k) & \longrightarrow & \mathcal{K}_n(S^k, p) & \longrightarrow & \mathcal{K}_{n-1}(p)
\end{array}$$

3. We now prove the theorem for a pair (X, X') by induction on the number of cells of X. We have already done the case where X has one cell, so let us assume that $\Phi_n(Y, Y')\colon \mathcal{H}_n(Y, Y') \to \mathcal{K}_n(Y, Y')$ is an isomorphism for all $n \geq 0$ and for any cell complex Y with at most $N-1$ cells. Let X be a cell complex with N cells and X' a subcomplex. If $\dim X = k$, pick a specific k-cell of X and let Z denote the complement of this cell. Then Z has $N-1$ cells, and there is a relative homeomorphism

$$f\colon (B^k, S^{k-1}) \to (X, Z).$$

We now consider the diagram:

$$\begin{array}{ccc}
\mathcal{H}_n(B^k, S^{k-1}) & \xrightarrow{\mathcal{H}_n(f)} & \mathcal{H}_n(X, Z) \\
\downarrow & & \downarrow \\
\mathcal{K}_n(B^k, S^{k-1}) & \xrightarrow{\mathcal{K}_n(f)} & \mathcal{K}_n(X, Z)
\end{array}$$

The horizontal maps are isomorphisms and the left-hand vertical map is an isomorphism for all $n \geq 0$, and thus the same is true of the right-hand vertical map. Next, apply the Five Lemma to this diagram:

$$
\begin{array}{ccccccccc}
\mathcal{H}_{n+1}(X,Z) & \xrightarrow{\delta} & \mathcal{H}_n(Z) & \longrightarrow & \mathcal{H}_n(X) & \longrightarrow & \mathcal{H}_n(X,Z) & \longrightarrow & \mathcal{H}_{n-1}(Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{K}_{n+1}(X,Z) & \xrightarrow[\varepsilon]{} & \mathcal{K}_n(Z) & \longrightarrow & \mathcal{K}_n(X) & \longrightarrow & \mathcal{K}_n(X,Z) & \longrightarrow & \mathcal{K}_{n-1}(Z)
\end{array}
$$

Thus $\Phi_n(X)\colon \mathcal{H}_n(X) \to \mathcal{K}_n(X)$ is an isomorphism for all $n \geq 0$. Similarly $\Phi_n(X')\colon \mathcal{H}_n(X') \to \mathcal{K}_n(X')$ is an isomorphism for every $n \geq 0$. Finally, we apply the Five Lemma a fifth time to the diagram:

$$
\begin{array}{ccccccccc}
\mathcal{H}_n(X') & \longrightarrow & \mathcal{H}_n(X) & \longrightarrow & \mathcal{H}_n(X,X') & \xrightarrow{\delta} & \mathcal{H}_{n-1}(X') & \longrightarrow & \mathcal{H}_{n-1}(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{K}_n(X') & \longrightarrow & \mathcal{K}_n(X) & \longrightarrow & \mathcal{K}_n(X,X') & \xrightarrow[\varepsilon]{} & \mathcal{K}_{n-1}(X') & \longrightarrow & \mathcal{K}_{n-1}(X)
\end{array}
$$

Thus $\Phi_n(X,X')\colon \mathcal{H}_n(X,X') \to \mathcal{K}_n(X,X')$ is also an isomorphism. This completes the proof. ∎

Remark 22.2. If we knew our homology theories arose from taking the homology of a chain complex, one could now invoke Theorem 16.22 to deduce the same result for any cell complex (rather than just finite ones). In general, the excision axiom and the additivity axiom allows one to prove (roughly speaking) that a homology theory does indeed commute with filtered colimits, but this is a somewhat involved argument.

We now embark upon some more homological algebra. Our journey will culminate at the end of the next lecture with the famous *Acyclic Models Theorem*, which will allow us to give new and simpler proofs of various statements from the course (eg. the proof that singular homology satisfies the homotopy axiom). We begin with the following lemma, whose proof is similar to the last part of Problem F.6.

Lemma 22.3. *Let F be a free abelian group. Suppose $g\colon B \to C$ is a surjective homomorphism of abelian groups and $h\colon F \to C$ is a homomorphism. Then there exists a homomorphism $f\colon F \to B$ such that $gf = h$.*

$$
\begin{array}{ccccc}
 & & F & & \\
 & \overset{f}{\swarrow} & \downarrow h & & \\
B & \xrightarrow[g]{} & C & \longrightarrow & 0
\end{array}
$$

Proof. Let X be a basis of F. For each $x \in X$, choose $b_x \in B$ such that $g(b_x) = h(x)$. By Lemma 7.2 there exists a unique homomorphism $f\colon F \to B$ with the property that $f(x) = b_x$ for all $x \in X$. Then both gf and h agree on X, and hence $gf = h$ as desired. ∎

This result implies the following statement.

Proposition 22.4. *Suppose we have a commutative diagram of abelian groups where the bottom row is exact, that the top row satisfies $ji = 0$, and that A is free abelian.*

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ A' & \xrightarrow[i']{} & B' & \xrightarrow[j']{} & C' \end{array}$$

Then there exists a homomorphism $f\colon A \to A'$ making the first square commute:

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C \\ {\scriptstyle f}\,\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ A' & \xrightarrow[i']{} & B' & \xrightarrow[j']{} & C' \end{array}$$

Proof. We claim that $\operatorname{im} gi \subseteq \operatorname{im} i'$. Indeed, by exactness $\operatorname{im} i' = \ker j'$, and thus it suffices to show that $j'gi = 0$. But $j'gi = hji$ by commutativity. Since $ji = 0$ by assumption, the claim follows. This means we have a diagram:

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow{\scriptstyle gi} & & \\ A' & \xrightarrow[i']{} & \operatorname{im} i' & \longrightarrow & 0 \end{array}$$

Now apply Lemma 22.3 to obtain the desired homomorphism $f\colon A \to A'$. ∎

Proposition 22.4 has a rather surprising consequence, which we now explain. Suppose $C_\bullet$ and $D_\bullet$ are two chain complexes. A chain map $f\colon C_\bullet \to D_\bullet$ induces maps $H_n(f)\colon H_n(C_\bullet) \to H_n(D_\bullet)$ for each $n \geq 0$. But when can we go the other way?

Namely, suppose we start with a homomorphism $h\colon H_0(C_\bullet) \to H_0(D_\bullet)$. We are interested in obtaining criterion for the existence of a chain map $f\colon C_\bullet \to D_\bullet$ such that

$$H_0(f) = h. \tag{22.1}$$

Let us say that f is a chain map **over** h if (22.1) holds. Here are two more definitions.

Definition 22.5. A chain complex $C_\bullet$ is called **free** if each group C_n is a free abelian group.

Definition 22.6. A chain complex $C_\bullet$ is **non-negative** if $C_n = 0$ for all $n < 0$. Thus the singular chain complex is always non-negative. A non-negative chain complex $C_\bullet$ is **acyclic in positive degrees** if $H_n(C_\bullet) = 0$ for all $n > 0$.

The next result is sometimes called the *Comparison Theorem* in homological algebra[1].

[1]This is actually a weaker version than the usual "Comparison Theorem" where the top complex is assumed to a complex of *projectives* rather than a free complex.

Theorem 22.7. *Suppose $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ are two non-negative chain complexes. Assume that $C_\bullet$ is free and that $D_\bullet$ is acyclic in positive degrees. Then given any homomorphism $h\colon H_0(C_\bullet) \to H_0(D_\bullet)$, there always exists a chain map $f\colon C_\bullet \to D_\bullet$ over h. Moreover if f and g are two chain maps over h then f and g are chain homotopic.*

Remark 22.8. In particular, if $C_\bullet$ is free and $D_\bullet$ is acyclic in positive degrees, then a chain map $f\colon C_\bullet \to D_\bullet$ is determined up to chain homotopy by the map $H_0(f)$.

Proof. Since $C_n = 0$ for all $n < 0$, $H_0(C_\bullet)$ is the cokernel of $\partial\colon C_1 \to C_0$, and similarly for $H_0(D_\bullet)$. Denote by $\varepsilon\colon C_0 \to H_0(C_\bullet)$ and $\varepsilon'\colon D_0 \to H_0(D_\bullet)$ the two quotient maps. Note that the extended complex

$$\cdots \to D_2 \xrightarrow{\partial'} D_1 \xrightarrow{\partial'} D_0 \xrightarrow{\varepsilon'} H_0(D_\bullet) \to 0$$

is an acyclic complex (its homology is zero in every degree.) Our goal is to find maps $f_n\colon C_n \to D_n$ such that the entire diagram below commutes:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\partial} & C_0 & \xrightarrow{\varepsilon} & H_0(C_\bullet) & \longrightarrow & 0 \\
 & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle h} & & \\
\cdots & \longrightarrow & D_2 & \xrightarrow[\partial']{} & D_1 & \xrightarrow[\partial']{} & D_0 & \xrightarrow[\varepsilon']{} & H_0(D_\bullet) & \longrightarrow & 0
\end{array}$$

We argue by induction on $n \geq 0$. For the case $n = 0$, we use Lemma 22.3 with the diagram

$$\begin{array}{ccccc}
 & {\scriptstyle f_0}\swarrow & C_0 & & \\
 & & \downarrow{\scriptstyle h\circ\varepsilon} & & \\
D_0 & \xrightarrow[\varepsilon']{} & H_0(D_\bullet) & \longrightarrow & 0
\end{array}$$

For the inductive step, we apply Proposition 22.4 to the diagram

$$\begin{array}{ccccc}
C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\
D_{n+1} & \xrightarrow[\partial']{} & D_n & \xrightarrow[\partial']{} & D_{n-1}
\end{array}$$

Now suppose we are given two such chain maps f and g over h. We wish to construct maps $P_n\colon C_n \to D_{n+1}$ for $n \geq -1$ such that

$$\partial' P_n + P_{n-1}\partial = f_n - g_n.$$

Define $P_{-1} = 0$. Now consider the diagram:

$$\begin{array}{ccccc}
C_0 & \xrightarrow{\mathrm{id}} & C_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_0 - g_0} & & \downarrow{\scriptstyle 0} \\
D_1 & \xrightarrow[\partial']{} & D_0 & \longrightarrow & H_0(D_\bullet)
\end{array}$$

Proposition 22.4 applies to give us the desired map $P_0\colon C_0 \to D_1$. Abbreviate $k_n = f_n - g_n - P_{n-1}\partial$. For the inductive step we use the diagram

$$\begin{array}{ccccc} C_n & \xrightarrow{\text{id}} & C_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle k_n} & & \downarrow{\scriptstyle 0} \\ D_{n+1} & \xrightarrow[\partial']{} & D_n & \xrightarrow{\partial'} & D_{n-1} \end{array}$$

This diagram commutes, since

$$\begin{aligned} \partial' k_n &= \partial'(f_n - g_n) - (\partial' P_{n-1})\partial \\ &= \partial'(f_n - g_n) - (f_{n-1} - g_{n-1} - P_{n-2}\partial)\partial \\ &= \partial'(f_n - g_n) - (f_{n-1} - g_{n-1})\partial \\ &= 0 \end{aligned}$$

as $\partial^2 = 0$ and $f - g$ is a chain map. Thus we can apply Proposition 22.4 again to get a map $P_n\colon C_n \to D_{n+1}$. This completes the proof. ■

Corollary 22.9. *Suppose $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ are two non-negative chain complexes. Assume that $C_\bullet$ and $D_\bullet$ are both free and acyclic in positive degrees. Assume we are given an isomorphism $h\colon H_0(C_\bullet) \to H_0(D_\bullet)$. Then every chain map f over h is a chain equivalence.*

Proof. We apply Theorem 22.7 twice to obtain chain maps $f\colon C_\bullet \to D_\bullet$ over h and $g\colon D_\bullet \to C_\bullet$ over h^{-1}. Then $g \circ f\colon C_n \to C_n$ is a chain map over $h^{-1} \circ h = \text{id}_{H_0(C_\bullet)}$. But another obvious chain map over $\text{id}_{H_0(C_\bullet)}$ is $\text{id}_{C_\bullet}$. By the last statement of Theorem 22.7, $g \circ f$ is chain homotopic to $\text{id}_{C_\bullet}$. Similarly $f \circ g$ is chain homotopic to $\text{id}_{D_\bullet}$. Thus f is a chain equivalence as desired. ■

LECTURE 23

The Acyclic Models Theorem

In this final lecture we state and prove the *Acyclic Models Theorem*. The main ideas were all contained in Theorem 22.7 and Corollary 22.9 from the last lecture. The formalism below will seem somewhat complicated, but all we are really doing is carrying out the constructions above in a more general setting.

DEFINITION 23.1. Let C be a category. A family of **models** for C is simply an indexed *subset* $\mathcal{M} = \{M_\lambda \mid \lambda \in \Lambda\} \subseteq \mathrm{obj}(\mathsf{C})$.

DEFINITION 23.2. Let C be a category with family of models $\mathcal{M} = \{M_\lambda \mid \lambda \in \Lambda\}$. Suppose $T\colon \mathsf{C} \to \mathsf{Ab}$ is a functor. A **T-model set** $\mathcal{X}$ is a choice of element $x_\lambda \in T(M_\lambda)$ for each λ:

$$\mathcal{X} = \{x_\lambda \in T(M_\lambda) \mid \lambda \in \Lambda\}.$$

DEFINITION 23.3. Let C be a category with a family of models $\mathcal{M} = \{M_\lambda \mid \lambda \in \Lambda\}$. Suppose $T\colon \mathsf{C} \to \mathsf{Ab}$ is a functor. We say that T is **free with basis in $\mathcal{M}$** if:

1. $T(C)$ is a free abelian group for every $C \in \mathrm{obj}(\mathsf{C})$,
2. There is a T-model set $\mathcal{X} = \{x_\lambda \in T(M_\lambda) \mid \lambda \in \Lambda\}$ such that for every object in C the set
$$\{T(f)(x_\lambda) \mid f \in \mathrm{Hom}(M_\lambda, C),\ \lambda \in \Lambda\}$$
is a basis for $T(C)$.

We call $\mathcal{X}$ a **model basis** for T.

EXAMPLE 23.4. Fix $n \geq 0$. Consider a family of models for Top consisting of just one model $\mathcal{M} = \{\Delta^n\}$. Consider now the functor $C_n\colon \mathsf{Top} \to \mathsf{Ab}$ that assigns a topological space X the free abelian group of singular n-chains $C_n(X)$. We claim that C_n is free with basis in $\{\Delta^n\}$. The first condition is by definition, so we need only verify the second. For this, recall (see Lecture 13) we denote by $\ell_n\colon \Delta^n \to \Delta^n$ the identity map, thought of as a singular n-simplex in Δ^n. Then the set $\{\ell_n\}$ is a C_n-model set. We claim that $\{\ell_n\}$ is a model basis. Indeed, if $\sigma\colon \Delta^n \to X$ is any singular n-simplex then (thinking of σ as a continuous map from Δ^n to X), we have

$$C_n(\sigma)(\ell_n) = \sigma_n^{\#}\ell_n = \sigma$$

(see (13.4)). Since $C_n(X)$ has basis the singular n-simplices in X, we have that

$$\{C_n(\sigma)(\ell_n) \mid \sigma\colon \Delta^n \to X \text{ continuous}\}$$

is a basis for $C_n(X)$.

Proposition 23.5. *Let C be a category with family of models $\mathcal{M} = \{M_\lambda \mid \lambda \in \Lambda\}$. Suppose $S, T \colon \mathsf{C} \to \mathsf{Ab}$ are functors, and assume T is free with basis in $\mathcal{M}$. Let*

$$\{x_\lambda \in T(M_\lambda) \mid \lambda \in \Lambda\}$$

denote a model basis for T. Choose elements $y_\lambda \in S(M_\lambda)$ for each $\lambda \in \Lambda$, and set

$$\mathcal{Y} := \{y_\lambda \in S(M_\lambda) \mid \lambda \in \Lambda\}$$

Then there exists a unique natural transformation $\Phi \colon T \to S$ such that

$$\Phi(M_\lambda)(x_\lambda) = y_\lambda, \qquad \forall\, \lambda \in \Lambda.$$

The following picture might help you remember the statement:

$$\begin{array}{ccc} T & \overset{\Phi}{\dashrightarrow} & S \\ \uparrow & & \uparrow \\ \mathcal{X} & \xrightarrow[x_\lambda \mapsto y_\lambda]{} & \mathcal{Y} \end{array}$$

Proof. Let us first check that Φ is unique if it exists. For fixed $\lambda \in \Lambda$ and a fixed object $C \in \mathrm{obj}(\mathsf{C})$, we obtain a commutative diagram for every morphism $f \colon M_\lambda \to C$:

$$\begin{array}{ccc} T(M_\lambda) & \xrightarrow{T(f)} & T(C) \\ {\scriptstyle \Phi(M_\lambda)}\downarrow & & \downarrow{\scriptstyle \Phi(C)} \\ S(M_\lambda) & \xrightarrow[S(f)]{} & S(C) \end{array}$$

Thus if $x_\lambda \in \mathcal{X}$ we have by the hypothesis on Φ that

$$\Phi(C) \circ T(f)(x_\lambda) = S(f) \circ \Phi(M_\lambda)(x_\lambda) = S(f)(y_\lambda).$$

Since the family $\{T(f)(x_\lambda)\}$ forms a basis of (and hence generates) $T(C)$, it follows that each homomorphism $\Phi(C)$ is uniquely determined. Since C was an arbitrary object, it follows that Φ is unique. Now let us construct Φ. Again, fix an object C of C. We first define $\Phi(C)$ on basis elements $\{T(f)(x_\lambda)\}$ by declaring that $\Phi(C)\big(T(f)(x_\lambda)\big) := S(f)(y_\lambda)$. Then since $T(C)$ is free abelian, by Lemma 7.2 there is a unique homomorphism $\Phi(C) \colon T(C) \to S(C)$ that extends this map. It remains to show that Φ is a natural transformation. For this, take a morphism $g \colon A \to B$ in C. We need to prove the following diagram commutes:

$$\begin{array}{ccc} T(A) & \xrightarrow{T(g)} & T(B) \\ {\scriptstyle \Phi(A)}\downarrow & & \downarrow{\scriptstyle \Phi(B)} \\ S(A) & \xrightarrow[S(g)]{} & S(B) \end{array}$$

Since $T(A)$ is free abelian, it suffices to evaluate both sides on a typical basis element $T(f)(x_\lambda)$ for some $\lambda \in \Lambda$ and $f \colon M_\lambda \to A$. Then

$$S(g) \circ \Phi(A)\big(T(f)(x_\lambda)\big) = S(g)S(f)y_\lambda = S(g \circ f)(y_\lambda).$$

But also going the other way round:

$$\Phi(B) \circ T(g)\big(T(f)(x_\lambda)\big) = \Phi(B) \circ T(g \circ f)(x_\lambda) = S(g \circ f)(y_\lambda).$$

Thus Φ is indeed a natural transformation, and this completes the proof. ■

We now prove a generalisation of Proposition 22.4 to this setting.

Proposition 23.6. *Let* C *be a category with family of models* $\mathcal{M}$*. Suppose we are given six functors*

$$T_i,\ S_i \colon \mathsf{C} \to \mathsf{Ab}, \qquad i = 1, 2, 3.$$

together with six natural transformations as pictured below:

$$\begin{array}{ccccc} T_1 & \xrightarrow{\Phi_1} & T_2 & \xrightarrow{\Phi_2} & T_3 \\ & & \downarrow{\scriptstyle \Theta_1} & & \downarrow{\scriptstyle \Theta_2} \\ S_1 & \xrightarrow[\Psi_1]{} & S_2 & \xrightarrow[\Psi_2]{} & S_3 \end{array}$$

Assume that:

1. *Assume that for every object* $C \in \operatorname{obj}(\mathsf{C})$ *the composition* $\Phi_2(C) \circ \Phi_1(C) \colon T_1(C) \to T_3(C)$ *is the zero homomorphism.*
2. *The bottom row is exact on* $\mathcal{M}$*, in the sense that for every model* $M \in \mathcal{M}$ *one has* $\operatorname{im} \Psi_1(M) = \ker \Psi_2(M)$.
3. *The diagram commutes for every object* C *of* C.
4. T_1 *is free with basis in* $\mathcal{M}$.

Then there exists a natural transformation $\Upsilon \colon T_1 \to S_1$ *such that the first square commutes for every object of* C.

$$\begin{array}{ccccc} T_1 & \xrightarrow{\Phi_1} & T_2 & \xrightarrow{\Phi_2} & T_3 \\ {\scriptstyle \Upsilon}\,\vdots & & \downarrow{\scriptstyle \Theta_1} & & \downarrow{\scriptstyle \Theta_2} \\ S_1 & \xrightarrow[\Psi_1]{} & S_2 & \xrightarrow[\Psi_2]{} & S_3 \end{array}$$

The trickiest thing in Proposition 23.6 is the statement. It is easiest to see this as a direct generalisation of Proposition 22.4 where we use functors instead of maps. Indeed, if C had exactly one object and only the identity morphism, then Proposition 23.6 would reduce to Proposition 22.4. But in general Proposition 23.6 is *much stronger*: the important bit is that we only require the bottom row to be exact *on* $\mathcal{M}$. If you think back to Example 23.4, this can often be a massive simplification if $\mathcal{M}$ is very small compared to $\operatorname{obj}(\mathsf{C})$.

Proof. Let $\mathcal{X} = \{x_\lambda \in T_1(M_\lambda) \mid \lambda \in \Lambda\}$ denote a model basis for T_1. Then for each $\lambda \in \Lambda$ we have a commutative diagram in Ab that satisfies the hypotheses of Proposition 22.4:

$$\begin{array}{ccccc} T_1(M_\lambda) & \longrightarrow & T_2(M_\lambda) & \longrightarrow & T_3(M_\lambda) \\ & & \downarrow & & \downarrow \\ S_1(M_\lambda) & \longrightarrow & S_2(M_\lambda) & \longrightarrow & S_3(M_\lambda) \end{array}$$

Thus by Proposition 22.4 we obtain a homomorphism $T_1(M_\lambda) \to S_1(M_\lambda)$:

$$\begin{array}{ccccc} T_1(M_\lambda) & \longrightarrow & T_2(M_\lambda) & \longrightarrow & T_3(M_\lambda) \\ \downarrow & & \downarrow & & \downarrow \\ S_1(M_\lambda) & \longrightarrow & S_2(M_\lambda) & \longrightarrow & S_3(M_\lambda) \end{array}$$

Set $y_\lambda \in S_1(M_\lambda)$ denote the image of x_λ under this map. By Proposition 23.5 we obtain a natural transformation $\Upsilon \colon T_1 \to S_1$ such that $\Upsilon(M_\lambda)(x_\lambda) = y_\lambda$ for each $\lambda \in \Lambda$. It remains to check that the desired diagram commutes. For this consider

$$z_\lambda := \Psi_1(M_\lambda)(y_\lambda), \qquad \mathscr{Z} := \{z_\lambda \in S_2(M_\lambda) \mid \lambda \in \Lambda\}.$$

Then both $\Theta_1 \circ \Phi_1$ and $\Psi_1 \circ \Upsilon$ are natural transformations $T_1 \to S_2$ that send x_λ to z_λ. The uniqueness part of Proposition 23.5 then implies that $\Theta_1 \circ \Phi_1 = \Psi_1 \circ \Upsilon$. This completes the proof. ■

The main result of today's lecture is basically the "models" version of Theorem 22.7. This means we need to study functors with values in Comp. So suppose $T_\bullet \colon \mathsf{C} \to \mathsf{Comp}$ is a functor. Thus for each $C \in \operatorname{obj}(\mathsf{C})$ we obtain a chain complex $T_\bullet(C)$. Given $n \in \mathbb{Z}$, let $T_n \colon \mathsf{C} \to \mathsf{Ab}$ denote the functor given by $C \mapsto T_n(C)$. As with the case of a single chain complex, we say that $T_\bullet$ is **non-negative** if $T_n(C) = 0$ for all $n < 0$, and we say an object $C \in \operatorname{obj}(\mathsf{C})$ is **T-acyclic in positive degrees** if $H_n(T_\bullet(C)) = 0$ for all $n > 0$.

A natural transformation $\Phi \colon T_\bullet \to S_\bullet$ between two functors $T_\bullet, S_\bullet \colon \mathsf{C} \to \mathsf{Comp}$ is usually called a **natural chain map**. There is an analogous notion of a natural chain homotopy.

Definition 23.7. Suppose $T_\bullet, S_\bullet \colon \mathsf{C} \to \mathsf{Comp}$ are two functors and $\Phi, \Psi \colon T_\bullet \to S_\bullet$ are two natural chain maps (i.e. natural transformations). A **natural chain homotopy** between Φ and Ψ is a sequence of natural transformations $\Upsilon_n \colon T_n \to S_{n+1}$ such that

$$\partial' \Upsilon_n + \Upsilon_{n-1} \partial = \Phi_n - \Psi_n$$

for every n. Here ∂ is the boundary operator of $T_\bullet$ and ∂' is the boundary operator of $S_\bullet$. Explicitly, this means for every object C of C, if we denote by ∂_C the boundary operator of $T_\bullet(C)$ and ∂'_C the boundary operator of $S_\bullet(C)$, we have

$$\partial'_C \Upsilon_n(C) + \Upsilon_{n-1}(C) \partial_C = \Phi_n(C) - \Psi_n(C)$$

as homomorphisms $T_n(C) \to S_n(C)$.

Similarly, a **natural chain equivalence** $\Phi \colon T_\bullet \to S_\bullet$ is a natural chain map with the property that there exists another natural chain map $\Psi \colon S_\bullet \to T_\bullet$ such that $\Psi \circ \Phi$ is naturally chain homotopic to $\mathrm{id}_{T_\bullet}$ and $\Phi \circ \Psi$ is naturally chain homotopic to $\mathrm{id}_{S_\bullet}$.

Now we introduce the functor-valued version of (22.1). Suppose $T_\bullet, S_\bullet \colon \mathsf{C} \to \mathsf{Comp}$ are two functors. Then $H_0 \circ T_\bullet$ and $H_0 \circ S_\bullet$ are two functors $\mathsf{C} \to \mathsf{Ab}$, given by

$C \mapsto H_0(T_\bullet(C))$ and $C \mapsto H_0(S_\bullet(C))$ respectively. Suppose we are given a natural transformation

$$\Theta\colon H_0 \circ T_\bullet \to H_0 \circ S_\bullet.$$

We can then ask the question: when does there exist a natural chain map $\Phi\colon T_\bullet \to S_\bullet$ such that $H_0(\Phi) = \Theta$? (This is of course a direct generalisation of asking when (22.1) held last lecture). The *Acyclic Models Theorem* gives an answer.

THEOREM 23.8 (The Acyclic Models Theorem). *Let C be a category with models $\mathcal{M}$. Assume that $S_\bullet, T_\bullet\colon \mathsf{C} \to \mathsf{Comp}$ are non-negative functors. Assume that for all $n \geq 0$, T_n is free with basis contained in $\mathcal{M}$. Assume that each model $M \in \mathcal{M}$ is $S_\bullet$-acyclic in positive degrees. If $\Theta\colon H_0 \circ T_\bullet \to H_0 \circ S_\bullet$ is a natural transformation then there exists a natural chain map $\Phi\colon T_\bullet \to S_\bullet$ over Θ. Moreover any two such natural chain maps are naturally chain homotopic.*

COROLLARY 23.9. *Assume instead that for all $n \geq 0$, both S_n and T_n are free with basis contained in $\mathcal{M}$, and that each model $M \in \mathcal{M}$ is both $S_\bullet$-acyclic in positive degrees and $T_\bullet$-acyclic in positive degrees. Then if $\Theta\colon H_0 \circ T_\bullet \to H_0 \circ S_\bullet$ is a natural equivalence then every natural chain map Φ over Θ is a natural chain equivalence.*

Again, by far the hardest part of this theorem is understanding the statement! The proof is basically identical to the proof of Theorem 22.7 and Corollary 22.9.

Proof of Theorem 23.8 and Corollary 23.9. We prove both the two results in three steps.

1. As in the proof of Theorem 22.7, our goal is to construct natural transformations $\Phi_n\colon T_n \to S_n$ such that the following diagram commutes.

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\partial} & T_2 & \xrightarrow{\partial} & T_1 & \xrightarrow{\partial} & T_0 & \longrightarrow & H_0(T_\bullet) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \Phi_2} & & \Big\downarrow{\scriptstyle \Phi_1} & & \Big\downarrow{\scriptstyle \Phi_0} & & \Big\downarrow{\scriptstyle \Theta} & & \\
\dots & \xrightarrow[\partial']{} & S_2 & \xrightarrow[\partial']{} & S_1 & \xrightarrow[\partial']{} & S_0 & \longrightarrow & H_0(S_\bullet) & \longrightarrow & 0
\end{array}$$

For $n = 0$, we have the following picture:

$$\begin{array}{ccccc}
T_0 & \longrightarrow & H_0(T_\bullet) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \Theta} & & \Big\downarrow{\scriptstyle 0} \\
S_0 & \longrightarrow & H_0(S_\bullet) & \longrightarrow & 0
\end{array}$$

The bottom row is exact because $H_0(S_\bullet(C))$ is the cokernel of $S_1(C) \to S_0(C)$. Thus Proposition 23.6 gives us a natural transformation $\Phi_0\colon T_0 \to S_0$ such that the diagram commutes:

$$\begin{array}{ccccc}
T_0 & \longrightarrow & H_0(T_\bullet) & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \Phi_0} & & \Big\downarrow{\scriptstyle \Theta} & & \Big\downarrow{\scriptstyle 0} \\
S_0 & \longrightarrow & H_0(S_\bullet) & \longrightarrow & 0
\end{array}$$

Now we inductively define $\Phi_n \colon T_n \to S_n$ for $n \geq 1$. Indeed, if we have constructed Φ_n then we have a diagram

$$\begin{array}{ccccc} T_{n+1} & \xrightarrow{\partial} & T_n & \xrightarrow{\partial} & T_{n-1} \\ & & \downarrow{\scriptstyle \Phi_n} & & \downarrow{\scriptstyle \Phi_{n-1}} \\ S_{n+1} & \xrightarrow[\partial']{} & S_n & \xrightarrow[\partial']{} & S_{n-1} \end{array}$$

By assumption the bottom row is exact for every model M, and thus as T_{n+1} is free, Proposition 23.6 applies to give us the desired $\Phi_{n+1} \colon T_{n+1} \to S_{n+1}$.

2. Now suppose we have two such maps $\Phi, \Psi \colon T_\bullet \to S_\bullet$. We need to find natural transformations $\Upsilon_n \colon T_n \to S_{n+1}$ for all $n \geq -1$ such that

$$\partial' \Upsilon_n + \Upsilon_{n-1} \partial = \Phi_n - \Psi_n.$$

We define $\Upsilon_{-1} = 0$ and proceed inductively. Let $\Xi_n := \Phi_n - \Psi_n$. Then we have a diagram:

$$\begin{array}{ccccc} T_0 & \xrightarrow{\mathrm{id}} & T_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Xi_0} & & \downarrow{\scriptstyle 0} \\ S_1 & \xrightarrow[\partial']{} & S_0 & \longrightarrow & H_0(S_\bullet) \end{array}$$

Again, Proposition 23.6 applies to give us the desired map $\Upsilon_0 \colon T_0 \to S_1$. For the inductive step we use the diagram

$$\begin{array}{ccccc} T_n & \xrightarrow{\mathrm{id}} & T_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Xi_n - \Upsilon_{n-1} \circ \partial} & & \downarrow{\scriptstyle 0} \\ S_{n+1} & \xrightarrow[\partial']{} & S_n & \longrightarrow & S_{n-1} \end{array}$$

We need to show this diagram commutes to apply Proposition 23.6. But by induction:

$$\begin{aligned} \partial'(\Xi_n - \Upsilon_{n-1}\partial) &= \partial' \Xi_n - (\partial \Upsilon_{n-1})\partial \\ &= \partial' \Xi_n - (\Xi_{n-1} - \Upsilon_{n-2}\partial)\partial \\ &= \partial' \Xi_n - \Xi_{n-1}\partial \\ &= 0 \end{aligned}$$

as $\partial^2 = 0$ and $\Xi_\bullet$ is a chain map.

3. Finally we prove Corollary 23.9. In this case Θ is a natural isomorphism, and hence there exists a natural transformation $\Pi \colon H_0 \circ S_\bullet \to H_0 \circ T_\bullet$ such that $\Pi \circ \Theta = \mathrm{id}_{H_0 \circ T}$ and $\Theta \circ \Pi = \mathrm{id}_{H_0 \circ S}$. We then have two natural chain maps $\Phi : T_\bullet \to S_\bullet$ and $\Psi \colon S_\bullet \to T_\bullet$ over from Θ and Π respectively. This gives us two natural chain maps over $\Pi \circ \Theta$: the identity $\mathrm{id}_{T_\bullet} \colon T_\bullet \to T_\bullet$ and $\Psi \circ \Phi \colon T_\bullet \to T_\bullet$. By what we have already proved, these two natural chain maps are naturally chain homotopic. Similarly $\Phi \circ \Psi \colon S_\bullet \to S_\bullet$ is naturally chain homotopic to $\mathrm{id}_{S_\bullet} \colon S_\bullet \to S_\bullet$. Thus $\Phi \colon T_\bullet \to S_\bullet$ is natural chain equivalence. This completes the proof. ∎

With all this heavy lifting out of the way, let us now reap the benefits. Recall Proposition 8.5:

PROPOSITION 23.10. *Let X be a topological space and define inclusions $\imath, \jmath\colon X \hookrightarrow X \times I$ by*

$$\imath(x) := (x, 0), \qquad \jmath(x) := (x, 1).$$

Then

$$H_n(\imath) = H_n(\jmath), \qquad \forall\, n \geq 0.$$

We can now give a cute easy proof.

Proof. We give Top models $\mathcal{M} = \{\Delta^n \mid n \geq 0\}$. Then by Example 23.4, for all $n \geq 0$ the singular chain functor $C_\bullet\colon \mathsf{Top} \to \mathsf{Comp}$ has the property that C_n is free with basis in $\mathcal{M}_n := \{\Delta^n\} \subset \mathcal{M}$. Define another functor $S_\bullet\colon \mathsf{Top} \to \mathsf{Comp}$ by $S_\bullet(X) = C_\bullet(X \times I)$. Since $\Delta^n \times I$ is convex, by Corollary 13.3, every model Δ^n is $S_\bullet$-acyclic in positive degrees (note this argument is not circular, see Remark 13.4.) Now the Acyclic Models Theorem tells us that in order to deduce that $\imath_\#, \jmath_\#\colon C_\bullet(X) \to C_\bullet(X \times I)$ are naturally chain homotopic (and hence in particular induce the same map on homology, cf. Proposition 10.24) it suffices to show that $H_0(\imath) = H_0(\jmath)$. But this is trivial. ■

Here is an even simpler application of the Acyclic Models Theorem.

PROPOSITION 23.11. *Let $\Phi, \Psi\colon C_\bullet \to C_\bullet$ denote natural transformations from the singular chain functor to itself. Assume that $\Phi_0 = \Psi_0$. Then there is a natural chain homotopy from Φ to Ψ.*

Proof. Apply the Acyclic Models Theorem with $S_\bullet = T_\bullet$ both equal to the singular chain functor, and with models $\{\Delta^n \mid n \geq 0\}$ as above and with $\Theta = H_0(\Phi) = H_0(\Psi)$. ■

We conclude with a much nicer proof of Theorem 13.11:

THEOREM 23.12. *Let X be a topological space and consider the barycentric division operator $\mathrm{Sd}\colon C_\bullet(X) \to C_\bullet(X)$. Then Sd is naturally chain homotopic to the identity (and hence induces an isomorphism on homology).*

Proof. Immediate from Proposition 23.11 because Sd is a natural (this is (13.3)) and Sd_0 is the identity. ■

This is the end of Algebraic Topology I. See you next semester!

PROPOSITION 23.13. *Let C be a category with models $\mathcal{M}$. Assume that $S_\bullet, T_\bullet\colon \mathsf{C} \to \mathsf{Comp}$ are non-negative functors. Assume that for all $n \geq 0$, T_n is free with basis contained in $\mathcal{M}$. Assume that each model $M \in \mathcal{M}$ is $S_\bullet$-acyclic in positive degrees. Assume that $\Phi_0\colon S_0 \to T_0$ is a natural transformation.*

LECTURE 24

Tensor products and a taste of Tor

Welcome to Algebraic Topology II! We have lots to get through this semester, so rather than a long and fancy introduction, this time we'll get straight on to our first topic: *universal coefficients*.

Unfortunately before the exciting topology can start, we need a rather lengthy algebraic prelude on tensor products and the Tor functor, which will take all of today's lecture.

DEFINITION 24.1. Let A and B be two abelian groups. Their **tensor product** is the abelian group $A \otimes B$, which has:

- *Generators:* all ordered pairs (a, b) where $a \in A$ and $b \in B$.
- *Relations* if $a, a' \in A$ and $b, b' \in B$ then

$$(a + a', b) = (a, b) + (a', b), \qquad \text{and} \qquad (a, b + b') = (a, b) + (a, b').$$

More formally, $A \otimes B$ is the quotient F/N where F is the free abelian group with basis $A \times B$ and N is the subgroup of F generated by all the relations. We denote the coset $(a, b) + N$ by $a \otimes b$.

Thus a typical element x of $A \otimes B$ can be written as a sum

$$x = \sum_i m_i \, a_i \otimes b_i, \qquad a_i \in A, \; b_i \in B, \; m_i \in \mathbb{Z}.$$

Actually one can always dispense with the m_i, since for any $m \in \mathbb{Z}$ and $a \in A$, $b \in B$, as element of $A \otimes B$ one has

$$m(a \otimes b) = (ma) \otimes b = a \otimes (mb),$$

as can be easily seen from the relations.

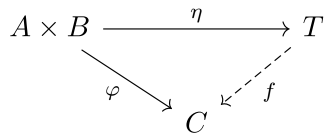
This definition of $A \otimes B$ is rather concrete, but it is presumably not clear to most of you what the point is. We now show that the tensor product can also be specified via a universal property.

DEFINITION 24.2. Let A, B, C be abelian groups. A **bilinear function** $\varphi \colon A \times B \to C$ is a function such that for all $a, a' \in A$ and all $b, b' \in B$,

$$\varphi(a + a', b) = \varphi(a, b) + \varphi(a', b), \qquad \text{and} \qquad \varphi(a, b + b') = \varphi(a, b) + \varphi(a, b').$$

As an example, the natural map $u \colon A \times B \to A \otimes B$ that sends $(a, b) \mapsto a \otimes b$ is bilinear.

Definition 24.3. Suppose we are given three abelian groups A, B, T and a bilinear map $\eta\colon A \times B \to T$. Consider the following universal property: Then we require that if C is any abelian group and $\varphi\colon A \times B \to C$ is a bilinear map, then there exists a *unique* homomorphism $f\colon T \to C$ such that the following diagram commutes:



As with all universal properties[1], if such a pair (T, η) exist, they are unique up to isomorphism. Let us verify that $(A \otimes B, u)$ does indeed solve this universal property.

Lemma 24.4. *The tensor product $A \otimes B$ together with the bilinear map $u\colon A \times B \to A \otimes B$, $u(a, b) = a \otimes b$, satisfies the universal property from Definition 24.3.*

Proof. Let $\varphi\colon A \times B \to C$ be a bilinear function. Recall $A \otimes B = F/N$, where F is free abelian with basis $A \times B$. We first extend $\varphi\colon A \times B \to C$ by linearity to a map $\tilde{\varphi}\colon F \to C$ (cf. Lemma 7.2). The relations that generate N are such that $N \subset \ker \tilde{\varphi}$, and hence $\tilde{\varphi}$ factors to define a homomorphism $f\colon F/N \to C$ such that $(a, b) + N \mapsto \tilde{\varphi}(a, b) = \varphi(a, b)$, that is, $f(a \otimes b) = \varphi(a, b)$. Moreover the map f is unique, since the set of all the $a \otimes b$ generate $A \otimes B$. ■

Being able to use universal properties makes the next result very transparent.

Proposition 24.5. *Let $f\colon A \to A'$ and $g\colon B \to B'$ be two homomorphisms. Then there is a unique homomorphism $A \otimes B \to A' \otimes B'$, denoted by $f \otimes g$, with the property that*

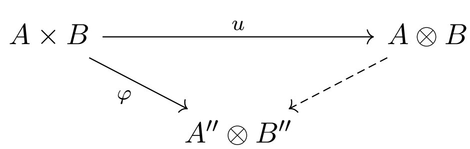
$$(f \otimes g)(a \otimes b) = fa \otimes gb, \qquad \forall\, a \in A,\ b \in B.$$

Moreover if $f'\colon A' \to A''$ and $g'\colon B' \to B''$ are two other homomorphisms then $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$.

Proof. The function $\varphi\colon A \times B \to A' \otimes B'$ defined by $\varphi(a, b) := fa \otimes gb$ is bilinear. Thus by Lemma 24.4 there is a unique homomorphism $A \otimes B \to A' \otimes B'$ which maps $a \otimes b \mapsto \varphi(a, b) = fa \otimes gb$. This is our desired homomorphism $f \otimes g$. This proves the first part. For the second part, we define a bilinear map $\varphi\colon A \times B \to A'' \otimes B''$ by

$$\varphi(a, b) = (f'(fa)) \otimes g'(gb)).$$

Then observe that both $(f' \otimes g') \circ (f \otimes g)$ and $(f' \circ f) \otimes (g' \circ g)$ fit on the dashed line to make the following diagram commute:



Thus by the uniqueness part of Lemma 24.4, we must have $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$. This completes the proof. ■

[1] To check how much you forgot over the Winter Vacation: prove that there is a most one pair (T, f) satisfying the universal property from Definition 24.3.

This means we can view the tensor product as a functor.

COROLLARY 24.6. *Let A be an abelian group. There is a functor $T\colon \mathsf{Ab} \to \mathsf{Ab}$ such that $T(B) = A \otimes B$ and if $f\colon B \to C$ then $T(f) = \mathrm{id}_A \otimes f\colon A \otimes B \to A \otimes C$.*

Proof. Let $f\colon B \to C$ and $g\colon C \to D$. By the second part of Proposition 24.5, we have

$$(\mathrm{id}_A \otimes g) \circ (\mathrm{id}_A \otimes f) = \mathrm{id}_A \otimes (g \circ f),$$

which shows that T preserves compositions. The fact that $\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A\otimes B}$ is obvious. ■

We normally denote the functor T by $A \otimes \square$. Similarly, given a fixed abelian group B, there is a functor $\square \otimes B\colon \mathsf{Ab} \to \mathsf{Ab}$ that sends $A \mapsto A \otimes B$, and sends morphisms $f\colon A \to C$ to $f \otimes \mathrm{id}_B\colon A \otimes B \to C \otimes B$.

We briefly mentioned additive functors last semester. Let me remind you of the definition.

DEFINITION 24.7. A functor $T\colon \mathsf{Ab} \to \mathsf{Ab}$ is said to be **additive** if $T(f + g) = T(f) + T(g)$ for any two morphisms $f, g\colon A \to B$.

An additive functor has the nice property that $T(0) = 0$, where 0 denotes either the zero group or the zero homomorphism.

The next result summarises some more properties of the tensor product. The proofs of parts (1)-(4) are all trivial. The proofs of (5) and (6) are slightly harder, and they are relegated to Problem Sheet L.

PROPOSITION 24.8.

1. *There is an isomorphism $A \otimes B \cong B \otimes A$ taking $a \otimes b$ to $b \otimes a$. The functors $A \otimes \square$ and $\square \otimes A$ are isomorphic.*
2. *The functors $A \otimes \square$ and $\square \otimes A$ are additive.*
3. *If $f\colon B \to B$ is multiplication by an integer m, $fb = mb$ for all $b \in B$, then $\mathrm{id}_A \otimes f\colon A \otimes B \to A \otimes B$ is also multiplication for m (and similarly for $f \otimes \mathrm{id}_A\colon B \otimes A \to B \otimes A$.)*
4. *For any abelian group A, the map $\mathbb{Z} \otimes A \to A$ given by $m \otimes a \mapsto ma$ is an isomorphism. Denoting this map by $\Phi(A)$, the resulting family Φ defines a natural equivalence $\mathbb{Z} \otimes \square \to \mathrm{id}_{\mathsf{Ab}}$ in Ab.*
5. *If A is an abelian group and $\{B_\lambda \mid \lambda \in \Lambda\}$ is a (possibly uncountable) family of abelian groups then there is an isomorphism*

$$A \otimes \bigoplus_{\lambda\in\Lambda} B_\lambda \cong \bigoplus_{\lambda\in\Lambda} (A \otimes B_\lambda).$$

6. *If F and F' are free abelian groups then so is $F \otimes F'$.*

Now for an algebraic definition.

Definition 24.9. Let $T\colon \mathsf{Ab} \to \mathsf{Ab}$ be an additive functor. We say that T is **exact** if given any exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

the corresponding sequence

$$T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$$

is also exact. We say that T is **left exact** if given any exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C,$$

the corresponding sequence

$$0 \to T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$$

is also exact (That is, a left exact functor T preserves an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ if the first map f is injective). Similarly we say that T is **right exact** if if given any exact sequence of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

the corresponding sequence

$$T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \to 0$$

is also exact. (That is, a right exact functor T preserves an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ if the second map g is surjective).

Thus an exact functor is both left exact and right exact.

Proposition 24.10. *The functors $A \otimes \Box$ and $\Box \otimes A$ are both right exact.*

Proof. We will prove $A \otimes \Box$ is right exact. The proof for $\Box \otimes A$ is almost identical. Suppose $B \xrightarrow{f} C \xrightarrow{g} D \to 0$ is exact. We must show that

$$A \otimes B \xrightarrow{\mathrm{id}_A \otimes f} A \otimes C \xrightarrow{\mathrm{id}_A \otimes g} A \otimes D \to 0$$

is also exact. There are three things to check:

1. $\mathrm{im}(\mathrm{id}_A \otimes f) \subseteq \ker(\mathrm{id}_A \otimes g)$.
2. $\ker(\mathrm{id}_A \otimes g) \subseteq \mathrm{im}(\mathrm{id}_A \otimes f)$.
3. $\mathrm{id}_A \otimes g$ is surjective.

The proof of (1) is easy:

$$(\mathrm{id}_A \otimes g) \circ (\mathrm{id}_A \otimes f) = (\mathrm{id}_A \otimes gf) = (\mathrm{id}_A \otimes 0) = 0,$$

where we used the second statement of Proposition 24.5 and the fact that $A \otimes \Box$ is additive.

The proof of (2) is rather trickier. For this, let us denote by $E := \operatorname{im}(\mathrm{id}_A \otimes f)$. Then the map $\mathrm{id}_A \otimes g\colon A \otimes C \to A \otimes D$ induces a map $h\colon (A \otimes C)/E \to A \otimes D$ given by

$$a \otimes c + E \mapsto a \otimes gc$$

(this is well defined by part (1)). By definition, one has $\mathrm{id}_A \otimes g = h \circ p$, where $p\colon A \otimes C \to (A \otimes C)/E$ is the quotient map. We will prove that h is an isomorphism. Then

$$\operatorname{im}(\mathrm{id}_A \otimes f) = E = \ker p = \ker(h \circ p) = \ker(\mathrm{id}_A \otimes g).$$

We will use the universal property of the tensor product to find an inverse to h. Given $d \in D$, since g is surjective there exists $c \in C$ such that $gc = d$. If c' is another such element of C such that $gc' = d$ then $c - c' \in \ker g = \operatorname{im} f$, and thus there exists $b \in B$ such that $fb = c - c'$. Thus for any $a \in A$,

$$a \otimes c - a \otimes c' = a \otimes (c - c') = (\mathrm{id}_A \otimes f)(a \otimes b) \in \operatorname{im}(\mathrm{id}_A \otimes f) = E.$$

This means that there is a well defined map $\varphi\colon A \times D \to (A \otimes C)/E$ given by

$$\varphi(a, d) = a \otimes c + E,$$

where c is any element of C such that $gc = d$. The function φ is obviously bilinear, and hence by the universal property there exists a homomorphism $j\colon A \otimes D \to (A \otimes C)/E$ such that $j(a \otimes d) = a \otimes c + E$ (where as before, c is any element of C such that $gc = d$.) Then by definition, $j \circ h = \mathrm{id}_{(A \otimes C)/E}$ and $h \circ j = \mathrm{id}_{A \otimes D}$. This proves (2).

Finally, if $\sum a_i \otimes d_i$ is an element of $A \otimes D$, since g is surjective we can choose $c_i \in C$ such that $gc_i = d_i$. Then

$$(\mathrm{id}_A \otimes g)\left(\sum a_i \otimes c_i\right) = \sum a_i \otimes d_i,$$

which proves (3). ∎

Now let us define free resolutions.

Definition 24.11. Suppose A is an abelian group. A **free resolution** of A is an exact sequence of the form

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \to 0,$$

where each F_i is a free abelian group. A **short free resolution**[2] of A is a free resolution where each F_i for $i \geq 2$ is the zero group, that is, a sequence of the form

$$0 \to K \to F \to A \to 0$$

where K and F are both free.

Let us show that short resolutions always exist.

[2]This terminology is non-standard.

Proposition 24.12. *Let A be an abelian group. Then there exists a short free resolution of A.*

Proof. Let F be the free abelian group with basis the elements of A. There is a surjective homomorphism $F \to A$ obtained by extending by linearity (Lemma 7.2) the map that sends each basis element to itself. Let K denote the kernel of this map. Then K is a subgroup of a free abelian group, and hence is also a free abelian group[3], and by construction, $0 \to K \to F \to A \to 0$ is exact. ■

We now define the Tor functor.

Definition 24.13. Let A be an abelian group. Let $0 \to K \xrightarrow{f} F \to A \to 0$ be a short free resolution of A. Given any other abelian group B, we define

$$\operatorname{Tor}(A, B) := \ker(f \otimes \mathrm{id}_B).$$

Thus $\operatorname{Tor}(A, B)$ measures the failure of $\square \otimes B$ to be left exact on the sequence $0 \to K \to F \to A \to 0$.

You should all immediately be asking: is this well defined? That is, does the value of $\operatorname{Tor}(A, B)$ depend on the choice of short free resolution? The answer (luckily!) is no. We will prove this next lecture using the Comparison Theorem (Proposition 22.7) from last semester.

[3] This is a non-trivial fact!

LECTURE 25

The Universal Coefficients Theorem

This lecture we'll begin by showing how to make Tor into a functor. This requires a few preliminaries.

Definition 25.1. Suppose $(C_\bullet, \partial)$ is a chain complex and A is an abelian group. Let us denote by $C_\bullet \otimes A$ the chain complex whose nth group is $C_n \otimes A$, and whose boundary operator is $\partial \otimes \mathrm{id}_A$. The fact that this is a chain complex (i.e. that $(\partial \otimes \mathrm{id}_A) \circ (\partial \otimes \mathrm{id}_A) = 0$) follows from additivity of $\square \otimes A$ (part (2) of Proposition 24.8). In this way we can view $\square \otimes A$ also as a functor $\mathsf{Comp} \to \mathsf{Comp}$.

Lemma 25.2. *Let $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ be two chain complexes, and let A be an abelian group.*

1. *Suppose $f \colon C_\bullet \to C'_\bullet$ is a chain map. Let A be an abelian group. Then $f \otimes \mathrm{id}_A \colon C_\bullet \otimes A \to C'_\bullet \otimes A$ is also a chain map.*
2. *Suppose $f \colon C_\bullet \to C'_\bullet$ and $g \colon C_\bullet \to C'_\bullet$ are two chain maps which are chain homotopic. Then the chain maps $f \otimes \mathrm{id}_A$ and $g \otimes \mathrm{id}_A$ are also chain homotopic.*

Proof. For the first statement, note that

$$(f \otimes \mathrm{id}_A) \circ (\partial \otimes \mathrm{id}_A) = (f \circ \partial) \otimes \mathrm{id}_A = (\partial' \circ f) \otimes \mathrm{id}_A = (\partial' \otimes \mathrm{id}_A) \circ (f \otimes \mathrm{id}_A).$$

For the second, if P is a chain homotopy from f to g, that is, $\partial' P + P\partial = f - g$, then

$$\begin{aligned}(\partial' \otimes \mathrm{id}_A) \circ (P \otimes \mathrm{id}_A) + (P \otimes \mathrm{id}_A) \circ (\partial \otimes \mathrm{id}_A) &= \partial' P \otimes \mathrm{id}_A + P\partial \otimes \mathrm{id}_A \\ &= (\partial' P + P\partial) \otimes \mathrm{id}_A \\ &= (f - g) \otimes \mathrm{id}_A \\ &= f \otimes \mathrm{id}_A - g \otimes \mathrm{id}_A,\end{aligned}$$

so that $P \otimes \mathrm{id}_A$ is a chain homotopy between $f \otimes \mathrm{id}_A$ and $g \otimes \mathrm{id}_A$. ■

Now let us rephrase the definition of Tor in rather fancier language. Suppose $0 \to K \xrightarrow{f} F \to A \to 0$ is a short free resolution. Let us define a (rather stupid) chain complex $(C_\bullet, \partial)$ by setting:

$$C_n := \begin{cases} F, & n = 0, \\ K, & n = 1, \\ 0, & n \neq 0, 1. \end{cases} \tag{25.1}$$

and defining the boundary map $\partial\colon C_1 \to C_0$ to be $f\colon K \to F$. Then this chain complex has the property that

$$H_0(C_\bullet) = F/\operatorname{im} f \cong A.$$

Note this chain complex is both free and acyclic in positive degrees. Now let us tensor this chain complex with B, forming a new complex $(C_\bullet \otimes B, \partial \otimes \mathrm{id}_B)$ (this complex is not free if B is not free.) This new complex has the property that

$$H_1(C_\bullet \otimes B) = \ker(\partial \otimes \mathrm{id}_B\colon C_1 \otimes B \to C_0 \otimes B) = \ker(f \otimes \mathrm{id}_B) = \mathrm{Tor}(A, B).$$

Fix an abelian group B. We will show that $\mathrm{Tor}(\square, B)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a functor. We have already defined $\mathrm{Tor}(\square, B)$ on objects (i.e. abelian groups), so it remains to explain what it does to morphisms. Thus suppose $h\colon A \to A'$ is a homomorphism. We wish to define a homomorphism

$$\mathrm{Tor}(h, B)\colon \mathrm{Tor}(A, B) \to \mathrm{Tor}(A', B).$$

Let $0 \to K \to F \to A \to 0$ and $0 \to K' \to F' \to A' \to 0$ be two short free resolutions of A and A' respectively, and denote by $C_\bullet$ and $C'_\bullet$ the corresponding chain complexes, as described in (25.1), so that

$$H_0(C_\bullet) = A, \qquad H_0(C'_\bullet) = A'.$$

We can therefore think of $h\colon A \to A'$ as a map $H_0(C_\bullet) \to H_0(C'_\bullet)$. Now we invoke Theorem 22.7, which tells us there exists a chain map $g\colon C_\bullet \to C'_\bullet$ with $H_0(g) = h$.

Tensoring with B, we get a chain map $g \otimes \mathrm{id}_B\colon : C_\bullet \otimes B \to C'_\bullet \otimes B$ by the first part of Lemma 25.2. Now pass to the first homology group to get a map

$$H_1(g \otimes \mathrm{id}_B)\colon H_1(C_\bullet \otimes B) \to H_1(C'_\bullet \otimes B).$$

Since $H_1(C_\bullet \otimes B) = \mathrm{Tor}(A, B)$ and $H_1(C'_\bullet \otimes B) = \mathrm{Tor}(A', B)$, we can think of this a map

$$\mathrm{Tor}(h, B) := H_1(g \otimes \mathrm{id}_B)\colon \mathrm{Tor}(A, B) \to \mathrm{Tor}(A', B).$$

We still haven't addressed the question as to why Tor is well defined (i.e. that it doesn't depend on the choice of short free resolution.) This is proved in the same way as the argument above: we start with two short free resolutions $0 \to K \to F \to A \to 0$ and $0 \to K' \to F' \to A \to 0$ of the same group A. Denote as before the two chain complexes by $C_\bullet$ and $C'_\bullet$. Now take h to be the identity.

Invoking Theorem 22.7 again, we obtain a chain map $g\colon C_\bullet \to C'_\bullet$ with $H_0(g) = \mathrm{id}_A$. Moreover, by Corollary 22.9, this chain map g is a chain equivalence. By the second part of Lemma 25.2, the tensored map $g \otimes \mathrm{id}_B$ is also a chain equivalence. Thus in particular $H_1(g \otimes \mathrm{id}_A)$ is an isomorphism, which implies that

$$H_1(C_\bullet \otimes B) \cong H_1(C'_\bullet \otimes B).$$

Thus $\mathrm{Tor}(A, B)$ is indeed independent of the choice of short free resolution.

We have therefore proved:

Proposition 25.3. *For each fixed abelian group B, $\mathrm{Tor}(\square, B)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a functor.*

Remark 25.4. In a similar vein, we can also fix the first variable of Tor: if A is any abelian group then $\mathrm{Tor}(A, \square)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a functor, and the value of $\mathrm{Tor}(A, \square)$ on B is isomorphic to the value of $\mathrm{Tor}(\square, B)$ on A (this is not so easy to prove though!) I will leave if to you as an exercise to guess how to define the induced map $\mathrm{Tor}(A, h)\colon \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, B')$ for a given homomorphism $h\colon B \to B'$.

The next result is on Problem Sheet L.

Lemma 25.5. *Suppose B is a torsion-free abelian group. Then $\square \otimes B$ and $B \otimes \square$ are exact functors.*

The following theorem summarises the main properties of Tor.

Theorem 25.6 (Properties of Tor).

1. *If either A or B are torsion-free abelian groups then $\mathrm{Tor}(A, B) = 0$.*
2. *If*[1] *$T(A)$ denotes the torsion subgroup of A then for any abelian group B one has $\mathrm{Tor}(A, B) = \mathrm{Tor}(T(A), B)$.*
3. *If $0 \to B \to B' \to B'' \to 0$ is a short exact sequence, then there is an exact sequence*

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, B') \to \mathrm{Tor}(A, B'') \to A \otimes B \to A \otimes B' \to A \otimes B'' \to 0. \tag{25.2}$$

Similarly if $0 \to A \to A' \to A'' \to 0$ is a short exact sequence, then there is an exact sequence

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A', B) \to \mathrm{Tor}(A'', B) \to A \otimes B \to A' \otimes B \to A'' \otimes B \to 0. \tag{25.3}$$

4. *For any two abelian groups A, B, $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$.*
5. *If B is an abelian group and $\{A_\lambda \mid \lambda \in \Lambda\}$ is a (possibly uncountable) family of abelian groups then there is an isomorphism*

$$\mathrm{Tor}\Big(\bigoplus_{\lambda \in \Lambda} A_\lambda, B\Big) \cong \bigoplus_{\lambda \in \Lambda} \mathrm{Tor}(A_\lambda, B).$$

and similarly

$$\mathrm{Tor}\Big(B, \bigoplus_{\lambda \in \Lambda} A_\lambda\Big) \cong \bigoplus_{\lambda \in \Lambda} \mathrm{Tor}(B, A_\lambda).$$

6. *For any $m \in \mathbb{N}$ and any abelian group B,*

$$\mathrm{Tor}(\mathbb{Z}_m, B) \cong \{b \in B \mid mb = 0\}.$$

[1]This is the reason for the name "Tor".

Proof. If A is free then we can choose a silly short free resolution: $0 \to 0 \to A \to A \to 0$. Then clearly $\mathrm{Tor}(A, B) = 0$ for any abelian group B. If B is torsion-free then by Lemma 25.5, for any short free resolution $0 \to K \to F \to A$ of A, the sequence $0 \to K \otimes B \to F \otimes B \to A \otimes B \to 0$ is exact, so that $\mathrm{Tor}(A, B) = 0$ in this case too. This proves part (1) in the case where A is free and the case where B is torsion-free. We have not yet done the case where A is merely torsion-free; we will do this after proving part (4).

Let us skip part (2) for now and prove the first statement of part (3). Suppose $0 \to B \to B' \to B'' \to 0$ is exact. Let $0 \to K \to F \to A \to 0$ be a short free resolution of A. Let $C_\bullet$ denote the chain complex as defined in (25.1), so that $\mathrm{Tor}(A, B) = H_1(C_\bullet \otimes B)$ and similarly for the other two. Then since K and F are free, by Lemma 25.5 again, the following diagram has exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K \otimes B & \longrightarrow & K \otimes B' & \longrightarrow & K \otimes B'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F \otimes B & \longrightarrow & F \otimes B' & \longrightarrow & F \otimes B'' & \longrightarrow & 0
\end{array}$$

This means that
$$0 \to C_\bullet \otimes B \to C_\bullet \otimes B' \to C_\bullet \otimes B'' \to 0$$
is a short exact sequence of chain complexes. The desired sequence (25.2) is then simply the last six terms of the long exact sequence in homology (Theorem 11.5) associated to this short exact sequence. This proves the first statement of part (3). We will prove the second statement of part (3) after we have proved part (4).

Suppose $0 \to K \to F \to A \to 0$ is a short free resolution of A. Then for any abelian group B, since K and F are free, from part (1) we know that $\mathrm{Tor}(B, K) = \mathrm{Tor}(B, F) = 0$. We then apply (25.2) to the exact sequence $0 \to K \to F \to A \to 0$ to obtain an exact sequence

$$0 \to 0 \to 0 \to \mathrm{Tor}(B, A) \to B \otimes K \to B \otimes F \to B \otimes A \to 0.$$

By definition of $\mathrm{Tor}(A, B)$, the bottom row of the next diagram is exact:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Tor}(B, A) & \longrightarrow & B \otimes K & \longrightarrow & B \otimes F & \longrightarrow & B \otimes A & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Tor}(A, B) & \longrightarrow & K \otimes B & \longrightarrow & F \otimes B & \longrightarrow & A \otimes B & \longrightarrow & 0
\end{array}$$

The vertical arrows are isomorphisms by natural commutativity of the tensor product (part (1) of Proposition 24.8). Since the squares commute and the two rows are exact there is a map $\mathrm{Tor}(B, A) \to \mathrm{Tor}(A, B)$, which then by the Five Lemma (Proposition 11.3) is an isomorphism. This proves part (4).

Now we can go back and finish the proof of part (1): if A is torsion-free then for any abelian group B, using part (4) we have $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A) = 0$, since we have already shown that $\mathrm{Tor}(\square, A)$ vanishes on torsion-free groups.

We can also now prove part (2). Note that for any abelian group A, if $T(A)$ denotes the torsion subgroup then $A/T(A)$ is torsion-free. Thus by part (1) (which we have now completely proved) we have $\mathrm{Tor}(A/T(A), B) = 0$ for any abelian group B. We now apply part (3) to the short exact sequence $0 \to T(A) \to A \to A/T(A) \to 0$. The first three terms of (25.2) become

$$0 \to \mathrm{Tor}(T(A), B) \to \mathrm{Tor}(A, B) \to 0.$$

This proves part (2). The second sequence (25.3) in part (3) also easily follows from (25.2), given part (4).

The proof of part (5) is easy: if $0 \to K_\lambda \to F_\lambda \to A_\lambda \to 0$ is a short free resolution of A_λ then

$$0 \to \bigoplus_{\lambda \in \Lambda} K_\lambda \to \bigoplus_{\lambda \in \Lambda} F_\lambda \to \bigoplus_{\lambda \in \Lambda} A_\lambda \to 0$$

is a short free resolution of $\bigoplus_{\lambda \in \Lambda} A_\lambda$. This proves the first statement of part (5), and the second statement then follows by applying part (4).

Finally, the proof of part (6) is also easy. For this we use the short free resolution $0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}_m \to 0$ of $\mathbb{Z}_m$. The desired result then follows by applying part (3) of Proposition 24.8. This finally finishes the proof of the theorem. ■

REMARK 25.7. Theorem 25.6 allows one to compute the value of Tor in some easy situations. For instance, $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_d$, where d is the greatest common divisor of m and n.

Finally some topology:

DEFINITION 25.8. Let X be a topological spaces and let A be an abelian group. Let $C_\bullet(X)$ denote the singular chain complex, and set $C_\bullet(X; A) := C_\bullet(X) \otimes A$. Thus a typical n-chain in $C_n(X; A)$ has the form $\sum a_i \otimes \sigma_i$, where $a_i \in A$ and $\sigma_i \colon \Delta^n \to X$ is a singular n-simplex in X. We define the **nth singular homology of X with coefficients in A** to be the group

$$H_n(X; A) := H_n(C_\bullet(X; A)).$$

In the same way, if $X' \subseteq X$ is a subspace we define the chain complex[2] $C_\bullet(X, X'; A)$ and the relative homology with coefficients in A.

It follows from part (4) of Proposition 24.8 that taking $A = \mathbb{Z}$ recovers the normal singular homology groups:

$$H_n(X, X'; \mathbb{Z}) = H_n(X, X').$$

REMARK 25.9. Let A be an abelian group. We define a **homology theory** $(\mathcal{H}_\bullet, \delta)$ **with coefficients in** A in exactly the same way as we defined a (normal) homology theory in Definition 21.9, apart from the dimension axiom is replaced by:

- If $\{*\}$ is a one-point space then $\mathcal{H}_n(*) = 0$ for all $n > 0$ and $\mathcal{H}_0(*) = A$.

Singular homology with coefficients in A is then an example of a homology theory with coefficients in A.

[2]Pay attention to the difference between the comma and the semi-colon in $H_n(X, X'; A)$!

We will see shortly why homology with coefficients is useful. For instance, taking $A = \mathbb{Z}_2$ is often particularly pleasant, as this allows one to ignore all $\pm$ signs that crop up in formulae. On Problem Sheet L you will see that taking $\mathbb{Z}_2$ coefficients gives one an easier way to prove Theorem 15.12, that an odd map has an odd degree.

Another useful choice is $A = \mathbb{R}$; this gives homology with **real coefficients**. This theory is particularly useful in differential geometry (when X is a manifold). Since $\mathbb{R}$ is torsion-free, we will see shortly that one always has

$$H_n(X;\mathbb{R}) = H_n(X) \otimes \mathbb{R}.$$

Here is the main theorem of today's lecture:

Theorem 25.10 (The Universal Coefficients Theorem). *Let X be a topological space and let A be an abelian group. Then for every $n \geq 0$ there is an exact sequence*

$$0 \to H_n(X) \otimes A \xrightarrow{\omega} H_n(X;A) \to \operatorname{Tor}(H_{n-1}(X), A) \to 0, \tag{25.4}$$

where ω is the map $\langle c \rangle \otimes a \mapsto \langle c \otimes a \rangle$. Moreover this sequence splits, and hence

$$H_n(X;A) \cong H_n(X) \otimes A \oplus \operatorname{Tor}(H_{n-1}(X), A). \tag{25.5}$$

Remark 25.11. The splitting of the sequence (25.4) is not natural, and hence the isomorphism (25.5) is also not natural (cf. Remark 12.17.)

In fact we will prove a more general statement:

Theorem 25.12 (The Universal Coefficients Theorem II). *Let $(C_\bullet, \partial)$ denote a free chain complex and let A denote an abelian group. Then for every $n \geq 0$, there is an exact sequence*

$$0 \to H_n(C_\bullet) \otimes A \xrightarrow{\omega} H_n(C_\bullet \otimes A) \to \operatorname{Tor}(H_{n-1}(C_\bullet), A) \to 0,$$

where ω is the map $\langle c \rangle \otimes a \mapsto \langle c \otimes a \rangle$. Moreover this sequence splits, and hence

$$H_n(C_\bullet \otimes A) \cong H_n(C_\bullet) \otimes A \oplus \operatorname{Tor}(H_{n-1}(C_\bullet), A).$$

We will need the following lemma, whose proof is again on Problem Sheet L. It tells us that split exact sequences are *always* preserved by additive functors.

Lemma 25.13. *If $T\colon \mathsf{Ab} \to \mathsf{Ab}$ is an additive functor and $0 \to A \to B \to C \to 0$ is any split exact sequence, then $0 \to T(A) \to T(B) \to T(C) \to 0$ is also a split exact sequence.*

Proof of Theorem 25.12. The proof is even more tedious and long-winded than the proof of Theorem 25.6, and we will break the proof up into four steps.

1. Let $B_n \subseteq Z_n \subseteq C_n$ denote the boundaries and cycles respectively. Let us also denote by $i\colon Z_n \hookrightarrow C_n$ the inclusion. Then we have an exact sequence

$$0 \to Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \to 0 \tag{25.6}$$

Since B_{n-1} is a subgroup of the free abelian group C_{n-1}, B_{n-1} is itself free, and hence by part (3) of Problem F.6, the exact sequence (25.6) splits. To keep the notation simple, for the rest of the proof we write id for id_A. Now tensor with A to obtain another split exact sequence

$$0 \to Z_n \otimes A \xrightarrow{i\otimes\mathrm{id}} C_n \otimes A \xrightarrow{\partial\otimes\mathrm{id}} B_{n-1} \otimes A \to 0, \tag{25.7}$$

where we are using Lemma 25.13. Next, we can view $Z_\bullet$ as a subcomplex of $C_\bullet$ where all the boundary operators are zero. Define a chain complex $B_\bullet^+$ to be the chain complex whose nth group $B_n^+ := B_{n-1}$ and again with all the boundary operators zero. We can then think of ∂ as defining a chain map $C_\bullet \to B_\bullet^+$. Thus we can assemble the short exact sequences (25.7) into a short exact sequence of chain complexes:

$$0 \to Z_\bullet \otimes A \xrightarrow{i\otimes\mathrm{id}} C_\bullet \otimes A \xrightarrow{\partial\otimes\mathrm{id}} B_\bullet^+ \otimes A \to 0.$$

2. Now consider the long exact sequence in homology associated to this short exact sequence. Since $Z_\bullet$ and $B_\bullet^+$ have zero boundary operators, we have

$$H_n(Z_\bullet \otimes A) = Z_n \otimes A, \qquad H_n(B_\bullet^+ \otimes A) = B_{n-1} \otimes A,$$

and hence if δ is the connecting homomorphism of the long exact sequence, we can write it as:

$$\dots B_n \otimes A \xrightarrow{\delta} Z_n \otimes A \to H_n(C_\bullet \otimes A) \to B_{n-1} \otimes A \xrightarrow{\delta} Z_{n-1} \otimes A \to \dots$$

Thus for every n there is an exact sequence

$$0 \to Z_n \otimes A / \operatorname{im}\delta \xrightarrow{\omega} H_n(C_\bullet \otimes A) \to \ker\delta \to 0, \tag{25.8}$$

where ω is the map induced by $H_n(i \otimes \mathrm{id})$:

$$\omega\colon z \otimes a + \operatorname{im}\delta \mapsto H_n(i \otimes \mathrm{id})(z \otimes a) = \langle z \otimes a \rangle$$

(recall i is just an inclusion.) Let us identify the connecting homomorphism δ. From Theorem 11.5, we have for a generator $b \otimes a \in B_{n-1} \otimes A$ that

$$\delta(b \otimes a) = (i \otimes \mathrm{id})^{-1}(\partial \otimes \mathrm{id})(\partial \otimes \mathrm{id})^{-1}(b \otimes a),$$

which is just $b \otimes a$ again, only now regarded as an element of $Z_{n-1} \otimes A$. Thus if $j\colon B_\bullet \hookrightarrow Z_\bullet$ is the inclusion, then $\delta = j \otimes \mathrm{id}$. This means we can rewrite (25.8) as

$$0 \to (Z_n \otimes A)/\operatorname{im}(j \otimes \mathrm{id}) \xrightarrow{\omega} H_n(C_\bullet \otimes A) \to \ker(j \otimes \mathrm{id}) \to 0. \tag{25.9}$$

3. The definition of homology gives exact sequences

$$0 \to B_{n-1} \xrightarrow{j} Z_{n-1} \to H_{n-1}(C_\bullet) \to 0.$$

In fact, this is a short free resolution, since (as we have already observed), both B_{n-1} and Z_{n-1} are free. Thus $\mathrm{Tor}(H_{n-1}(C_\bullet), A) = \ker(j \otimes \mathrm{id})$. Next, apply (25.3) to this short exact sequence, and use the fact that $\operatorname{Tor}(Z_{n-1}, A) = 0$ as Z_{n-1} is free to obtain exact sequences

$$0 \to \operatorname{Tor}(H_{n-1}(C_\bullet), A) \to B_{n-1} \otimes A \xrightarrow{j \otimes \mathrm{id}} Z_{n-1} \otimes A \to H_{n-1}(C_\bullet) \otimes A \to 0.$$

This tells us that (replacing $n-1$ with n) that

$$(Z_n \otimes A)/\operatorname{im}(j \otimes \mathrm{id}) = \operatorname{coker}(j \otimes \mathrm{id}) = H_n(C_\bullet) \otimes A.$$

Thus we can rewrite (25.9) as

$$0 \to H_n(C_\bullet) \otimes A \xrightarrow{\omega} H_n(C_\bullet \otimes A) \to \operatorname{Tor}(H_{n-1}(C_\bullet), A) \to 0, \tag{25.10}$$

which is what we were trying to prove.

4. It remains to show that (25.10) splits. For this, let $r \colon C_n \to Z_n$ denote a splitting of (25.6) (such an r exists by part (1) of Problem F.6). Then the composition

$$\ker(\partial \otimes \mathrm{id}) \subseteq C_\bullet \otimes A \xrightarrow{r \otimes \mathrm{id}} Z_n \otimes A \to H_n(C_\bullet) \otimes A$$

maps $\operatorname{im}(\partial \otimes \mathrm{id})$ to zero and hence induces a map $\rho \colon H_n(C_\bullet \otimes A) \to H_n(C_\bullet) \otimes A$ with $\rho \circ \omega$ the identity on $H_n(C_\bullet) \otimes A$. This finally completes the proof. ■

The Algebraic Künneth Theorem

In the next two lectures we will show how to compute the homology of a product $X \times Y$ in terms of the homology of X and the homology of Y. The answer is *not* as simple as one might naively guess (it involves Tor terms!). This lecture we prove a purely algebraic statement about the homology of the tensor product of two free chain complexes; next lecture we will apply this to the singular chain complex.

We first generalise Definition 25.1 and tensor two entire chain complexes together (rather than just a chain complex and a single group).

DEFINITION 26.1. Let $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ be two non-negative chain complexes. The **tensor product** chain complex $(C_\bullet \otimes C'_\bullet, \Delta)$ is the chain complex whose nth group is

$$(C_\bullet \otimes C'_\bullet)_n := \bigoplus_{i+j=n} C_i \otimes C'_j,$$

and the boundary operator Δ is defined by

$$\Delta(c_i \otimes c'_j) := \partial c_i \otimes c'_j + (-1)^i c_i \otimes \partial' c'_j, \qquad c_i \in C_i,\ c_j \in C'_j.$$

Note $C_\bullet \otimes C'_\bullet$ is also non-negative.

The $(-1)^i$ in the second term is included to ensure that $\Delta \circ \Delta = 0$. Indeed, if $c_i \in C_i$ and $c_j \in C'_j$, then since $\partial c_i \in C_{i-1}$ we have

$$\begin{aligned}
\Delta \circ \Delta(c_i \otimes c_j) &= \Delta\left(\partial c_i \otimes c'_j + (-1)^i c_i \otimes \partial' c'_j\right) \\
&= \partial^2 c_i \otimes c'_j + (-1)^{i-1} \partial c_i \otimes \partial' c'_j + (-1)^i \partial c_i \otimes \partial' c'_j + (-1)^{2i} c_i \otimes (\partial')^2 c'_j \\
&= 0 + (-1)^i \left(\partial c_i \otimes \partial' c'_j - \partial c_i \otimes \partial' c'_j\right) \\
&= 0.
\end{aligned}$$

DEFINITION 26.2. Let $C_\bullet, C'_\bullet, D_\bullet, D'_\bullet$ be four non-negative chain complexes. Let $f \colon C_\bullet \to D_\bullet$ and $g \colon C'_\bullet \to D'_\bullet$ be two chain maps. We define a chain map

$$f \otimes g \colon C_\bullet \otimes C'_\bullet \to D_\bullet \otimes D'_\bullet$$

given by

$$(f \otimes g)_n = \sum_{i+j=n} f_i \otimes g_j.$$

The verification that this is indeed a chain map is trivial, and I will leave it to you. The following result is slightly trickier, and is on Problem Sheet M.

Lemma 26.3. *Let $C_\bullet, C'_\bullet, D_\bullet, D'_\bullet$ be four non-negative chain complexes. Assume we are given four chain maps*

$$f, f' \colon C_\bullet \to D_\bullet, \qquad g, g' \colon C'_\bullet \to D'_\bullet,$$

and assume that f and f' are chain homotopic and g and g' are chain homotopic. Then $f \otimes g$ is chain homotopic to $f' \otimes g'$.

An immediate corollary is then:

Corollary 26.4. *Let $C_\bullet, C'_\bullet, D_\bullet, D'_\bullet$ be four non-negative chain complexes. If $C_\bullet$ is chain equivalent to $C'_\bullet$ and $D_\bullet$ is chain equivalent to $D'_\bullet$ then $C_\bullet \otimes D_\bullet$ is chain equivalent to $C'_\bullet \otimes D'_\bullet$.*

We now aim to prove the *Algebraic Künneth Theorem*, which tells us how to compute the homology of a tensor product complex $C_\bullet \otimes D_\bullet$ under the additional assumption that both $C_\bullet$ and $D_\bullet$ are free. The reason for the "algebraic" is that next lecture we will prove a "topological" version of the same result.

Theorem 26.5 (The Algebraic Künneth Theorem). *Let $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ be two non-negative free chain complexes. Then for every $n \geq 0$, there is an exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(D_\bullet) \xrightarrow{\omega} H_n(C_\bullet \otimes D_\bullet) \to \bigoplus_{k+l=n-1} \operatorname{Tor}\big(H_k(C_\bullet), H_l(D_\bullet)\big) \to 0,$$

where ω is the map $\langle c_i \rangle \otimes \langle d_j \rangle \mapsto \langle c_i \otimes d_j \rangle$. Moreover this sequence splits, and hence

$$H_n(C_\bullet \otimes D_\bullet) \cong \left(\bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(D_\bullet) \right) \oplus \left(\bigoplus_{k+l=n-1} \operatorname{Tor}\big(H_k(C_\bullet), H_l(D_\bullet)\big) \right).$$

The proof of Theorem 26.5 requires two preliminary results.

Proposition 26.6. *Let $E_\bullet$ be a non-negative chain complex where every boundary operator is the zero map. Let $(D_\bullet, \partial)$ be any non-negative chain complex. For $i \geq 0$, let $D^i_\bullet$ denote the chain complex whose nth group is $D^i_n := D_{n-i}$ and with boundary operator $\partial \colon D^i_n \to D^i_{n-1}$ equal to $\partial \colon D_{n-i} \to D_{n-i-1}$. Then*

$$H_n(E_\bullet \otimes D_\bullet) \cong \bigoplus_{i \geq 0} H_n(E_i \otimes D^i_\bullet)$$

(here the chain complex $E_i \otimes D^i_\bullet$ is just the chain complex $D^i_\bullet$ tensored with the abelian group E_i, as in Definition 25.1.)

Proof. Since $E_\bullet$ has zero boundary operators, the boundary operator Δ of $E_\bullet \otimes D_\bullet$ is given in degree n by

$$\Delta(e_i \otimes d_{n-i}) = (-1)^i e_i \otimes \partial d_{n-i} = (-1)^i (\mathrm{id}_{E_i} \otimes \partial)(e_i \otimes d_{n-i}),$$

where $e_i \in E_i$, $d_{n-i} \in D_{n-i}$. Thus

$$\begin{aligned} H_n(E_\bullet \otimes D_\bullet) &= \bigoplus_{i \geq 0} \frac{\ker(\mathrm{id}_{E_i} \otimes \partial|_{D_{n-i}})}{\mathrm{im}(\mathrm{id}_{E_i} \otimes \partial|_{D_{n+1-i}})} \\ &= \bigoplus_{i \geq 0} H_n(E_i \otimes D^i_\bullet). \end{aligned}$$

■

LEMMA 26.7. *Let* $0 \to C_\bullet \xrightarrow{f} C'_\bullet \xrightarrow{g} C''_\bullet \to 0$ *be a short exact sequence of non-negative chain complexes. Let* $D_\bullet$ *be a non-negative free chain complex. Then*

$$0 \to C_\bullet \otimes D_\bullet \xrightarrow{f \otimes \mathrm{id}} C'_\bullet \otimes D_\bullet \xrightarrow{g \otimes \mathrm{id}} C''_\bullet \otimes D_\bullet \to 0$$

is another short exact sequence of chain complexes.

The proof of Lemma 26.7 is relegated to Problem Sheet M.

Proof of the Künneth Theorem 26.5. The proof is broken into three steps. The third step uses the Universal Coefficients Theorem 25.12.

1. Let $H_\bullet$ denote the chain complex whose nth group is $H_n = H_n(C_\bullet)$, and with all its boundary operators zero. In this first step, we will set up notation and then define a homomorphism $\varphi\colon H_n(C_\bullet \otimes D_\bullet) \to H_n(H_\bullet \otimes D_\bullet)$. In the second step we will prove that φ is an isomorphism. This allows us to reduce the problem of computing $H_n(C_\bullet \otimes D_\bullet)$ to that of $H_n(H_\bullet \otimes D_\bullet)$, and for this Proposition 26.6 is applicable.

We will use the same notation as in the proof of the Universal Coefficients Theorem 25.12: let $B_n \subseteq Z_n \subseteq C_n$ denote the boundaries and cycles of C_n, and let $B^+_\bullet$ and $Z_\bullet$ denote the chain complexes with groups $B^+_n = B_{n-1}$ and Z_n respectively, both with zero boundary operators. Let $i\colon Z_n \to C_n$ and $j\colon B_n \to Z_n$ denote inclusions, and finally let $p\colon Z_n \to H_n$ denote the projection $c \mapsto \langle c \rangle$. Thus we have two short exact sequences of chain complexes:

$$0 \to Z_\bullet \xrightarrow{i} C_\bullet \xrightarrow{\partial} B^+_\bullet \to 0$$

and

$$0 \to B_\bullet \xrightarrow{j} Z_\bullet \xrightarrow{p} H_\bullet \to 0,$$

where in the first sequence we think of the boundary operator ∂ as a chain map $C_\bullet \to B^+_\bullet$. Since each term of $D_\bullet$ is free abelian, by Lemma 26.7, these sequences remain exact when we tensor with $D_\bullet$:

$$0 \to Z_\bullet \otimes D_\bullet \xrightarrow{i \otimes \mathrm{id}} C_\bullet \otimes D_\bullet \xrightarrow{\partial \otimes \mathrm{id}} B^+_\bullet \otimes D_\bullet \to 0 \tag{26.1}$$

and

$$0 \to B_\bullet \otimes D_\bullet \xrightarrow{j \otimes \mathrm{id}} Z_\bullet \otimes D_\bullet \xrightarrow{p \otimes \mathrm{id}} H_\bullet \otimes D_\bullet \to 0 \tag{26.2}$$

Now let $r\colon C_n \to Z_n$ denote a splitting of $0 \to Z_n \to C_n \to B_{n-1} \to 0$ (such an r exists as B_{n-1} is free abelian, as discussed at the end of the proof of Theorem 25.12 last lecture.) Define $\mu\colon C_\bullet \to H_\bullet$ by $\mu = p \circ r$, that is,

$$\mu c = \langle rc \rangle.$$

We claim that $\mu\colon C_\bullet \to H_\bullet$ is a chain map. Indeed, if $c \in C_{n+1}$ then $\mu \circ \partial c = \langle r\partial c\rangle = \langle \partial c\rangle = 0$, since $\partial c \in B_n \subseteq Z_n$ and $r|_{Z_n} = \mathrm{id}_{Z_n}$. Since $H_\bullet$ has zero boundary operators, this shows that μ is indeed a chain map.

We then define

$$\varphi\colon H_n(C_\bullet \otimes D_\bullet) \to H_n(H_\bullet \otimes D_\bullet), \qquad \varphi := H_n(\mu \otimes \mathrm{id}).$$

2. In the second step, we will show that φ is an isomorphism. For this we take the long exact sequences in homology associated to the short exact sequences of chain complexes (26.1) and (26.2), and then "stick them together" as follows:

$$\begin{array}{ccccccccc} H_{n+1}(B^+_\bullet \otimes D_\bullet) & \xrightarrow{\delta} & H_n(Z_\bullet \otimes D_\bullet) & \longrightarrow & H_n(C_\bullet \otimes D_\bullet) & \longrightarrow & H_n(B^+_\bullet \otimes D_\bullet) & \xrightarrow{\delta} & H_{n-1}(Z_\bullet \otimes D_\bullet) \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \downarrow \\ H_n(B_\bullet \otimes D_\bullet) & \longrightarrow & H_n(Z_\bullet \otimes D_\bullet) & \longrightarrow & H_n(H_\bullet \otimes D_\bullet) & \xrightarrow{\delta'} & H_{n-1}(B_\bullet \otimes D_\bullet) & \longrightarrow & H_{n-1}(Z_\bullet \otimes D_\bullet) \end{array}$$

Here δ and δ' are the two connecting homomorphisms. The vertical maps are all isomorphisms: the second and fifth are just the identity maps, and the first and fourth come from the fact that by definition of $B^+_\bullet$, one has

$$H_{n+1}(B^+_\bullet \otimes D_\bullet) \cong H_n(B_\bullet \otimes D_\bullet).$$

You can probably all now guess how we're going to prove that φ is an isomorphism (the Five Lemma). The only problem is in the "stick them together" above: one cannot just bung two exact sequences together and hope that the resulting diagram commutes! So we need to check this by hand.

In the following we denote by Δ the boundary operators in all the tensored complexes. Let us show the last square commutes. Let $b^+_i \otimes d_{n-i} \in B^+_i \otimes D_{n-i}$ be a non-zero cycle in $B^+_i \otimes D_{n-i} = B_{i-1} \otimes D_{n-i}$. Then since $B^+_\bullet$ has zero boundary operators, we have

$$0 = \Delta(b^+_i \otimes d_{n-i}) = (-1)^i b^+_i \otimes \partial' d_{n-i}.$$

Since $B^+_i \otimes D_{n-i-1}$ is free abelian by Problem L.2, it follows that $\partial' d_{n-i} = 0$, and hence d_{n-i} is a cycle in D_{n-i}. Now choose $c_i \in C_i$ such that $\partial c_i = b^+_i$. By definition of the connecting homomorphism δ, we have

$$\delta\langle b^+_i \otimes d_{n-i}\rangle = \langle \Delta(c_i \otimes d_{n-i})\rangle.$$

However

$$\Delta(c_i \otimes d_{n-i}) = \partial c_i \otimes d_{n-i} + (-1)^i c_i \otimes \partial' d_{n-i} = b^+_i \otimes d_{n-i},$$

since $\partial' d_{n-i} = 0$. This shows that the last square commutes on a set of generators for $H_n(B^+_\bullet \otimes D_\bullet)$. Exactly the same argument (with n replaced by $n+1$) shows the first square commutes, and the proof of the second square is similarly routine.

The third square is a little trickier. If $c_i \otimes d_{n-i}$ is a cycle in $C_i \otimes D_{n-i}$ then

$$\Delta(c_i \otimes d_{n-i}) = \partial c_i \otimes d_{n-i} + (-1)^i c_i \otimes \partial' d_{n-i} = 0.$$

Applying $r \otimes \mathrm{id}$ to this equation we obtain:

$$\partial c_i \otimes d_{n-i} = -(-1)^i r c_i \otimes \partial' d_{n-i}, \tag{26.3}$$

since $r(\partial c_i) = \partial c_i$. The connecting homomorphism δ' is given by $(j \otimes \mathrm{id})^{-1} \circ \Delta \circ (p \otimes \mathrm{id})^{-1}$ (cf. (11.3)). Thus going anti-clockwise round the third square sends $\langle c_i \otimes d_{n-i} \rangle$ to $(-1)^i \langle r c_i \otimes \partial' d_{n-i} \rangle$. Meanwhile going clockwise sends $\langle c_i \otimes d_{n-i} \rangle$ to $\langle \partial c_i \otimes d_{n-i} \rangle$. Thus (26.3) tells us that the third square does *not* commute, but it does however commute *up to a sign*.

Luckily, as I invite you to check, the proof of the Five Lemma (Proposition 11.3) goes through without change if one only assumes that the squares commute up to a sign, and so we can still conclude that φ is an isomorphism.

3. We now complete the proof. Using what we just proved together with Proposition 26.6, we see that

$$H_n(C_\bullet \otimes D_\bullet) \cong H_n(H_\bullet \otimes D_\bullet) \cong \bigoplus_{i \geq 0} H_n(H_i(C_\bullet) \otimes D^i_\bullet), \tag{26.4}$$

since $H_\bullet$ has zero boundary operators. By the Universal Coefficients Theorem 25.12, there are split exact sequences for all n and i:

$$0 \to H_i(C_\bullet) \otimes H_n(D^i_\bullet) \xrightarrow{\omega} H_n(H_i(C_\bullet) \otimes D^i_\bullet) \to \mathrm{Tor}\big(H_i(C_\bullet), H_{n-1}(D^i_\bullet)\big) \to 0,$$

which we can rewrite as

$$0 \to H_i(C_\bullet) \otimes H_{n-i}(D_\bullet) \xrightarrow{\omega} H_n(H_i(C_\bullet) \otimes D^i_\bullet) \to \mathrm{Tor}\big(H_i(C_\bullet), H_{n-i-1}(D_\bullet)\big) \to 0,$$

Now take the direct sum of this last sentence over all $i \geq 0$ and use the fact that the first term vanishes for $i > n$ and the last term term vanishes for $i > n-1$ to obtain a split exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(D_\bullet) \xrightarrow{\omega} \bigoplus_{q \geq 0} H_n(H_q(C_\bullet) \otimes D^q_\bullet) \to \bigoplus_{k+l=n-1} \mathrm{Tor}\big(H_k(C_\bullet), H_l(D_\bullet)\big) \to 0.$$

Finally, combining this with (26.4) completes the proof. ■

LECTURE 27

The Eilenberg-Zilber Theorem

In this lecture we state and prove the *Eilenberg-Zilber Theorem* which allows us to apply the Algebraic Künneth Theorem 26.5 from last lecture to the singular homology of a product of two spaces.

We begin however with a digression about chain equivalences[1] that we will need in the course of the proof of the Eilenberg-Zilber Theorem. Our first result is a partial converse to Corollary 10.26.

PROPOSITION 27.1. *Let $(C_\bullet, \partial)$ be a free chain complex. Then $(C_\bullet, \partial)$ is acyclic if and only if it has a contracting chain homotopy.*

Proof. Sufficiency was proved in Corollary 10.26. For the converse, assume that $H_n(C_\bullet) = 0$ for all n. Since $B_n = Z_n$, we have the following short exact sequence for every n:

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial} Z_{n-1} \to 0$$

Since Z_{n-1} is free abelian, this sequence splits, so let $r\colon Z_{n-1} \to C_n$ be such that $\partial \circ r_n = \mathrm{id}$. Observe that $\mathrm{id}_{C_n} - r_{n-1}\partial$ has image in Z_n. Indeed, if $c \in C_n$ then

$$\partial(c - \partial r_{n-1} c) = \partial c - \partial r_{n-1} \partial c = \partial c - \partial c = 0.$$

Now define

$$Q_n\colon C_n \to C_{n+1}, \qquad Q_n = r_n(\mathrm{id} - r_{n-1}\partial).$$

Then

$$\begin{aligned} \partial Q_n + Q_{n-1}\partial &= \partial r_n(\mathrm{id} - r_{n-1}\partial) + r_{n-1}(\mathrm{id} - r_{n-2}\partial)\partial \\ &= \mathrm{id} - r_{n-1}\partial + r_{n-1}\partial - 0 \\ &= \mathrm{id}. \end{aligned}$$

■

DEFINITION 27.2. Let $f\colon (C_\bullet, \partial) \to (D_\bullet, \partial')$ be a chain map. Given $n \in \mathbb{Z}$, define an abelian group

$$\mathrm{Cone}_n(f) := C_{n-1} \oplus D_n.$$

Define a map $\partial^f\colon \mathrm{Cone}_n(f) \to \mathrm{Cone}_{n-1}(f)$ by

$$\partial^f(c, d) = (-\partial c, fc + \partial' d).$$

[1]This material originally appeared as Question 6 in the Algebraic Topology I Exam in January 2018! I am including this again here for those of you that either (a) did not take the exam, (b) took the exam but got it wrong, or (c) took the exam, got it right, but then forgot everything thirty seconds after the exam and have absolutely no recollection of it anymore...

In matrix form,

$$\partial^f = \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix}.$$

From this it is clear that $\partial^f \circ \partial^f = 0$, since

$$\partial^f \circ \partial^f = \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix} \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix} = \begin{pmatrix} -\partial^2 & 0 \\ -f\partial + \partial' f & (\partial')^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We call this the **mapping cone** of f.

Observe that if both $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ are free chain complexes then so is $\big(\mathrm{Cone}_\bullet(f), \partial^f\big)$.

Here is an easy result about mapping cones.

Proposition 27.3. *Let $f\colon (C_\bullet, \partial) \to (D_\bullet, \partial')$ be a chain map between two free chain complexes. Assume that $\big(\mathrm{Cone}_\bullet(f), \partial^f\big)$ is acyclic. Then f is a chain equivalence.*

Proof. Since $\big(\mathrm{Cone}_\bullet(f), \partial^f\big)$ is free and acyclic, by Proposition 27.1 it has a contracting homotopy Q. Let us suggestively write Q in the form:

$$Q = \begin{pmatrix} p & g \\ r & -p' \end{pmatrix},$$

so that

$$\begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix} \begin{pmatrix} p & g \\ r & -p' \end{pmatrix} + \begin{pmatrix} p & g \\ r & -p' \end{pmatrix} \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

This gives us four equations:

$$\begin{pmatrix} -\partial p - p\partial + gf & -\partial g + g\partial' \\ \text{mess} & fg - \partial' p' - p'\partial' \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{id} \end{pmatrix}$$

Then $g\colon D_\bullet \to C_\bullet$ satisfies $-\partial g + g\partial' = 0$ and hence is a chain map. Moreover we obtain $p\partial + \partial p = gf - \mathrm{id}$ and $p'\partial' + \partial' p' = fg - \mathrm{id}$, which shows that f is a chain equivalence. ∎

Our next result fits the mapping cone into a long exact sequence.

Proposition 27.4. *Let $f\colon (C_\bullet, \partial) \to (D_\bullet, \partial')$ be a chain map. Then there is an exact sequence*

$$\cdots \to H_{n+1}\big(\mathrm{Cone}_\bullet(f)\big) \to H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(D_\bullet) \to H_n\big(\mathrm{Cone}_\bullet(f)\big) \to \dots$$

Proof. Let $C_\bullet^+$ denote the same chain complex as $C_\bullet$ but the groups shifted by one: $C_n^+ := C_{n-1}$ and the boundary operator $\partial^+ := -\partial$. Then there is a short exact sequence of chain complexes

$$0 \to D_\bullet \xrightarrow{i} \mathrm{Cone}_\bullet(f) \xrightarrow{p} C_\bullet^+ \to 0,$$

where $i\colon d \mapsto (0, d)$ and $p\colon (c, d) \mapsto c$. This gives us a long exact sequence in homology:

$$\dots H_{n+1}\big(\mathrm{Cone}_\bullet(f)\big) \to H_{n+1}(C_\bullet^+) \xrightarrow{\delta} H_n(D_\bullet) \to H_n\big(\mathrm{Cone}_\bullet(f)\big) \to \dots$$

It is clear that $H_{n+1}(C_\bullet^+) = H_n(C_\bullet)$, and it remains to see that under this identification the connecting homomorphism δ is just $H_n(f)$. For this we recall that for a cycle $c \in C_n$, one has $\partial^f(c, 0) = (-\partial c, fc) = (0, fc) = i(fc)$, and hence from the definition of δ (cf. (11.3))

$$\delta\colon \langle c\rangle \mapsto \langle i^{-1}\partial^f p^{-1}(c)\rangle = \langle fc\rangle = H_n(f)\langle c\rangle.$$

This completes the proof. ■

We can now use the mapping cone construction to obtain a partial converse to Proposition 10.24.

PROPOSITION 27.5. *Let $(C_\bullet, \partial)$ and $(D_\bullet, \partial')$ be two free chain complexes. Let $f\colon C_\bullet \to D_\bullet$ denote a chain map. Then f is a chain equivalence if and only if $H_n(f)\colon H_n(C_\bullet, \partial) \to H_n(D_\bullet, \partial')$ is an isomorphism for every $n \in \mathbb{Z}$.*

Proof. Necessity was proved in Proposition 10.24. For sufficiency, we use the exact sequence from Proposition 27.4. Since $H_n(f)$ is an isomorphisms, we must have $H_n\big(\mathrm{Cone}_\bullet(f)\big) = 0$ for all n. Thus $\mathrm{Cone}_\bullet(f)$ is acyclic, and hence by Proposition 27.3 we see that f is a chain equivalence as desired. ■

The next result allows us to use the Künneth Theorem to compute the homology of the product of two spaces. This is our first example of a theorem which can be proved directly using a rather lengthy and horrible argument, but has a nice short proof using the Acyclic Models Theorem 23.8.

THEOREM 27.6 (Eilenberg-Zilber). *For topological spaces X and Y, there exists a natural chain equivalence*

$$\Omega\colon C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$$

which is unique up to chain homotopy. Thus for all $n \geq 0$, we have

$$H_n(X \times Y) \cong H_n(C_\bullet(X) \otimes C_\bullet(Y)).$$

We will call any such map Ω an **Eilenberg-Zilber morphism**. It is *not* unique, but it is unique up to chain homotopy. In Lecture 31 we will write down an explicit formula for one such Ω. Before starting the proof of Theorem 27.6, we need one more lemma.

LEMMA 27.7. *Define a map*

$$\theta\colon C_0(X \times Y) \to C_0(X) \otimes C_0(Y)$$

by setting $\theta(x, y) := x \otimes y$. Then θ induces a natural (in X and Y) isomorphism

$$H_0(X \times Y) \to H_0(C_\bullet(X) \otimes C_\bullet(Y)).$$

Proof. Naturality is clear: if $f\colon X\to X'$ and $g\colon Y\to Y'$ then

$$\theta(f,g)_\#(x,y)=\theta(f(x),g(y))=f(x)\otimes g(y)=(f_\#\otimes g_\#)(x\otimes y)=(f_\#\otimes g_\#)\theta(x,y).$$

To show that there is a well-defined map in homology, we need only show that θ maps a boundary to a boundary. Let us denote the boundary operator in $C_\bullet(X)$ by ∂_X, the boundary operator in $C_\bullet(Y)$ by ∂_Y, the boundary operator in $C_\bullet(X\times Y)$ by $\partial_{X\times Y}$, and finally the boundary operator in $C_\bullet(X)\otimes C_\bullet(Y)$ by Δ, so that

$$\Delta(c\otimes c')=\partial_X c\otimes c'+(-1)^n c\otimes\partial_Y c',\qquad \forall\, c\otimes c'\in C_n(X)\otimes C_m(Y).$$

Now suppose $\sigma\colon\Delta^1\to X\times Y$ is a singular 1-simplex in $X\times Y$. Let $\sigma(0)=(x_0,y_0)$ and $\sigma(1)=(x_1,y_1)$. Then $\partial_{X\times Y}\sigma=(x_1,y_1)-(x_0,y_0)$. The path components of $X\times Y$ are of the form $X'\times Y'$ where X' is a path component of X and Y' is a path component of Y, and thus there exist singular 1-simplices $\tau\colon\Delta^1\to X$ and $\rho\colon\Delta^1\to Y$ such that

$$\tau(0)=x_0,\quad \tau(1)=x_1,\qquad \rho(0)=y_0,\quad \rho(1)=y_1.$$

Consider now the singular 1-chain $c:=\tau\otimes y_1+x_0\otimes\rho$ in $C_\bullet(X)\otimes C_\bullet(Y)$. We compute:

$$\begin{aligned}\Delta c=&\,\partial_X\tau\otimes y_1+(-1)^1\tau\otimes\partial_Y y_1\\ &+\partial_X x_0\otimes\rho+(-1)^0 x_0\otimes\partial_Y\rho\\ =&\,x_1\otimes y_1-x_0\otimes y_1+0+0+x_0\otimes y_1-x_0\otimes y_0\\ =&\,x_1\otimes y_1-x_0\otimes y_0.\end{aligned}$$

Thus

$$\theta(\partial_{X\times Y}\sigma)=x_1\otimes y_1-x_0\otimes y_0=\Delta c.$$

Thus θ maps boundaries to boundaries and hence induces a homomorphism $H_0(X\times Y)\to H_0(C_\bullet(X)\otimes C_\bullet(Y))$. It is clear this map is an isomorphism, since the map $x\otimes y\mapsto(x,y)$ induces a homomorphism $H_0(C_\bullet(X)\otimes C_\bullet(Y))\to H_0(X\times Y)$ which inverts it. ■

Now let us prove the theorem.

Proof of the Eilenberg-Zilber Theorem 27.6. We will apply the Acyclic Models Theorem from Lecture 23. Let $\mathsf{Top}\times\mathsf{Top}$ denote the category with objects all ordered pairs (X,Y) of topological spaces and morphisms all ordered pairs of continuous maps. (Note: we do *not* require Y to be a subspace of X; this is *not* the same as the category Top^2.)

Now we define a family of models $\mathcal{M}$ for $\mathsf{Top}\times\mathsf{Top}$. Let

$$\mathcal{M}:=\left\{(\Delta^i,\Delta^j)\mid i,j\geq 0\right\}.$$

We define two functors

$$S_\bullet,T_\bullet\colon\mathsf{Top}\times\mathsf{Top}\to\mathsf{Comp}$$

by

$$S_\bullet(X,Y) := C_\bullet(X \times Y), \qquad T_\bullet(X,Y) := C_\bullet(X) \otimes C_\bullet(Y).$$

We claim that for all $n \geq 0$, both S_n and T_n are free with basis contained in $\mathcal{M}$, and moreover that every model (Δ^i, Δ^j) is both $S_\bullet$-acyclic in positive degrees and $T_\bullet$-acyclic in positive degrees.

Let's start with $S_\bullet$. Let $d_i \colon \Delta^i \to \Delta^i \times \Delta^i$ denote the diagonal map $x \mapsto (x,x)$. Thus $d_i \in C_i(\Delta^i \times \Delta^i) = S_i(\Delta^i, \Delta^i)$. We claim that $\mathcal{X}_i := \{d_i\}$ is an S_i-model basis. Indeed, if (X,Y) is any object in $\mathsf{Top} \times \mathsf{Top}$ and if $\sigma \colon \Delta^i \to X \times Y$ is any singular i-simplex in $S_i(X,Y) = C_i(X \times Y)$ then we can write $\sigma = (\sigma_X \times \sigma_Y) \circ d_i$, where $\sigma_X = p_X \circ \sigma$ and $\sigma_Y = p_Y \circ \sigma$, and $p_X \colon X \times Y \to X$ and $p_Y \colon X \times Y \to Y$ are projections. Conversely, given any pair of singular i-simplices $\tau \colon \Delta^i \to X$ and $\tau' \colon \Delta^i \to Y$, the composition $(\tau \times \tau') \circ d_i$ is a singular i-simplex in $X \times Y$. Thus $\{d_i\}$ is indeed a model basis for S_i.

Next, since $\Delta^i \times \Delta^j$ is convex, it follows that any model $(\Delta^i, \Delta^j) \in \mathcal{M}$ is $S_\bullet$-acyclic in positive degrees. (This is Corollary 13.3.)

Now let us move onto $T_\bullet$. By Problem L.2 (and its solution), for any $(X,Y) \in \mathsf{Top} \times \mathsf{Top}$, $C_i(X) \otimes C_j(Y)$ is free abelian with basis

$$\{\sigma \otimes \tau \mid \sigma \colon \Delta^i \to X,\ \tau \colon \Delta^j \to Y\}.$$

By Example 23.4, the functor C_i is free with model basis $\{\ell_i\}$, with $\ell_i \colon \Delta^i \to \Delta^i$ the identity map (thought of as a singular i-simplex in Δ^i). It follows that T_n is free with basis contained in $\mathcal{M}$: a T_n-model basis is

$$\{\ell_i \otimes \ell_j \mid i + j = n\}.$$

The proof that each model is $T_\bullet$-acyclic in positive degrees is much harder, and this is the reason we first carried out the digression above. By Corollary 13.3 again, combined with Proposition 27.5 above, we see that $C_\bullet(\Delta^i)$ is *chain equivalent* to the chain complex

$$\dots 0 \to 0 \to 0 \to \underset{\text{in degree } 0}{\mathbb{Z}} \to 0 \to 0 \to 0 \to \dots$$

Thus by Corollary 26.4, $T_\bullet(\Delta^i, \Delta^j) = C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j)$ is chain equivalent to the chain complex

$$\dots 0 \to 0 \to 0 \to \underset{\text{in degree } 0}{\mathbb{Z} \otimes \mathbb{Z}} \to 0 \to 0 \to 0 \to \dots$$

Thus in particular $H_n(T_\bullet(\Delta^i, \Delta^j)) = 0$ for all $n > 0$. (One can also prove this used the Algebraic Künneth Theorem 26.5 from the last lecture.)

We have now verified that the hypotheses of the Acyclic Models Theorem and its corollary (Corollary 23.9) are satisfied. To complete the proof we need only define a natural equivalence $\Theta \colon H_0 \circ S_\bullet \to H_0 \circ T_\bullet$. But this was precisely the content of Lemma 27.7—we can take Θ to be the map induced by θ. Thus we can apply Corollary 23.9 to obtain a natural chain equivalence $\Omega \colon S_\bullet \to T_\bullet$, which is unique up to chain homotopy and which satisfies $H_0(\Omega) = \Theta$. This completes the proof. ■

REMARK 27.8. An Eilenberg-Zilber morphism $\Omega\colon C_\bullet(X\times Y)\to C_\bullet(X)\otimes C_\bullet(Y)$ is necesarily given by $\Omega_0(x,y)=x\otimes y$ in degree 0. This is a seemingly stronger statement than Theorem 27.6, where told us merely that the induced map $H_0(\Omega)$ is the same as the map induced by θ from Lemma 27.7.

However, this follows immediately from the naturality of Ω. To see this, observe that if $\{*\}$ is a space with one point then $\Omega\colon C_\bullet(\{*\}\times\{*\})\to C_\bullet(\{*\})\otimes C_\bullet(\{*\})$ is the map $*\mapsto *\otimes *$ in degree zero (there is no choice about that!) Now suppose X and Y are arbitrary topological spaces, and let $(x,y)\in X\times Y$ be an arbitrary point. Consider maps $p\colon\{*\}\to X$ and $q\colon\{*\}\to Y$ such that $p(*)=x$ and $q(*)=y$. Then since Ω is natural,

$$\Omega_0(x,y)=\Omega_0(p_\#,q_\#)(*,*)=(p_\#,q_\#)\Omega_0(*,*)=(p_\#,q_\#)(*\otimes *)=x\otimes y.$$

Putting the pieces together, we obtain our desired result, which is usually known as the *Künneth Formula* (in contrast to the Algebraic Künneth Theorem proved last lecture.)

COROLLARY 27.9 (The Künneth Formula). *Let X and Y be topological spaces. Then for every $n\geq 0$ there is a split exact sequence*

$$0\to\bigoplus_{i+j=n}H_i(X)\otimes H_j(Y)\to H_n(X\times Y)\to\bigoplus_{k+l=n-1}\operatorname{Tor}\big(H_k(X),H_l(Y)\big)\to 0.$$

Thus

$$H_n(X\times Y)\cong\left(\bigoplus_{i+j=n}H_i(X)\otimes H_j(Y)\right)\oplus\left(\bigoplus_{k+l=n-1}\operatorname{Tor}\big(H_k(X),H_l(Y)\big)\right).$$

This gives us (yet another) way to compute the homology of $S^n\times S^m$ (recall we already saw two ways to do this in Problem I.6 and the discussion just after Corollary 20.9). We obtain immediately:

EXAMPLE 27.10. Let $m,n\geq 1$. If $m\neq n$ then

$$H_i(S^m\times S^n)=\begin{cases}\mathbb{Z}, & i=0,m,n,m+n,\\ 0, & \text{otherwise.}\end{cases}$$

If $m=n$ then

$$H_i(S^m\times S^n)=\begin{cases}\mathbb{Z}, & i=0,2m,\\ \mathbb{Z}\oplus\mathbb{Z}, & i=m,\\ 0, & \text{otherwise.}\end{cases}$$

However now we can also compute more complicated spaces, such as $\mathbb{R}P^m\times\mathbb{R}P^n$. A collection of examples for you to try is on Problem Sheet M.

LECTURE 28

Cochain complexes and cohomology

The last four lectures have all centred on how the homology functor interacts with the tensor product functor $\square \otimes A$. We now repeat this theme, only instead of the functor $\square \otimes A$ we use the functor $\mathrm{Hom}(\square, A)$.

To begin with though, consider the functor $\mathrm{Hom}(A, \square) \colon \mathsf{Ab} \to \mathsf{Ab}$ (i.e. with the $\square$ in the second position instead.) This is defined as one might expect: to an abelian group B it associates the abelian group $\mathrm{Hom}(A, B)$, and if $f \colon B \to B'$ is a homomorphism between two abelian groups then

$$\mathrm{Hom}(A, f) \colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$$

is defined by sending $g \colon A \to B$ to $f \circ g \colon A \to B'$. It is routine to see that this is an additive well-defined functor.

However, when we try this with the functor $\mathrm{Hom}(\square, A)$, we come across a problem. In this case if $f \colon B \to B'$ is a homomorphism then there is a natural induced map that sends a homomorphism $g \colon B' \to A$ to $g \circ f \colon B \to A$. Denoting this homomorphism by $\mathrm{Hom}(f, A)$, we have

$$\mathrm{Hom}(f, A) \colon \mathrm{Hom}(B', A) \to \mathrm{Hom}(B, A).$$

But this goes the "wrong" way round! This means that $\mathrm{Hom}(\square, A)$ is *not* a functor (at least as we have defined functors so far). Luckily, this can be easily rectified, by taking a slightly more liberal-minded approach to the definition of a functor.

DEFINITION 28.1. Let C be a category. The **opposite category** is the category C^{op} with

$$\mathrm{obj}(\mathsf{C}^{\mathrm{op}}) := \mathrm{obj}(\mathsf{C}),$$

and morphism sets given by,

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathsf{C}}(B, A), \qquad A, B \in \mathrm{obj}(\mathsf{C}).$$

The composition $\circ^{\mathrm{op}}$ in C^{op} is defined by

$$f \circ^{\mathrm{op}} g := g \circ f,$$

where $\circ$ is the composition in C. This makes sense, i.e. it defines a map

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) \times \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(B, C) \to \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, C).$$

One easily checks that C^{op} is a well-defined category; the identity element in $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, A)$ is just the identity element in $\mathrm{Hom}_{\mathsf{C}}(A, A)$, and associativity of $\circ^{\mathrm{op}}$ follows from associativity of $\circ$.

We can use the notion of the opposite category to extend the definition of a functor.

DEFINITION 28.2. Let C and D be categories. A **contravariant functor** $T\colon \mathsf{C} \to \mathsf{D}$ is simply a functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$. Let us spell out exactly what this means: A contravariant functor associates to each $A \in \mathrm{obj}(\mathsf{C})$ an object $T(A) \in \mathrm{obj}(\mathsf{D})$, and to each morphism $A \xrightarrow{f} B$ in C a morphism $T(B) \xrightarrow{T(f)} T(A)$ in D which satisfies the following two axioms:

1. If $A \xrightarrow{f} B \xrightarrow{g} C$ in C then $T(C) \xrightarrow{T(g)} T(B) \xrightarrow{T(f)} T(A)$ in D and
$$T(g \circ f) = T(f) \circ T(g).$$
2. $T(\mathrm{id}_A) = \mathrm{id}_{T(A)}$ for every $A \in \mathrm{obj}(\mathsf{C})$.

In other words, a contravariant functor is defined in exactly the same way as a normal functor, apart from the fact that it *reverses* the directions of the arrows.

REMARK 28.3. Contravariant functors are not really anything new, since they are just (normal) functors from the opposite category. In particular, up to remembering to reverse directions of arrows, all the abstract results we have proved about functors between categories continue to hold for contravariant functors too. As an easy test of the definitions, I invite you to explore what a natural transformation between two contravariant functors looks like.

EXAMPLE 28.4. With this new terminology, if A is an abelian group then $\mathrm{Hom}(\square, A)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a contravariant functor.

The type of functor we have studied up to now (i.e. with the arrows pointing the right way round) is sometimes called a **covariant functor**. When no confusion is possible, we will normally refer to both covariant and contravariant functors simply as "functors".

REMARK 28.5. If $T\colon \mathsf{C} \to \mathsf{C}$ is a contravariant functor from a given category to itself, then $T \circ T$ is a covariant functor (as reversing the arrows twice means they go in the right direction again). Thus $\mathrm{Hom}(\mathrm{Hom}(\square, A), A)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a functor. Taking $A = \mathbb{R}$ and restricting to real vector spaces, this is the functor that assigns to a vector space its double dual, cf. Theorem 21.5.

Let us see what happens when we apply the contravariant functor $\mathrm{Hom}(\square, A)$ to a chain complex $(C_\bullet, \partial)$. Applying $\mathrm{Hom}(\square, A)$ to $\partial\colon C_n \to C_{n-1}$ we get maps

$$d := \mathrm{Hom}(\partial, A)\colon \mathrm{Hom}(C_{n-1}, A) \to \mathrm{Hom}(C_n, A),$$

defined by

$$d\gamma(c) = \gamma(\partial c), \qquad \gamma \in \mathrm{Hom}(C_{n-1}, A),\ c \in C_n,$$

and hence we get a sequence

$$\cdots \longleftarrow \mathrm{Hom}(C_{n+1}, A) \xleftarrow{d} \mathrm{Hom}(C_n, A) \xleftarrow{d} \mathrm{Hom}(C_{n-1}, A) \longleftarrow \cdots$$

This doesn't immediately give us a chain complex, since the arrows are going the wrong way. But this is easily fixed. Let us define

$$\tilde{C}_{-n} := \mathrm{Hom}(C_n, A).$$

LEMMA 28.6. *$(\tilde{C}_\bullet, d)$ is a chain complex.*

Proof. We need only check that $d \circ d = 0$. But this is obvious: if $\gamma \in \tilde{C}_{-n} = \mathrm{Hom}(C_n, A)$ and $d\gamma = \delta$ then for any $c \in C_{n+2}$ we have

$$d\delta(c) = \delta(\partial c) = \gamma(\partial^2 c) = 0.$$

Thus $d\delta$ is zero[1] in $\mathrm{Hom}(C_{n+2}, A) = \tilde{C}_{-n-2}$. ■

Nevertheless, negative indices are annoying.

So we introduce a notational "trick". Set:

$$C^n := \tilde{C}_{-n}.$$

Then d is a map $C^n \to C^{n+1}$. This gives us the notion of a **cochain complex**.

DEFINITION 28.7. A **cochain complex** is a sequence of abelian groups and homomorphisms

$$\dots \longrightarrow C^{n-1} \xrightarrow{d} C^n \xrightarrow{d} C^{n+1} \longrightarrow \dots$$

for $n \in \mathbb{Z}$ which satisfies

$$d^2 = 0, \qquad \forall\, n \in \mathbb{Z}.$$

We refer to the entire complex as $(C^\bullet, d)$ or sometimes just $C^\bullet$. The maps d are called the **differentials**[2] of the cochain complex.

DEFINITION 28.8. The fact that $d^2 = 0$ means that if we define

$$Z^n = Z^n(C^\bullet) = \ker d \colon C^n \to C^{n+1}$$

and

$$B^n = B^n(C^\bullet) = \operatorname{im} d \colon C^{n-1} \to C^n$$

then

$$B^n \subseteq Z^n.$$

We call elements of Z_n n-**cocycles** and elements of B^n n-**coboundaries**. We define the n**th cohomology group** of the cochain complex $C^\bullet$ to be the quotient group

$$H^n = H^n(C^\bullet) := Z^n(C^\bullet) \Big/ B^n(C^\bullet).$$

If[3] $\gamma \in Z^n$ then we will continue to the use the notation $\langle \gamma \rangle$ to denote the class in H^n.

[1] This argument is implicitly using that $\mathrm{Hom}(\Box, A)$ is an additive functor: $d^2 = \mathrm{Hom}(\partial, A)^2 = \mathrm{Hom}(\partial^2, A) = \mathrm{Hom}(0, A) = 0$; the last equality is only true due to additivity.

[2] The name "differential" is used (instead of "coboundary operator") because of *differential forms* in differential geometry, which gives rise to the *de Rham cohomology* of a manifold.

[3] I will usually use Greek letters for elements of cochain complexes, to help differentiate them from elements of chain complexes.

It is important to realise that (like with contravariant functors) we are not really doing anything new here: if $(C^\bullet, d)$ is a cochain complex then setting $\tilde{C}_{-n} := C^n$ gives us a chain complex, and the homology in degree $-n$ is the same as the cohomology in degree n:

$$H_{-n}(\tilde{C}_\bullet) = H^n(C^\bullet).$$

For this reason, we will *not* introduce the category of cochain complexes, since it is just the same as the category of chain complexes, modulo our trick of replacing n by $-n$. More importantly, this means that all the basic homological algebra results we already proved for chain complexes continue to hold for cochain complexes, without needing to reprove them. For instance, Theorem 11.5 still holds, which we will need at the end of the lecture[4].

As far as this course is concerned, the most important example of a cochain complex is the following:

DEFINITION 28.9. Let X be a topological space and let A be an abelian group. The **singular cochain complex of X with coefficients in A** is the cochain complex $C^\bullet(X; A)$ where $C^n(X; A) := \mathrm{Hom}(C_n(X), A)$. We denote by $Z^n(X)$ and $B^n(X)$ the cocycles and coboundaries of this complex.

The **singular cohomology of X with coefficients in A** is the cohomology of this complex. Taking $A = \mathbb{Z}$, we obtain the **singular cohomology $H^\bullet(X)$** of X.

PROPOSITION 28.10. *Singular cohomology with coefficients in A defines a contravariant functor* $\mathsf{Top} \to \mathsf{Ab}$.

Proof. If $f\colon X \to Y$ is a continuous map, then we can define

$$f^\#\colon C^n(Y; A) \to C^n(X; A), \qquad f^\#(\gamma)(\sigma) := \gamma(f_\#(\sigma)) = \gamma(f \circ \sigma),$$

for $\gamma\colon C_n(Y) \to A$ and $\sigma\colon \Delta^n \to X$ a singular n-simplex in X. If $\gamma \in Z^n(Y)$ then we claim that $f^\#\gamma \in Z^n(X)$. Indeed,

$$d(f^\#\gamma)(\sigma) = (f^\#\gamma)(\partial\sigma) = \gamma(f^\#\partial\sigma) \overset{(*)}{=} \gamma(\partial f_\#\sigma) = (d\gamma)(f^\#\sigma) = 0,$$

where $(*)$ used the fact that we already know that $f_\#$ is a chain map $C_\bullet(X) \to C_\bullet(Y)$ (Proposition 7.20). Similarly if $\gamma \in B^n(Y)$ then $f^\#\gamma \in B^n(X)$. Thus $f^\#$ induces a map $H^n(f)\colon H^n(Y; A) \to H^n(X; A)$. One easily sees that $H^n(\mathrm{id}_X) = \mathrm{id}_{H^n(X)}$ and that if $f\colon X \to Y$ and $g\colon Y \to Z$ then

$$H^n(g \circ f) = H^n(f) \circ H^n(g)\colon H^n(Z; A) \to H^n(X; A).$$

■

Let us now investigate the analogue of Eilenberg-Steenrod axioms for cohomology.

[4] **Warning:** Not everything works in exactly the same way though! For instance, since $\square \otimes B$ is only right exact, tensoring a cochain complex is slightly different to tensoring a chain complex. We will investigate this more next lecture.

Proposition 28.11 (The dimension axion for cohomology). *Let X be a one-point space. Then for any abelian group A,*

$$H^n(X;A) = \begin{cases} A, & n = 0, \\ 0, & n \geq 0. \end{cases}$$

Proof. Recall from the proof of the dimension axiom for homology (see the solution to Problem D.3) that if X is a one-point space then $C_n(X) \cong \mathbb{Z}$ for all $n \geq 0$, and that $\partial\colon C_n(X) \to C_{n-1}(X)$ is an isomorphism when n is even and positive, and zero if n is odd. Dualising, it follows immediately that $H^n(X;A) = 0$ for all $n \geq 1$.

So let us look at $H^0(X;A)$. We have

$$C_1(X) \xrightarrow{\partial=0} C_0(X) \xrightarrow{0} 0.$$

Thus applying $\mathrm{Hom}(\square, A)$ we get

$$0 \to \mathrm{Hom}(C_0(X), A) \xrightarrow{d=0} \mathrm{Hom}(C_1(X), A),$$

and hence

$$H^0(X;A) = \ker(d\colon \mathrm{Hom}(C_0(X), A) \to \mathrm{Hom}(C_1(X), A) = \mathrm{Hom}(C_0(X), A) \cong \mathrm{Hom}(\mathbb{Z}, A).$$

But $\mathrm{Hom}(\mathbb{Z}, A) \cong A$, since a homomorphism $\varphi\colon \mathbb{Z} \to A$ is uniquely determined by $\varphi(1) \in A$. ■

Theorem 28.12. *If $f, g\colon X \to Y$ are homotopic then they induce the same homomorphism $H^n(Y;A) \to H^n(X;A)$ for all $n \geq 0$.*

Proof. Recall from Lecture 8 that the main step in the proof of the homotopy axiom was to show that the following claim: If X is a topological space and we define inclusions $\imath, \jmath\colon X \hookrightarrow X \times I$ by

$$\imath(x) := (x, 0), \qquad \jmath(x) := (x, 1).$$

Then there exists a chain homotopy $P\colon C_n(X) \to C_{n+1}(X \times I)$ such that

$$\partial P + P\partial = \jmath_\# - \imath_\#$$

(this was Proposition 8.5.) Applying the functor $\mathrm{Hom}(\square, A)$ to P, we get a map $Q := \mathrm{Hom}(P, A)$. One checks that Q satisfies.

$$dQ + Qd = \jmath^\# - \imath^\#.$$

This allows us to finish the proof in exactly the same way as we did in Theorem 8.9: Let $F\colon X \times I \to Y$ be a homotopy from f to g. Then

$$f = F \circ \imath, \qquad g = F \circ \jmath.$$

Thus as H^n is a contravariant functor, we have

$$H^n(f) = H^n(F \circ \imath) = H^n(\imath) \circ H^n(F) = H_n(\jmath) \circ H^n(F) = H_n(g).$$

■

Now let us define relative cohomology and prove the analogue of the long exact sequence axiom. First, let us note the following result, whose proof is on Problem Sheet N.

PROPOSITION 28.13. *Let A be an abelian group. The contravariant functor* $\operatorname{Hom}(\square, A)$ *is left exact*[5]*. That is, if $B \xrightarrow{f} B' \xrightarrow{g} B'' \to 0$ is exact then*

$$0 \to \operatorname{Hom}(B'', A) \xrightarrow{\operatorname{Hom}(g,A)} \operatorname{Hom}(B', A) \xrightarrow{\operatorname{Hom}(f,A)} \operatorname{Hom}(B, A)$$

is exact.

In fact, we will need the following result, whose proof is an immediate consequence of Lemma 25.13 (cf. the solution to Problem L.4.)

LEMMA 28.14. *If $T \colon \mathsf{Ab} \to \mathsf{Ab}$ is an additive contravariant functor and $0 \to A \to B \to C \to 0$ is any split exact sequence, then $0 \to T(C) \to T(B) \to T(A) \to 0$ is also a split exact sequence.*

We then use:

PROPOSITION 28.15. *Let $X' \subseteq X$ and let A be an abelian group. Then for every $n \geq 0$ there is a short exact sequence of abelian groups:*

$$0 \to \operatorname{Hom}\big(C_n(X)/C_n(X'), A\big) \to \operatorname{Hom}(C_n(X), A) \to \operatorname{Hom}(C_n(X'), A) \to 0.$$

Thus there is also a short exact sequence of complexes:

$$0 \to \operatorname{Hom}\big(C_\bullet(X)/C_\bullet(X'), A\big) \to C^\bullet(X; A) \to C^\bullet(X'; A) \to 0.$$

Proof. The group $C_n(X)/C_n(X')$ is a free abelian group: a basis is given by all cosets of the form $\sigma + C_n(X')$ where $\sigma \colon \Delta^n \to X$ has $\operatorname{im} \sigma \nsubseteq X'$. Thus by Problem F.6, the sequence $0 \to C_n(X') \to C_n(X) \to C_n(X)/C_n(X') \to 0$ is a split exact sequence. By Lemma 28.14, the sequence is still split exact after applying $\operatorname{Hom}(\square, A)$, and then by Problem E.5 the sequence of complexes is also exact. ■

DEFINITION 28.16. Let $X' \subseteq X$ be a subspace, and let A be an abelian group. We define the **relative cohomology groups with coefficients in** A, written $H^n(X, X'; A)$ of the pair (X, X') to be the cohomology of the complex $C^\bullet(X, X'; A) = \operatorname{Hom}\big(C_\bullet(X)/C_\bullet(X'), A\big)$.

It now follows directly from Theorem 11.5 that there is a long exact sequence in cohomology.

THEOREM 28.17 (The exact sequence axiom for cohomology). *Let $X' \subseteq X$ and let A be an abelian group. Then there is an exact sequence*

$$\cdots \to H^n(X, X'; A) \to H^n(X; A) \to H^n(X'; A) \xrightarrow{\delta} H^{n+1}(X, X'; A) \ldots$$

Moreover the connecting homomorphisms $\delta \colon H^n(X'; A) \to H^{n+1}(X, X'; A)$ are natural.

[5] A contravariant functor $T \colon \mathsf{C} \to \mathsf{D}$ is said to be *left exact* if (when regarded as a normal functor), $T \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{D}$ is left exact. The notion of right exactness for contravariant functors is defined analogously.

The homotopy axiom extends to relative cohomology. The remaining axiom is excision. For this we will just state the result. The proof is an easy adaption of the proof of excision for homology (Theorem 14.8.)

Theorem 28.18 (The excision axiom for cohomology). *Assume that X_1, X_2 are subspaces of X such that $X = X_1^\circ \cup X_2^\circ$. Let A be an abelian group. Then the inclusion $\imath\colon (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2)$ induces an isomorphism in cohomology:*

$$H^n(\imath)\colon H^n(X, X_2; A) \to H^n(X_1, X_1 \cap X_2; A) \qquad \forall\, n \geq 0.$$

I will leave it up to you to formulate the precise axioms for a *cohomology theory with coefficients in A*, and check that we have now verified that singular cohomology with coefficients in A satisfies these axioms (modulo that, as with singular homology, we can't actually verify the weak equivalence axiom yet, as explained in item (2) of the remarks after Definition 21.9)

It is also easy to see that the analogue of the Mayer-Vietoris sequence (Theorem 14.9) continues to hold for cohomology; I will leave it up to you to formulate the precise statement.

LECTURE 29

Cohomological Universal Coefficients Theorems

In this lecture we prove three "Universal Coefficients" theorems for cohomology. The first relates cohomology and homology. This requires us to define a "cohomological" version of Tor, which (rather unhelpfully) is called Ext. The definition is essentially the same as Tor, we just replace $\square \otimes B$ with $\mathrm{Hom}(\square, B)$.

Definition 29.1. Let A be an abelian group, and let $0 \to K \xrightarrow{f} F \to A \to 0$ be a short free resolution of A. Let B be any abelian group. We apply the contravariant functor $\mathrm{Hom}(\square, B)$ and obtain an exact sequence

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(F, B) \xrightarrow{\mathrm{Hom}(f,B)} \mathrm{Hom}(K, B).$$

We define

$$\mathrm{Ext}(A, B) := \mathrm{coker}\,\mathrm{Hom}(f, B) = \mathrm{Hom}(K, B) \Big/ \mathrm{im}\,\mathrm{Hom}(f, B).$$

Thus $\mathrm{Ext}(A, B)$ measures the failure for $\mathrm{Hom}(\square, B)$ to be right exact on $0 \to K \to F \to A \to 0$. In a similar as with Tor, we can view $\mathrm{Ext}(A, B)$ as a (co)homology group. For this, as in (25.1), we first view the short free resolution as defining a chain complex $(C_\bullet, \partial)$ by setting:

$$C_n := \begin{cases} F, & n = 0, \\ K, & n = 1, \\ 0, & n \neq 0, 1. \end{cases}$$

and defining the boundary map $\partial \colon C_1 \to C_0$ to be $f \colon K \to F$. Then this chain complex has the property that

$$H_0(C_\bullet) = F / \mathrm{im}\, f \cong A.$$

Instead of applying $\square \otimes A$ as we did in Lecture 25, this time we apply $\mathrm{Hom}(\square, B)$ to obtain a cochain complex $\mathrm{Hom}(C_\bullet, B)$. This cochain complex has the property that

$$H^1(\mathrm{Hom}(C_\bullet, B)) = \mathrm{Ext}(A, B).$$

Before stating a result giving the main properties of Ext, we first need a definition,

Definition 29.2. An abelian group D is said to be **divisible** if for every $b \in D$ and every $n \in \mathbb{N}$, there exists $a \in D$ such that $na = b$.

Thus $\mathbb{Z}$ is not divisible, but $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$ are.

THEOREM 29.3 (Properties of Ext). *For a fixed abelian group A, $\mathrm{Ext}(A, \square)$ is a covariant functor from* Ab *to itself, and $\mathrm{Ext}(\square, A)$ is a contravariant functor. Morevoer:*

1. *If F is free group then $\mathrm{Ext}(F, B) = 0$ for any abelian group B. If D is a divisible group then $\mathrm{Ext}(B, D) = 0$ for any abelian group B.*
2. *If A is a finitely generated group with torsion subgroup $T(A)$ then $\mathrm{Ext}(A, \mathbb{Z}) \cong T(A)$.*
3. *If $0 \to A \to A' \to A'' \to 0$ is an exact sequence of abelian groups then for any abelian group B there is an exact sequence*

$$0 \to \mathrm{Hom}(A'', B) \to \mathrm{Hom}(A', B) \to \mathrm{Hom}(A, B) \to \mathrm{Ext}(A'', B) \to \mathrm{Ext}(A', B) \to \mathrm{Ext}(A, B) \to 0, \tag{29.1}$$

   Meanwhile if $0 \to B \to B' \to B'' \to 0$ is an exact sequence of abelian groups, then for any abelian group A there is an exact sequence

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B') \to \mathrm{Hom}(A, B'') \to \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B') \to \mathrm{Ext}(A, B'') \to 0, \tag{29.2}$$

4. *If B is an abelian group and $\{A_\lambda \mid \lambda \in \Lambda\}$ is a (possibly uncountable) family of abelian groups then there is an isomorphism*[1]

$$\mathrm{Ext}\Big(\bigoplus_{\lambda \in \Lambda} A_\lambda, B\Big) \cong \prod_{\lambda \in \Lambda} \mathrm{Ext}(A_\lambda, B).$$

   and similarly

$$\mathrm{Ext}\Big(B, \prod_{\lambda \in \Lambda} A_\lambda\Big) \cong \prod_{\lambda \in \Lambda} \mathrm{Ext}(B, A_\lambda).$$

5. *For any $m \in \mathbb{N}$ and any abelian group B,*

$$\mathrm{Ext}(\mathbb{Z}_m, B) \cong B/mB.$$

REMARK 29.4. **Warning:** The analogue of part (4) of Theorem 25.6 is false for Ext: in general $\mathrm{Ext}(A, B)$ is not the same as $\mathrm{Ext}(B, A)$! For example, $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m$ and $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}_m) = 0$.

I won't prove Theorem 29.3. The statements about $\mathrm{Ext}(A, \square)$ are analogous to the corresponding proofs in Theorem 25.6, modulo replacing tensor products with Hom everywhere. For instance, the fact that $\mathrm{Ext}(F, B) = 0$ for any free F starts from the short free resolution $0 \to 0 \to F \to F \to 0$, and the proof of (29.2) come from the long exact sequence in cohomology associated to a short exact sequence of cochain complexes, just as in the proof of (25.2) in Theorem 25.6. The corresponding statements about $\mathrm{Ext}(\square, A)$—that $\mathrm{Ext}(\square, D) \equiv 0$ whenever D is divisible, and the

[1]Recall that the difference between $\bigoplus_{\lambda \in \Lambda} A_\lambda$ and $\prod_{\lambda \in \Lambda} A_\lambda$ is that an element of the former only has finitely many non-zero elements, meanwhile an element of the latter is just an arbitrary element in the product. If Λ is finite the two coincide.

exact sequence (29.1)—are a little different though. For this one replaces short free resolutions with "short divisible resolutions", and I will leave this for you to investigate on Problem Sheet N.

Instead, let us move onto establishing a link between homology and cohomology. Let $C_\bullet$ be a chain complex, and let A be an abelian group. We define a map

$$\zeta\colon H^n(\mathrm{Hom}(C_\bullet, A)) \to \mathrm{Hom}(H_n(C_\bullet), A)$$

by

$$\zeta\langle\gamma\rangle\langle c\rangle := \gamma(c). \tag{29.3}$$

To check this is well-defined, we need to show that if γ, γ' are two cocycles with $\langle\gamma\rangle = \langle\gamma'\rangle$ in $H^n(\mathrm{Hom}(C_\bullet, A))$ and c, c' are two cycles with $\langle c\rangle = \langle c'\rangle$ in $H_n(C_\bullet)$ then

$$\gamma(c) = \gamma'(c').$$

Indeed, we can write $\gamma' = \gamma + d\delta$ and $c' = c + \partial a$. Then

$$\begin{aligned}\gamma'(c') &= (\gamma + d\delta)(c + \partial a)\\ &= \gamma(c) + d\delta(c) + \gamma(\partial a) + d\delta(\partial a)\\ &= \gamma(c) + \delta(\partial c) + d\gamma(a) + \delta(\partial^2 a)\\ &= \gamma(c),\end{aligned}$$

since $\partial c = 0$ as c is a cycle, and $d\gamma = 0$ as γ is a cocycle, and of course $\partial^2 a = 0$.

THEOREM 29.5 (The Dual Universal Coefficients Theorem I). *Let X be a topological space and let A be an abelian group. Then for every $n \geq 0$ there is an exact sequence*

$$0 \to \mathrm{Ext}(H_{n-1}(X), A) \to H^n(X; A) \xrightarrow{\zeta} \mathrm{Hom}(H_n(X), A) \to 0, \tag{29.4}$$

where ζ is the map from (29.3). Moreover this sequence splits, and hence

$$H^n(X; A) \cong \mathrm{Hom}(H_n(X), A) \oplus \mathrm{Ext}(H_{n-1}(X), A). \tag{29.5}$$

REMARK 29.6. The splitting of the sequence (29.4) is natural in A but not in X, and hence the same is true of the isomorphism (29.5).

Just as with the Universal Coefficients Theorem 25.10, this is an immediate corollary of a more general result about an arbitrary free chain complex.

THEOREM 29.7 (The Dual Universal Coefficients Theorem II). *Let $(C_\bullet, \partial)$ denote a free chain complex and let A denote an abelian group. Then for every $n \geq 0$, there is an exact sequence*

$$0 \to \mathrm{Ext}\big(H_{n-1}(C_\bullet), A\big) \to H^n(\mathrm{Hom}(C_\bullet, A)) \xrightarrow{\zeta} \mathrm{Hom}\big(H_n(C_\bullet), A\big) \to 0,$$

where ζ is the map from (29.3). Moreover this sequence splits, and hence

$$H^n(\mathrm{Hom}(C_\bullet, A)) \cong \mathrm{Hom}\big(H_n(C_\bullet), A\big) \oplus \mathrm{Ext}\big(H_{n-1}(C_\bullet), A\big).$$

The splitting is natural in A but not in $C_\bullet$.

We won't prove Theorem 29.7, however the interested reader can simply regurgitate the proof of Theorem 25.12, but replace all instances of tensor products with Hom instead, reversing arrows where appropriate, and use the properties of Ext (Theorem 29.3) instead of properties of Tor (Theorem 25.6).

Definition 29.8. A topological space X is said to be of **finite type** if each homology group $H_n(X)$ is finitely generated.

The following corollary is perhaps the most useful application of Theorem 29.5.

Corollary 29.9. *Let X be a topological space of finite type. Let $T_n(X) = T(H_n(X))$ denote the torsion subgroup of $H_n(X)$. Then for all $n \geq 0$*

$$H^n(X) \cong H_n(X)/T_n(X) \oplus T_{n-1}(X).$$

Proof. If A is any finitely generated group then $\mathrm{Hom}(A, \mathbb{Z}) \cong A/T(A)$ by Problem N.2. Thus for X of finite type $\mathrm{Hom}(H_n(X), \mathbb{Z}) \cong H_n(X)/T_n(X)$. Next, by part (2) of Theorem 29.3,

$$\mathrm{Ext}(H_{n-1}(X), \mathbb{Z}) \cong T_{n-1}(X).$$

The claim follows from (29.5). ■

We could now extend Theorem 29.5 to a Künneth-type result. Indeed, if $C_\bullet$ and $C'_\bullet$ are chain complexes, then there is a natural way to make $\mathrm{Hom}(C_\bullet, C'_\bullet)$ into a cochain complex. However, for our purposes this is not so useful, since there is no analogue of the Eilenberg-Zilber Theorem 27.6 in this setting. Instead, we will first prove a purely cohomological version of the Universal Coefficient Theorem for a topological space of finite type, and then use that to deduce a Künneth-type result.

We need two auxiliary results.

Proposition 29.10. *Let X be a topological space of finite type. There exists a non-negative free chain complex $E_\bullet$ such that every group E_n is finitely generated, and such that $E_\bullet$ is chain equivalent to the singular chain complex $C_\bullet(X)$.*

Remark 29.11. The assumption that X has finite type tells us that $H_n(X)$ is finitely generated for each n. But $C_n(X)$ is *never* finitely generated (unless X is a finite set), and thus Proposition 29.10 is non trivial.

Proof of Proposition 29.10. Let $p\colon Z_n(X) \to H_n(X)$ denote the map $c \mapsto \langle c \rangle$. Since $H_n(X)$ is finitely generated, there exists a finitely generated subgroup $F_n \subseteq Z_n(X)$ such that $p|_{F_n}\colon F_n \to H_n(X)$ is surjective. Note F_n is free. Let $F'_n := \ker p|_{F_n}$. Now set

$$E_n := F_n \oplus F'_{n-1}.$$

Then E_n is finitely generated and free. Define $\varepsilon\colon E_n \to E_{n-1}$ by $\varepsilon(c, c') := (c', 0)$. Then clearly $\varepsilon \circ \varepsilon = 0$, and thus $(E_\bullet, \varepsilon)$ is a free finitely generated chain complex. Moreover

$$H_n(E_\bullet) = \frac{\ker(\varepsilon\colon E_n \to E_{n-1})}{\mathrm{im}(\varepsilon\colon E_{n+1} \to E_n)} = F_n/F'_n = H_n(X). \tag{29.6}$$

Now let us build a chain map $f\colon (E_\bullet, \varepsilon) \to (C_\bullet(X), \partial)$. Since F'_n is free abelian, by Lemma 22.3 there exists a homomorphism $g\colon F'_n \to C_{n+1}(X)$ such that $\partial g(c') = c'$ for all $c' \in F'_n$. Define

$$f\colon E_n \to C_n(X), \qquad f(c, c') := c + g(c').$$

Then

$$\partial f(c, c') = \partial(c + g(c')) = \partial c + c' = c',$$

since $c \in F_n \subseteq Z_n(X)$, so $\partial c = 0$. Moreover

$$f\varepsilon(c, c') = f(c', 0) = c'.$$

Thus $f \circ \varepsilon = \partial \circ f$, and hence f is a chain map. Moreover using (29.6), the induced map in homology is given by

$$H_n(f)\colon H_n(E_\bullet) \to H_n(X), \qquad \langle c, 0\rangle \mapsto \langle c\rangle,$$

which is obviously an isomorphism. Since both $E_\bullet$ and $C_\bullet(X)$ are free, we can invoke Proposition 27.5 to conclude that f is a chain equivalence. This completes the proof. ■

Proposition 29.12. *If $E_\bullet$ is a free chain complex such that each group E_n is finitely generated, then for any abelian group A, there is an isomorphism of cochain complexes*

$$\operatorname{Hom}(E_\bullet, \mathbb{Z}) \otimes A \cong \operatorname{Hom}(E_\bullet, A).$$

Proof. Define

$$h\colon \operatorname{Hom}(E_n, \mathbb{Z}) \otimes A \to \operatorname{Hom}(E_n, A)$$

by

$$h(\gamma \otimes a)(c) = \gamma(c) \cdot a, \qquad \gamma \in \operatorname{Hom}(E_n, \mathbb{Z}),\ a \in A,\ c \in E_n,$$

note this makes sense as $\gamma(c) \in \mathbb{Z}$, and thus we can multiple a by $\gamma(c)$ in A. Then h is clearly a chain map. To prove that h is an isomorphism, we argue by induction on the rank of E_n. If the rank is 1, then $E_n \cong \mathbb{Z}$, and this follows from $\mathbb{Z} \otimes A \cong A$ and $\operatorname{Hom}(\mathbb{Z}, A) = A$. For the inductive step, we note that both $\operatorname{Hom}(\square, A)$ and $\square \otimes A$ commute with finite direct sums:

$$\operatorname{Hom}(B \oplus B', A) = \operatorname{Hom}(B, A) \oplus \operatorname{Hom}(B', A),$$

and

$$(B \oplus B') \otimes A \cong (B \otimes A) \oplus (B' \oplus A)$$

see part (5) of Proposition 24.8. ■

Remark 29.13. Note that if $C_\bullet$ and $C'_\bullet$ are chain equivalent complexes, then for any abelian group A the chain complexes $C_\bullet \otimes A$ and $C'_\bullet \otimes A$ are chain equivalent, and similarly the cochain complexes $\operatorname{Hom}(C_\bullet, A)$ and $\operatorname{Hom}(C'_\bullet, A)$ are also chain equivalent. The tensor products statement is a special case of Lemma 25.2, and the Hom statement is clear: if $f\colon C_\bullet \to C'_\bullet$ is a chain equivalence then so $\operatorname{Hom}(f, A)\colon \operatorname{Hom}(C'_\bullet, A) \to \operatorname{Hom}(C_\bullet, A)$.

Here is the promised "purely cohomological" version of the Universal Coefficients Theorem.

THEOREM 29.14 (The Universal Coefficients Theorem for Cohomology). *If X is a topological space of finite type, then for any abelian group A and any $n \geq 0$, there is an exact sequence*

$$0 \to H^n(X) \otimes A \xrightarrow{h} H^n(X; A) \to \mathrm{Tor}(H^{n+1}(X), A) \to 0,$$

where h is the map $\langle \gamma \rangle \otimes a \mapsto \langle \gamma \cdot a \rangle$, and (as in Proposition 29.12), $\gamma \cdot a \in \mathrm{Hom}(C_n(X), A)$ is defined by

$$(\gamma \cdot a)(\sigma) := \gamma(\sigma) \cdot a \in A, \qquad \sigma \colon \Delta^n \to X.$$

Moreover this sequence splits, and hence

$$H^n(X; A) \cong H^n(X) \otimes A \oplus \mathrm{Tor}(H^{n+1}(X), A).$$

Proof. By Proposition 29.10, there is a free chain complex $E_\bullet$ such that each E_n is finitely generated, and such that $H_n(X) \cong H_n(E_\bullet)$ for all n. Now set $E^\bullet := \mathrm{Hom}(E_\bullet, \mathbb{Z})$. Then $E^\bullet$ is a free cochain complex. But since cochain complexes are really just chain complexes in disguise (only with the indices flipped), we can apply the Universal Coefficient Theorem 25.12 directly[2] to $E^\bullet$ to obtain split short exact sequences

$$0 \to H^n(E^\bullet) \otimes A \to H^n(E^\bullet \otimes A) \to \mathrm{Tor}\big(H^{n+1}(E^\bullet), A\big) \to 0.$$

But using Remark 29.13, $H^n(E^\bullet) = H^n(\mathrm{Hom}(E_\bullet, \mathbb{Z})) = H^n(\mathrm{Hom}(C_\bullet(X), \mathbb{Z})) = H^n(X)$. Moreover by Proposition 29.12, one has:

$$E^\bullet \otimes A = \mathrm{Hom}(E_\bullet, \mathbb{Z}) \otimes A \underset{\text{isomorphic}}{\cong} \mathrm{Hom}(E_\bullet, A) \underset{\text{chain equivalent}}{\simeq} \mathrm{Hom}(C_\bullet(X), A).$$

Thus by Proposition 10.24, we have $H^n(E^\bullet \otimes A) \cong H^n(X; A)$. The result follows. ∎

We conclude this lecture by using Theorem 29.14 to prove a Künneth-type result for cohomology. To state the result, let us introduce the following notation: given topological spaces X and Y, and $\gamma \in C^i(X)$, $\delta \in C^j(Y)$, define $\gamma \otimes \delta \in \mathrm{Hom}(C_\bullet(X) \otimes C_\bullet(Y), \mathbb{Z})$ by

$$(\gamma \otimes \delta)(\sigma \otimes \tau) := \begin{cases} \gamma(\sigma)\delta(\tau), & \text{if } \sigma \in C_i(X), \ \tau \in C_j(Y), \\ 0, & \text{otherwise.} \end{cases}$$

Now let $\Omega \colon C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$ be an Eilenberg-Zilber morphism (cf. Theorem 27.6.) Then there is an induced map

$$\mathrm{Hom}(\Omega, \mathbb{Z}) \colon \mathrm{Hom}\big(C_\bullet(X) \otimes C_\bullet(Y), \mathbb{Z}\big) \to \mathrm{Hom}(C_\bullet(X \times Y), \mathbb{Z}) = C^\bullet(X \times Y),$$

[2]For the reader unhappy here: set $\tilde{E}_n := E^{-n} = \mathrm{Hom}(E_{-n}, \mathbb{Z})$. Then $\tilde{E}_\bullet$ is a non-positive free chain complex and $H_{-n}(\tilde{E}_\bullet) = H^n(E^\bullet)$. To apply Theorem 25.12, we work with $\tilde{E}_\bullet$, and then at the last stage switch back to $E^\bullet$ to get a cohomological statement.

and hence a map

$$H^n(\mathrm{Hom}(\Omega, \mathbb{Z}))\colon H^n\Big(\mathrm{Hom}\big(C_\bullet(X) \otimes C_\bullet(Y), \mathbb{Z}\big)\Big) \to H^n(X \times Y).$$

Thus there is a well-defined map

$$\eta\colon H^i(X) \otimes H^j(Y) \mapsto H^{i+j}(X \times Y)$$

given by

$$\eta\colon \langle\gamma\rangle \otimes \langle\delta\rangle \mapsto H^{i+j}(\mathrm{Hom}(\Omega, \mathbb{Z}))\langle\gamma \otimes \delta\rangle.$$

REMARK 29.15. If you are worried about why

$$\langle\gamma\rangle \otimes \langle\delta\rangle \mapsto \langle\gamma \otimes \delta\rangle$$

makes sense at the level of cohomology classes, see Lemma 31.11 in Lecture 31.

THEOREM 29.16 (The Künneth Formula for Cohomology). *Let X and Y be topological spaces of finite type. Then for every $n \geq 0$, there is a split short exact sequence*

$$0 \to \bigoplus_{i+j=n} H^i(X) \otimes H^j(Y) \xrightarrow{\eta} H^n(X \times Y) \to \bigoplus_{k+l=n+1} \mathrm{Tor}\big(H^k(X), H^l(Y)\big) \to 0.$$

Proof. By Proposition 29.10, there exist finitely generated free chain complexes $E_\bullet$ and $E'_\bullet$ which are chain equivalent to $C_\bullet(X)$ and $C_\bullet(Y)$. Then $E_\bullet \otimes E'_\bullet$ is chain equivalent to $C_\bullet(X) \otimes C_\bullet(Y)$ by Corollary 26.4, and we have a commutative diagram where the vertical maps are isomorphisms:

$$\begin{array}{ccc} H^i(X) \otimes H^j(Y) & \xrightarrow{\quad\eta\quad} & H^{i+j}(X \times Y) \\ \downarrow & & \downarrow \\ H^i(E_\bullet) \otimes H^j(E'_\bullet) & \longrightarrow & H^{i+j}\big(\mathrm{Hom}(E_\bullet \otimes E'_\bullet, \mathbb{Z})\big). \end{array}$$

Thus it suffices to prove the statement with the bottom row, rather than top row. But since $E_\bullet$ and $E'_\bullet$ are finitely generated in each degree, it follows that the chain complexes $\mathrm{Hom}(E_\bullet, \mathbb{Z})$ and $\mathrm{Hom}(E'_\bullet, \mathbb{Z})$ are free. Thus we can apply Künneth Theorem 26.5 to conclude. ■

LECTURE 30

The cup product and the cohomology ring

Up to now, we have always worked with coefficients A where A is some abelian group. The aim of this lecture is to show that if we take A to be a *commutative ring* R, then the singular cohomology with coefficients in R is also a ring. To begin with, let us check we all understand the definition of a ring.

DEFINITION 30.1. A **ring** R is an abelian group (where we use $+$ to denote addition in the group structure), together with an additional operation "$\cdot$", called *multiplication*, which is associative:

$$(r \cdot s) \cdot t = r \cdot (s \cdot t), \qquad \forall\, r, s, t \in R,$$

and which has multiplicative identity, that is, an element $1 \in R$ such that

$$r \cdot 1 = r = 1 \cdot r, \qquad \forall\, r \in R.$$

Moreover, multiplication should be *distributive* over addition:

$$r \cdot (a + b) = (r \cdot a) + (r \cdot b), \qquad (a + b) \cdot r = a \cdot r + b \cdot r, \qquad \forall\, a, b, r \in R.$$

Most of the time we will just write rs instead of $r \cdot s$. If 0 denotes the zero element for the group structure, then we do *not* take $0 \neq 1$ as part of the definition. Nevertheless it takes merely a moment of thought to note that if $0 = 1$ then $R = \{0\}$, in which case we call R the **zero ring**.

DEFINITION 30.2. A **ring homomorphism** $f \colon R \to S$ between two rings is a homorphism between R and S as abelian groups which also preserves the multiplicative structure:

$$f(rs) = f(r)f(s), \qquad f(1_R) = 1_S.$$

The category Rings of rings has objects the rings, and morphisms the ring homomorphisms, and composition the usual composition of homomorphisms.

There is a forgetful functor $\mathsf{Rings} \to \mathsf{Ab}$. A ring R is **commutative** if $rs = sr$ for all $r, s \in R$. Commutative rings form a full subcategory $\mathsf{ComRings}$ of Rings.

DEFINITION 30.3. A subset S of a ring R is a **subring** if it can be regarded as a ring with the addition and the multiplication restricted from R to S. Thus S is a subring if $1_R \in S$ and for any $r, s \in S$, all of $rs, r + s, r - s$ belong to S.

Subrings are the ring-theoretic analogue of subgroups. The analogue of a normal subgroup is an *ideal*[1]:

DEFINITION 30.4. A nonempty subset I of a ring R is then said to be a **left ideal** in R if, for any $s, t \in I$ and $r \in R$, all of $s+t, rs$ are in I. Equivalently, an additive subgroup I is a left ideal if $R \cdot I \subseteq I$. Similarly an additive subgroup I is a **right ideal** if $I \cdot R \subseteq I$. If I is both a left ideal and a right ideal, then I is said to be a **two-sided ideal**.

Thus every ideal in a commutative ring is a two-sided ideal. An ideal $I \subset R$ is said to be *proper* if $I \neq R$.

DEFINITION 30.5. If R is a ring and I is a two-sided ideal, then there is a well-defined equivalence relation on R given by $r \sim s$ if $r - s \in I$. The **quotient ring** R/I has as an elements the cosets $r+I$, with addition given by $(r+I)+(s+I) := (r+s)+I$ and multiplication by $(r+I)(s+I) = rs+I$. The natural quotient map $R \to R/I$ is then a surjective ring homomorphism.

DEFINITION 30.6. A **graded ring** R is a ring R with additive subgroups R^n, for $n \geq 0$, such that $R = \bigoplus_n R^n$, and such that $R^n \cdot R^m \subseteq R^{n+m}$. A **graded ring homomorphism** $f\colon R \to S$ between graded rings is a ring homomorphism with the property that $f(R^n) \subseteq S^n$ for all $n \geq 0$. This forms the category $\mathsf{GradedRings}$ of graded rings.

There is a forgetful functor $\mathsf{GradedRings} \to \mathsf{Rings}$. Conversely, if R is any ring, then R can be made into a graded ring by setting $R^0 = R$ and $R^n = 0$ for $n \geq 1$.

DEFINITION 30.7. A graded ring R is said to be **graded commutative**

$$r \cdot s = (-1)^{nm} s \cdot r, \qquad \forall\, r \in R^n,\ s \in R^m.$$

This gives rise to a full subcategory $\mathsf{ComGradedRings}$ of $\mathsf{GradedRings}$.

An element x in a graded ring is said to be **homogeneous** if $x \in R^n$ for some n. In this case one says that x has **degree** n. The zero element 0 therefore has degree n for any (and every) n, and the identity element 1 is always[2] homogeneous of degree zero. We use the convention that the degree of an element is *not defined* if the element is not homogeneous. An ideal I is said to be **homogeneous** if it is generated by homogeneous elements.

EXAMPLE 30.8. Let R be a commutative ring, and let $Q_1, \ldots, Q_k$ denote formal variables. The **polynomial ring** $R[Q_1, \ldots, Q_k]$ has elements all formal sums

$$x = \sum r_{i_1 i_2 \cdots i_k} \cdot Q_1^{i_1} Q_2^{i_2} \cdots Q_k^{i_k}, \qquad r_{i_1 i_2 \cdots i_k} \in R, \qquad i_j \geq 0,\ 1 \leq j \leq k.$$

Then $R[Q_1, \ldots, Q_n]$ is a graded ring, where

$$R^n := \left\{ \sum r_{i_1 i_2 \cdots i_k} \cdot Q_1^{i_1} Q_2^{i_2} \cdots Q_k^{i_k} \mid \sum_{j=1}^{k} i_j = n \right\}$$

[1]This analogy doesn't quite work: for instance, a proper ideal is not even a subring.
[2]Exercise: Why?

Thus R^n is generated by all monomials of total degree[3] n.

The following lemma is a trivial piece of algebra, whose proof I leave to you. We will need it in the proof of Theorem 30.20 at the end of the lecture.

Lemma 30.9. *If I is a homogeneous two-sided ideal in a graded ring, then the quotient ring R/I is again a graded ring. Indeed,*

$$R/I = \bigoplus_{n \geq 0} (R^n + I)/I.$$

Definition 30.10. Let X be a topological space and let R be a ring. The **total cohomology** of X with coefficients in R is given by

$$H^\star(X; R) := \bigoplus_{n \geq 0} H^n(X; R).$$

This is well-defined, since a ring is in particular an abelian group. I use the notation $H^\star$ instead of $H^\bullet$ to indicate we are taking the direct sum of all the cohomology groups (as opposed to considering the cohomology groups as a complex). Similarly we denote by $C^\star(X; R) := \bigoplus_n C^n(X; R)$ (as a direct sum, not a chain complex).

Our aim is to make $H^\star(X; R)$ into a graded ring when R is commutative. We shall do this by first making $C^\star(X; R)$ into a graded ring, and then showing that the ring structure descends to cohomology. For this, let us first recall the *face maps* from Lecture 7:

$$\varepsilon_i \colon \Delta^{n-1} \to \Delta^n, \qquad i = 0, 1, \dots, n$$

that maps the standard $(n-1)$-simplex Δ^{n-1} homeomorphically onto the ith face of Δ^n. Explicitly,

$$\varepsilon_0(s_0, s_1, \dots, s_{n-1}) = (0, s_0, s_1, \dots, s_{n-1}),$$

for $i = 0$, and for $1 \leq i \leq n-1$,

$$\varepsilon_i(s_0, s_1, \dots, s_{n-1}) = (s_0, s_1, \dots, s_{i-1}, 0, s_i, \dots, s_{n-1}),$$

and finally

$$\varepsilon_n(s_0, s_1, \dots, s_{n-1}) = (s_0, s_1, \dots, s_{n-1}, 0).$$

Where necessary we will write $\varepsilon_i^n \colon \Delta^{n-1} \to \Delta^n$.

Definition 30.11. If $0 \leq i \leq n$, we define the **ith front face** to be the map $F_i^n \colon \Delta^i \to \Delta^n$ by

$$F_i^n(s_0, s_1, \dots, s_i) = (s_0, s_1, \dots, s_i, 0, \dots, 0),$$

and we define the **ith back face** $B_n^i \colon \Delta^i \to \Delta^n$ by

$$B_i^n(s_0, s_1, \dots, s_i) = (0, \dots, 0, s_0, s_1, \dots, s_i).$$

[3]Our convention implies that only monomials get a degree; an arbitrary polynomial does *not* have a degree in the sense of graded rings.

As with the face maps, where possible we will just write F_i and B_i. Note that $F^n_n = B^n_n = \mathrm{id}_{\Delta^n}$, meanwhile F^n_0 has image $e_0 = (1, 0, \dots, 0)$ and B^n_0 has image $e_n = (0, \dots, 0, 1)$. The next lemma tells us how these maps compose with each other.

Lemma 30.12.

1. *One has* $\varepsilon^{n+1}_0 = B^{n+1}_n$ *and* $\varepsilon^{n+1}_{n+1} = F^{n+1}_n$.
2. *One has:*
$$B^n_{m+p} \circ B^{m+p}_p = B^n_p, \qquad F^n_{m+p} \circ F^{m+p}_p = F^n_p,$$
and
$$B^{n+m+p}_{m+p} \circ F^{m+p}_m = F^{n+m+p}_{n+m} \circ B^{n+m}_m.$$
3. *One has*
$$\varepsilon^{n+1}_i \circ F^n_m = \begin{cases} F^{n+1}_{m+1} \circ \varepsilon^{m+1}_i, & \text{if } i \le m, \\ F^{n+1}_m, & \text{if } i \ge m+1. \end{cases}$$
and
$$\varepsilon^{n+1}_i \circ B^n_m = \begin{cases} B^{n+1}_m, & \text{if } i \le n-m, \\ B^{n+1}_{m+1} \circ \varepsilon^{m+1}_{i+m-n}, & \text{if } i \ge n-m+1. \end{cases}$$

Proof. Evaluate both sides on a typical element of the desired simplex. ■

We now come to the key definition of this lecture.

Definition 30.13. Let X be a topological space and let R be a ring. If $\alpha \in C^n(X; R)$ and $\beta \in C^m(X; R)$, we define $\alpha \smile \beta \in C^{n+m}(X; R)$, the **cup product** of α and β by requiring that

$$(\alpha \smile \beta)(\sigma) = \alpha(\sigma \circ F_n) \cdot \beta(\sigma \circ B_m), \qquad \forall\, \sigma \colon \Delta^{n+m} \to X \tag{30.1}$$

(since $C_{n+m}(X)$ is free abelian with basis the singular $(n+m)$-simplices $\sigma \colon \Delta^{n+m} \to X$, equation (30.1) does indeed uniquely determine an element of $C^{n+m}(X; R)$. The expression makes sense: if $\sigma \colon \Delta^{n+m} \to X$ then $\sigma \circ F_n \colon \Delta^n \to X$ and $\sigma \circ B_m \colon \Delta^m \to X$, so that $\alpha(\sigma \circ F_n)$ and $\beta(\sigma \circ B_m)$ are well-defined elements of R which can then be multiplied to give another element of R.

Remark 30.14. Cup products are usually only defined for R a *commutative* ring. The reason for this will become apparent next lecture, when we show that if the coefficient ring is commutative, then the (graded) cohomology ring is graded commutative, see Remark 30.21 below. Nevertheless, everything we do in this lecture works fine for any (possibly non-commutative) ring, so for now we won't impose commutativity on R.

The cup products extends by linearity to define a function

$$C^\star(X; R) \times C^\star(X; R) \to C^\star(X; R)$$

by

$$\left(\sum_i \alpha_i \right) \smile \left(\sum_j \beta_j \right) := \sum_{i,j} \alpha_i \smile \beta_j.$$

Let us first check this gives us a ring structure.

PROPOSITION 30.15. *For any topological space X and any ring R, $C^\star(X;R)$ is a graded ring under the cup product.*

Proof. Suppose $\alpha \in C^n(X;R)$ and $\beta, \gamma \in C^m(X;R)$. We claim that $\alpha \smile (\beta+\gamma) = \alpha \smile \beta + \alpha \smile \gamma$. For this, take $\sigma\colon \Delta^{n+m} \to X$. Then

$$\begin{aligned}(\alpha \smile (\beta+\gamma))(\sigma) &= \alpha(\sigma \circ F_n) \cdot (\beta+\gamma)(\sigma \circ B_m) \\ &= \alpha(\sigma \circ F_n) \cdot \beta(\sigma \circ B_m) + \alpha(\sigma \circ F_n) \cdot \gamma(\sigma \circ B_m) \\ &= \alpha \smile \beta(\sigma) + \alpha \smile \gamma(\sigma).\end{aligned}$$

A similar computation shows that $(\alpha+\beta) \smile \gamma = \alpha \smile \gamma + \beta \smile \gamma$. To check associativity, suppose $\alpha \in C^n(X;R)$, $\beta \in C^m(X;R)$, and $\gamma \in C^p(X;R)$. Then if $\sigma\colon \Delta^{n+m+p} \to X$, one has

$$((\alpha \smile \beta) \smile \gamma)(\sigma) = \alpha(\sigma \circ F_{n+m} \circ F_n) \cdot \beta(\sigma \circ F_{n+m} \circ B_m) \cdot \gamma(\sigma \circ B_p),$$

and similarly

$$(\alpha \smile (\beta \smile \gamma))(\sigma) = \alpha(\sigma \circ F_n) \cdot \beta(\sigma \circ B_{m+p} \circ F_m) \cdot \gamma(\sigma \circ B_{m+p} \circ B_p).$$

By part (2) of Lemma 30.12, the right-hand side of both of these equations is equal to

$$\alpha(\sigma \circ F_n) \cdot \beta(\sigma \circ F_{n+m} \circ B_m) \cdot \gamma(\sigma \circ B_p).$$

Finally, define a cochain $\nu \in C^0(X;R)$ by

$$\nu(x) = 1_R, \qquad \forall\, x \in X, \tag{30.2}$$

and then extending by linearity (recall we identify singular 0-simplices in X with points in X). It is clear that $\nu \smile \alpha = \alpha = \alpha \smile \nu$ for any $\alpha \in C^n(X;R)$ and any $n \geq 0$. Thus $C^\star(X;R)$ is indeed a graded ring. ■

REMARK 30.16. The distributive laws give bilinearity of the cup product as a map

$$C^\star(X;R) \times C^\star(X;R) \to C^\star(X;R),$$

and hence we may regard cup product as a map on the tensor product

$$\smile\colon C^\star(X;R) \otimes C^\star(X;R) \to C^\star(X;R).$$

Now let us investigate the functorial properties of $\smile$.

PROPOSITION 30.17. *Let $f\colon X \to Y$. Then*

$$f^\#(\alpha \smile \beta) = f^\#\alpha \smile f^\#\beta,$$

and moreover $f^\#\nu_Y = \nu_X$, where the units ν_X, ν_Y are as defined in (30.2)

Proof. Take $\alpha \in C^n(Y;R)$ and $\beta \in C^m(Y;R)$ and $\sigma\colon \Delta^{n+m} \to X$. Then by definition

$$\begin{aligned} f^{\#}(\alpha \smile \beta)(\sigma) &= (\alpha \smile \beta)(f_{\#}\sigma) \\ &= (\alpha \smile \beta)(f \circ \sigma) \\ &= \alpha(f \circ \sigma \circ F_n) \cdot \beta(f \circ \sigma \circ B_m) \\ &= \alpha(f_{\#}(\sigma \circ F_n)) \cdot \beta(f_{\#}(\sigma \circ B_m)) \\ &= f^{\#}\alpha(\sigma \circ F_n) \cdot f^{\#}\beta(\sigma \circ B_m) \\ &= (f^{\#}\alpha \smile f^{\#}\beta)(\sigma). \end{aligned}$$

Next, if $x \in X$ then $f^{\#}\nu_Y(x) = \nu_Y(f(x)) = 1_R$. Since this holds for all $x \in X$, we must have $f^{\#}\nu_Y = \nu_X$. ∎

We thus have:

Corollary 30.18. *For a given ring R, there is a contravariant functor*

$$C^\star(\square; R)\colon \mathsf{Top} \to \mathsf{GradedRings}.$$

Unfortunately, the ring structure on $C^\star(X;R)$ is not very useful, as it is too "large", and almost impossible to compute. This ring structure does not restrict the homotopy axiom, and it is not graded commutative. However, as we will now see, the total cohomology $H^\star(X;R)$ also inherits a ring structure, and this structure is much nicer.

Proposition 30.19. *Let X be a topological space and let R be a ring. If $\alpha \in C^n(X;R)$ and $\beta \in C^m(X;R)$ then*

$$d(\alpha \smile \beta) = d\alpha \smile \beta + (-1)^n \alpha \smile d\beta. \tag{30.3}$$

Proof. Set $p := n+m+1$, so that both sides of (30.3) have degree p. Let $\sigma\colon \Delta^p \to X$ be a singular p-simplex. Let us start with the right-hand side of equation (30.3). We have

$$\begin{aligned} \left(d\alpha \smile \beta + (-1)^n \alpha \smile d\beta\right)(\sigma) &= d\alpha(\sigma \circ F_{n+1}) \cdot \beta(\sigma \circ B_m) \\ &\quad + (-1)^n \left(\alpha(\sigma \circ F_n) \cdot d\beta(\sigma \circ B_{m+1})\right) \\ &= \alpha(\partial(\sigma \circ F_{n+1})) \cdot \beta(\sigma \circ B_m) \\ &\quad + (-1)^n \left(\alpha(\sigma \circ F_n) \cdot \beta(\partial(\sigma \circ B_{m+1}))\right) \\ &= \left(\sum_{i=0}^{n+1} (-1)^i \alpha(\sigma \circ F_{n+1} \circ \varepsilon_i)\right) \cdot \beta(\sigma \circ B_m) \\ &\quad + (-1)^n \left(\alpha(\sigma \circ F_n) \cdot \left(\sum_{j=0}^{m+1} (-1)^j \beta(\sigma \circ B_{m+1} \circ \varepsilon_j)\right)\right). \end{aligned}$$

Now by part (1) of Lemma 30.12, one has $\sigma \circ F_{n+1} \circ \varepsilon_{n+1} = \sigma \circ F_{n+1} \circ F_n = \sigma \circ F_n$ and similarly $\sigma \circ B_{m+1} \circ \varepsilon_0 = \sigma \circ B_{m+1} \circ B_m = \sigma \circ B_m$. Thus the last term of the first sum cancels with the first term of the second sum, and thus the right-hand side is equal to

$$\sum_{i=0}^{n}(-1)^i\alpha(\sigma\circ F_{n+1}\circ\varepsilon_i)\cdot\beta(\sigma\circ B_m)+(-1)^n\sum_{j=1}^{m+1}(-1)^j\alpha(\sigma\circ F_n)\cdot\beta(\sigma\circ B_{m+1}\circ\varepsilon_j). \quad (30.4)$$

Next, starting with the left-hand side of (30.3), we have

$$\begin{aligned} d(\alpha\smile\beta)(\sigma) &= (\alpha\smile\beta)(\partial\sigma) \\ &= \sum_{i=0}^{p}(-1)^i(\alpha\smile\beta)(\sigma\circ\varepsilon_i) \\ &= \sum_{i=0}^{n}(-1)^i\alpha(\sigma\circ\varepsilon_i\circ F_n)\cdot\beta(\sigma\circ\varepsilon_i\circ B_m) \\ &\quad + \sum_{i=n+1}^{p}(-1)^i\alpha(\sigma\circ\varepsilon_i\circ F_n)\cdot\beta(\sigma\circ\varepsilon_i\circ B_m) \end{aligned}$$

Since $p-m=n+1$, part (3) of Lemma 30.12 tells us that this sum is equal to

$$\sum_{i=0}^{n}(-1)^i\alpha(\sigma\circ F_{n+1}\circ\varepsilon_i)\cdot\beta(\sigma\circ B_m)+\sum_{i=n+1}^{p}(-1)^i\alpha(\sigma\circ F_n)\cdot\beta(\sigma\circ B_{m+1}\circ\varepsilon_{i-n}). \quad (30.5)$$

The first sum in (30.5) is already the same as the first sum in (30.4). The second sum is too, if we set $j=i-n$. This completes the proof. ■

We can now prove the main result of today's lecture.

THEOREM 30.20. *For any ring R, there is a contravariant functor*

$$H^\star(\square;R)\colon \mathsf{hTop}\to\mathsf{GradedRings}.$$

Proof. Let $Z^\star=\bigoplus Z^n(X;R)$ and define $B^\star(X;R)$ similarly. If $\alpha\in Z^n$ and $\beta\in Z^m$ then $d\alpha=d\beta=0$ and hence $d(\alpha\smile\beta)=d\alpha\smile\beta+(-1)^n\alpha\smile d\beta=0$. Thus $\alpha\smile\beta\in Z^{n+m}$, and hence $Z^\star$ is a homogeneous subring of $C^\star(X;R)$.

Next, if $\alpha\in Z^n$ and $\beta\in B^m$ then $d\alpha=0$ and $\beta=d\gamma$ for some $\gamma\in C^{m-1}(X;R)$. Thus

$$\begin{aligned} \alpha\smile\beta &= \alpha\smile d\gamma \\ &= \pm((d(\alpha\smile\gamma)-d\alpha\smile\gamma) \\ &= \pm d(\alpha\smile\gamma). \end{aligned}$$

Thus $\alpha\smile\beta\in B^{n+m}$. Similarly $\beta\smile\alpha\in B^{n+m}$, and hence $B^\star$ is a two-sided homogeneous ideal in $Z^\star$. By Lemma 30.9, it follows that $H^\star(X;R)=Z^\star/B^\star$ is a graded ring, with

$$\langle\alpha\rangle\smile\langle\beta\rangle:=\langle\alpha\smile\beta\rangle.$$

If $f\colon X \to Y$ then if we denote by $H^\star(f)\colon H^\star(Y;R) \to H^\star(X;R)$ the map $H^\star(f) = \sum_{n\geq 0} H^n(f)$ then since $H^n(f)\langle\alpha\rangle = \langle f^\#\alpha\rangle$, it follows Proposition 30.17 that

$$H^\star(f)\left(\langle\alpha\rangle \smile \langle\beta\rangle\right) = H^\star(f)\langle\alpha\rangle \smile H^\star(f)\langle\beta\rangle,$$

and thus $H^\star(f)$ is a graded ring homomorphism. Moreover the homotopy axiom for cohomology (Theorem 28.12) shows that if $f \simeq g$ then (as graded ring homomorphisms) $H^\star(f) = H^\star(g)$. It follows easily that $H^\star$ is a contravariant functor as claimed. ■

Remark 30.21. We will prove next lecture that if R is commutative then $H^\star(\square;R)$ is actually a functor

$$H^\star(\square;R)\colon \mathsf{hTop} \to \mathsf{ComGradedRings}.$$

This is not as easy as it looks (we will prove it via an application of the Acyclic Models Theorem again!)

Remark 30.22. As in Remark 30.16, we can view the cup product as being a multiplication

$$H^\star(X;R) \otimes H^\star(X;R) \to H^\star(X;R).$$

This will also be important next lecture.

LECTURE 31

Diagonal approximations and the cross product

The goal of today's lecture is to prove that the cohomology ring is graded commutative whenever the coefficient ring is commutative.

THEOREM 31.1. *Let R be a commutative ring and let X be a topological space. Then*

$$\langle\alpha\rangle \smile \langle\beta\rangle = (-1)^{mn}\langle\beta\rangle \smile \langle\alpha\rangle, \qquad \forall\, \langle\alpha\rangle \in H^n(X;R),\ \langle\beta\rangle \in H^m(X;R). \tag{31.1}$$

Thus when R is commutative, the functor $H^\star(\square;R)$ takes values in $\mathsf{ComGradedRings}$*:*

$$H^\star(\square;R)\colon \mathsf{hTop} \to \mathsf{ComGradedRings}.$$

This will take us some time. We begin with a general definition, which we have already met in a different guise before.

DEFINITION 31.2. Let $(C_\bullet, \partial)$ be a non-negative chain complex. An **augmentation** of $(C_\bullet, \partial)$ is a surjective homomorphism

$$\zeta\colon C_0 \to \mathbb{Z}$$

such that the composition $\zeta \circ \partial\colon C_1 \to C_0 \to \mathbb{Z}$ is identically zero. We call $(C_\bullet, \partial, \zeta)$ an **augmented chain complex**.

A chain complex that admits an augmentation necessarily has a non-zero homology group $H_0(C_\bullet)$. Indeed, let $\tilde{\mathbb{Z}}_\bullet$ denote the chain complex with all groups zero, apart from the zeroth group, which is $\mathbb{Z}$:

$$\dots 0 \to 0 \to 0 \to \underset{\text{in degree } 0}{\mathbb{Z}} \to 0 \to 0 \to 0 \to \dots$$

We can regard an augmentation as a surjective chain map $\zeta\colon C_\bullet \to \tilde{\mathbb{Z}}_\bullet$. Thus we get an induced map in homology $H_0(\zeta)\colon H_0(C_\bullet) \to \mathbb{Z} = H_0(\tilde{\mathbb{Z}}_\bullet)$, which is again surjective.

An augmentation allows one to form the **reduced chain complex** $\tilde{C}_\bullet$, where $\tilde{C}_n = C_n$ for $n \geq 1$, and $\tilde{C}_0 = \ker\zeta$.

LEMMA 31.3. *The homology groups of the reduced complex satisfy*

$$H_n(C_\bullet) \cong \begin{cases} H_n(\tilde{C}_\bullet), & n \geq 1, \\ H_0(\tilde{C}_\bullet) \oplus \mathbb{Z}, & n = 0 \\ 0, & n \leq -1. \end{cases}$$

Proof. We have an exact sequence $0 \to \tilde{C}_0 \to C_0 \to \mathbb{Z} \to 0$. Since $\mathbb{Z}$ is free this sequence splits, and hence $C_0 \cong \tilde{C}_0 \oplus \mathbb{Z}$. It follows that $Z_n(C_\bullet) = Z_n(\tilde{C}_\bullet)$ for $n \neq 0$, $Z_0(C_\bullet) \cong Z_0(\tilde{C}_\bullet) \oplus \mathbb{Z}$, and $B_n(C_\bullet) = B_n(\tilde{C}_\bullet)$ for all $n \in \mathbb{Z}$. ∎

We have already met one example of an augmented chain complex many times.

Example 31.4. Let X be a non-empty topological space, and let $\zeta_X \colon C_0(X) \to \mathbb{Z}$ be the map defined by

$$\zeta_X(x) = 1, \qquad \forall\, x \in X,$$

and then extended by linearity. Then $(C_\bullet(X), \zeta_X)$ is an augmented chain complex, and the homology of the reduced complex is the reduced homology of X, cf. Remark 12.24.

If $(C_\bullet, \partial, \zeta)$ and $(C'_\bullet, \partial', \zeta')$ are two augmented chain complexes, we say that a chain map $f \colon C_\bullet \to C'_\bullet$ is **augmentation-preserving** if there is a commutative diagram:

$$\begin{array}{ccc} C_0 & \xrightarrow{\zeta} & \mathbb{Z} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1} \\ C'_0 & \xrightarrow[\zeta']{} & \mathbb{Z} \end{array}$$

Equivalently, this means

$$H_0(\zeta') \circ H_0(f) = H_0(\zeta) \colon H_0(C_\bullet) \to \mathbb{Z}.$$

Augmented chain complexes define a category $\mathsf{AugComp}$. Let us now present a version of the Acyclic Models Theorem for augmented complexes. We will denote a functor

$$\widehat{T}_\bullet \colon \mathsf{C} \to \mathsf{AugComp}$$

as a pair $\widehat{T}_\bullet = (T_\bullet, \zeta_T)$, where $T_\bullet \colon \mathsf{C} \to \mathsf{Comp}$ is a non-negative complex valued functor, and $\zeta_T(C) \colon T_0(C) \to \mathbb{Z}$ is an augmentation of the complex $T_\bullet(C)$ for each $C \in \mathrm{obj}(\mathsf{C})$. We say an object C is **totally $\widehat{T}_\bullet$-acyclic** if $H_n(\tilde{T}_\bullet(C)) = 0$ for all $n \geq 0$, where $\tilde{T}_\bullet(C)$ is the reduced complex associated to the augmentation $\zeta_T(C)$. Equivalently, $H_n(T_\bullet(C)) = 0$ for all $n \neq 0$, and $H_0(T_\bullet(C)) = \mathbb{Z}$.

Theorem 31.5 (The Augmented Acyclic Models Theorem). *Let C be a category with models $\mathcal{M}$. Assume that $\widehat{S}_\bullet, \widehat{T}_\bullet \colon \mathsf{C} \to \mathsf{AugComp}$ are two functors. Assume that for all $n \geq 0$, T_n is free with basis contained in $\mathcal{M}$. Assume that each model $M \in \mathcal{M}$ is totally $\widehat{S}_\bullet$-acyclic. Then there exists a natural augmentation preserving chain map $\Phi \colon T_\bullet \to S_\bullet$. Moreover any two such natural augmentation-preserving chain maps are naturally chain homotopic.*

The difference between Theorem 31.5 and the non-augmented one (Theorem 23.8) is that this time we do not need to start with a natural transformation $\Theta \colon H_0 \circ T_\bullet \to H_0 \circ S_\bullet$. (The price to pay for this is, of course, that this version only works for augmented complexes.)

Corollary 31.6. *If instead we assume that for all $n \geq 0$, both S_n and T_n are free with basis contained in $\mathcal{M}$, and that each model $M \in \mathcal{M}$ is both totally $\widehat{S}_\bullet$-acyclic and totally $\widehat{T}_\bullet$ acyclic, then every augmentation preserving natural chain map is a natural chain equivalence.*

Proof. Go back to the proof of the standard Acyclic Models Theorem 23.8, and replace every instance of $T_0 \to H_0 \circ T_\bullet \to 0$ and $S_0 \to H_0 \circ S_\bullet \to 0$ with the augmentations, so $\mathbb{Z}$ plays the role of H_0 and the identity map $\mathbb{Z} \to \mathbb{Z}$ plays the role of $\Theta\colon H_0 \circ T_\bullet \to H_0 \circ S_\bullet$. Explicitly, this means that our goal is to construct natural transformations $\Phi_n\colon T_n \to S_n$ such that the following diagram commutes.

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\partial} & T_2 & \xrightarrow{\partial} & T_1 & \xrightarrow{\partial} & T_0 & \xrightarrow{\zeta_T} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi_2} & & \downarrow{\scriptstyle \Phi_1} & & \downarrow{\scriptstyle \Phi_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\dots & \xrightarrow[\partial']{} & S_2 & \xrightarrow[\partial']{} & S_1 & \xrightarrow[\partial']{} & S_0 & \xrightarrow[\zeta_S]{} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

But now an inspection of the proof of Theorem 23.8 show that it goes through word for word in this new setting. Indeed, the only properties we used of $H_0 \circ T_\bullet$ and $H_0 \circ S_\bullet$ was that $S_1(M) \to S_0(M) \xrightarrow{\zeta_S(M)} \mathbb{Z} \to 0$ should be exact for every model M, which follows from the fact that every model is totally $\widehat{S}_\bullet$-acyclic.

Thus the standard Acyclic Models Theorem provides us with such a $\Phi\colon T_\bullet \to S_\bullet$, which moreover is unique up to natural chain homotopy. Finally, the fact that the initial square commutes:

$$\begin{array}{ccc}
T_0 & \longrightarrow & \mathbb{Z} \\
{\scriptstyle \Phi_0}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
S_0 & \longrightarrow & \mathbb{Z}
\end{array}$$

tells us that Φ is augmentation-preserving. This completes the proof. ■

Remark 31.7. We can use Corollary 31.6 to give a quicker proof of the Eilenberg-Zilber Theorem 27.6. Indeed, we consider augmentations

$$\zeta_{X,Y}\colon C_0(X \times Y) \to \mathbb{Z}, \qquad \zeta_{X,Y}(x, y) = 1, \qquad \forall\, (x, y) \in X \times Y,$$

and

$$\zeta'_{X,Y}\colon C_0(X) \otimes C_0(Y) \to \mathbb{Z}, \qquad \zeta'_{X,Y}(x \otimes y) = 1, \qquad \forall\, (x, y) \in X \times Y. \tag{31.2}$$

Then by assumption, an Eilenberg-Zilber morphism $\Omega\colon C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$ commutes with these augmentations. This means that the extra work we did constructing the Θ in the proof of Theorem 27.6 (the content of this was Lemma 27.7) was unnecessary[1].

What relevance does this have to the problem at hand? Here is another definition.

[1]Nevertheless, I thought it was mean to "remind" you of the Acyclic Models Theorem, which most of you have probably forgotten over the winter vacation by immediately starting off with a generalisation. Also, Lemma 27.7 will be helpful in the proof of Theorem 32.1 next lecture.

Definition 31.8. A **diagonal approximation** D is a natural chain map $C_\bullet(X) \to C_\bullet(X) \otimes C_\bullet(X)$ that satisfies

$$D_0(x) = x \otimes x, \qquad \forall\, x \in X,$$

The reason for the name is the following: if $\Omega\colon C_\bullet(X \times X) \to C_\bullet(X) \otimes C_\bullet(X)$ is an Eilenberg-Zilber morphism and $\delta\colon X \to X \times X$ is the diagonal map $\delta(x) = (x, x)$, then

$$\Omega \circ \delta_\#\colon C_\bullet(X) \to C_\bullet(X) \otimes C_\bullet(X)$$

is a diagonal approximation. Moreover, an application of the Augmented Acyclic Models Theorem shows that this is the *only* diagonal approximation (up to chain homotopy).

Theorem 31.9. *Any two diagonal approximations are naturally chain homotopic, and hence in particular induce the same homomorphisms in (co)homology.*

Proof. Firstly note that a diagonal approximation D could equivalently be defined as natural chain map $C_\bullet(X) \to C_\bullet(X) \otimes C_\bullet(X)$ that preserves the augmentations ζ_X from Example 31.4 and $\zeta'_{X,X}$ from (31.2). Indeed, it is clear that a diagonal approximation preserves these augmentations, and conversely by naturality any such D must satisfy $D_0(x) = x \otimes x$ in degree zero (argue as in Remark 27.8.)

Now let $\mathcal{M} = \{\Delta^n \mid n \geq 0\}$ denote our standard family of models in Top. We view the singular chain functor as a functor $C_\bullet\colon \mathsf{Top} \to \mathsf{AugComp}$, where we are using the standard augmentation from Example 31.4. Then $C_\bullet$ is free with basis in $\mathcal{M}$ (cf. Example 23.4)

Next, consider the functor $S_\bullet\colon \mathsf{Top} \to \mathsf{Comp}$ given by $S_\bullet(X) = C_\bullet(X) \otimes C_\bullet(X)$. We can regard $S_\bullet$ as a functor $\widehat{S}_\bullet\colon \mathsf{Top} \to \mathsf{AugComp}$ using the augmentation $\zeta'_{X,X}$ from (31.2). Moreover by arguing as at the end of the proof of the Eilenberg-Zilber Theorem 27.6, we see that each model in $\mathcal{M}$ is totally $\widehat{S}_\bullet$-acyclic.

The result now follows from Theorem 31.5. ■

It is still probably rather opaque to you why a diagonal approximation is (a) interesting and (b) of any use whatsoever in proving Theorem 31.1. For, this, note that if D is a diagonal approximation then applying $\mathrm{Hom}(\square, R)$ we obtain

$$\mathrm{Hom}(D, R)\colon \mathrm{Hom}(C_\bullet(X) \otimes C_\bullet(X), R) \to \mathrm{Hom}(C_\bullet(X), R),$$

which suddenly "almost" looks like a multiplication on cohomology. Roughly speaking, we will show that cup product is the unique multiplication on cohomology arising from a diagonal approximation. This uniqueness statement will then quickly imply our initial statement about graded commutativity, Theorem 31.1.

Let us mimic the construction we did in the proof of the Künneth Theorem for Cohomology (Theorem 29.16).

Definition 31.10. Let R be a ring. Suppose $A_\bullet$ and $B_\bullet$ are two non-negative chain complexes, and let $A^\bullet = \mathrm{Hom}(A_\bullet, R)$ and $B^\bullet = \mathrm{Hom}(B_\bullet, R)$ denote the dual cochain complexes. Let $C^\bullet = \mathrm{Hom}(A_\bullet \otimes B_\bullet, R)$ denote the dual cochain complex to $A_\bullet \otimes B_\bullet$.

Given $n, m \geq 0$, define $\mu\colon A^n \times B^m \to C^{n+m}$ by setting

$$\mu(\alpha, \beta)\left(\sum_{i=0}^{n+m} a_i \otimes b_{n+m-i}\right) := \alpha(a_n) \cdot \beta(b_m).$$

The map μ is clearly bilinear, and hence can be thought of as a map $A^n \otimes B^m \to C^{n+m}$. We claim that μ is also well-defined on the level of cohomology. For this, let us make the following computation. Take any element $\sum_{i=0}^{n+m+1} a_i \otimes b_{n+m+1-i}$ in C^{n+m+1} and evaluate:

$$\begin{aligned}(d\mu(\alpha \otimes \beta))\left(\sum_{i=0}^{n+m+1} a_i \otimes b_{n+m+1-i}\right) &= \mu(\alpha \otimes \beta)\left(\sum_{i=0}^{n+m+1} \left(\partial a_i \otimes b_{n+m+1-i} + (-1)^i a_i \otimes \partial b_{n+m+1-i}\right)\right) \\ &= \alpha(\partial a_{n+1}) \cdot \beta(b_m) + (-1)^n \alpha(a_n) \cdot \beta(\partial b_{m+1}) \\ &= (d\alpha)(a_{n+1}) \cdot \beta(b_m) + (-1)^n \alpha(a_n) \cdot (d\beta)(b_{m+1}) \\ &= (\mu(d\alpha \otimes \beta) + (-1)^n \mu(\alpha \otimes d\beta))\left(\sum_{i=0}^{n+m+1} a_i \otimes b_{n+m+1-i}\right).\end{aligned}$$

Thus we have shown that

$$d\mu(\alpha \otimes \beta) = \mu(d\alpha \otimes \beta) + (-1)^n \mu(\alpha \otimes d\beta) \tag{31.3}$$

This equation tells us that μ is well-defined on the level of cohomology. Indeed, suppose $d\alpha = d\beta = 0$. Then $d\mu(\alpha \otimes \beta) = 0$, and moreover in this case for any α' and β' one has that

$$\begin{aligned}\mu((\alpha + d\alpha') \otimes (\beta + d\beta')) - \mu(\alpha \otimes \beta) &= \mu(d\alpha' \otimes \beta) + \mu(\alpha \otimes d\beta') + \underbrace{\mu(d\alpha' \otimes d\beta')}_{=0} \\ &= \mu(d\alpha' \otimes \beta) + (-1)^n \mu(\alpha' \otimes \underbrace{d\beta}_{=0}) \\ &\quad + (-1)^n \left((-1)^n \mu(\alpha \otimes d\beta') + \mu(\underbrace{d\alpha}_{=0} \otimes \beta')\right) \\ &= d\mu(\alpha' \otimes \beta) + (-1)^n d\mu(\alpha \otimes \beta').\end{aligned}$$

Thus we conclude:

LEMMA 31.11. *The map*

$$H^n(A^\bullet) \otimes H^m(B^\bullet) \to H^{n+m}(C^\bullet), \qquad \langle\alpha\rangle \otimes \langle\beta\rangle \mapsto \langle\mu(\alpha \otimes \beta)\rangle$$

is well-defined.

We will slightly abuse notation and refer to this map also as μ (rather than $H^{n+m}(\mu)$) so as to keep the formulae to come not too cumbersome.

We return to the situation at hand.

Definition 31.12. The **cross product** is the operation

$$\times_\Omega := \mathrm{Hom}(\Omega, R) \circ \mu \colon C^\star(X;R) \otimes C^\star(Y;R) \to C^\star(X \times Y; R).$$

We write this as

$$\alpha \otimes \beta \mapsto \alpha \times_\Omega \beta.$$

By Lemma 31.11 (together with the fact that Ω is a chain map), the cross product induces a map on cohomology:

$$\times \colon H^\star(X;R) \otimes H^\star(Y;R) \to H^\star(X \times Y; R), \qquad \langle \alpha \rangle \otimes \langle \beta \rangle \mapsto \langle \alpha \times_\Omega \beta \rangle.$$

In cohomology we drop the subscript Ω on $\times_\Omega$, since the Eilenberg-Zilber morphism is unique up to chain homotopy, and hence different choices of Ω give rise to the same map on cohomology. Nevertheless, on the cochain level we *do* need to keep the subscript Ω, since this *does* depend on the choice of Ω.

Remark 31.13. In the case $R = \mathbb{Z}$, the induced map in cohomology agrees with the map η in the Künneth Theorem for Cohomology (Theorem 29.16).

Next lecture, we will explicitly construct a "special" choice of Eilenberg-Zilber morphism, which we call the **Alexander-Whitney choice of Eilenberg-Zilber morphism** and denote it by Ω_{AW}. This particular choice of Eilenberg-Zilber morphism gives rise to the cup product, as the next result shows.

Proposition 31.14. *For $\Omega = \Omega_{\mathrm{AW}}$, the composite*

$$C^\bullet(X;R) \otimes C^\bullet(X;R) \xrightarrow{\mu} \mathrm{Hom}(C_\bullet(X) \otimes C_\bullet(X), R) \xrightarrow{\mathrm{Hom}(\Omega_{\mathrm{AW}}, R)} C^\bullet(X \times X; R) \xrightarrow{\delta^\#} C^\bullet(X;R)$$

is precisely the cup product. Thus if $\alpha \in C^n(X;R)$ and $\beta \in C^m(X;R)$ then

$$\delta^\#(\alpha \times_{\Omega_{\mathrm{AW}}} \beta) = \alpha \smile \beta. \tag{31.4}$$

We will prove Proposition 31.14 next lecture, after we have constructed Ω_{AW}. There is one more ingredient needed for Theorem 31.1, whose proof is on Problem Sheet O.

Lemma 31.15. *Let $(C_\bullet, \partial)$ be a non-negative chain complex. Then the function*

$$\mathrm{twist} \colon C_\bullet \otimes C_\bullet \to C_\bullet \otimes C_\bullet$$

given by

$$\mathrm{twist}(c \otimes c') = (-1)^{nm} c' \otimes c, \qquad c \in C_n,\ c' \in C_m,$$

is a natural chain equivalence.

Let us finally add commutativity of R as a hypothesis:

Lemma 31.16. *If R is commutative then*

$$\mathrm{Hom}(\mathrm{twist}, R) \circ \mu(\alpha \otimes \beta) = (-1)^{nm} \mu(\beta \otimes \alpha), \qquad \forall\, \alpha \in C^n(X;R),\ \beta \in C^m(X;R).$$

Proof. Suppose $\sigma \in C_i(X)$ and $\tau \in C_j(X)$. Both sides clearly vanish on $\tau \otimes \sigma$ unless $i = n$ and $j = m$. In this case we compute

$$\begin{aligned}\operatorname{Hom}(\text{twist}, R)\mu(\alpha \otimes \beta)(\tau \otimes \sigma) &= \mu(\alpha \otimes \beta)((-1)^{nm}(\sigma \otimes \tau)) \\ &= (-1)^{nm}\alpha(\sigma) \cdot \beta(\tau) \\ &\overset{(\star)}{=} (-1)^{nm}\beta(\tau) \cdot \alpha(\sigma) \\ &= (-1)^{nm}\mu(\beta \otimes \alpha)(\tau \otimes \sigma),\end{aligned}$$

where $(\star)$ used commutativity of R. Thus both sides agree on a basis of $\operatorname{Hom}(C_\bullet(X) \otimes C_\bullet(X), R)$, and we conclude

$$\operatorname{Hom}(\text{twist}, R)\mu(\alpha \otimes \beta) = (-1)^{nm}\mu(\beta \otimes \alpha).$$

as desired. ■

We now have all the ingredients to prove Theorem 31.1 (modulo the construction of Ω_{AW} and the proof of Proposition 31.14, which we will do next lecture).

Proof of Theorem 31.1. We have already remarked that a diagonal approximation is given by $\Omega \circ \delta_\#$. It is then clear from the definition that twist $\circ\, \Omega \circ \delta_\#$ is another diagonal approximation. Thus by Theorem 31.9, the two chain maps are naturally chain equivalent.

Thus if we apply $\operatorname{Hom}(\square, R)$, we see that the for any choice of Eilenberg-Zilber morphism Ω the two compositions

$$C^\bullet(X;R) \otimes C^\bullet(X;R) \xrightarrow{\mu} \operatorname{Hom}(C_\bullet(X) \otimes C_\bullet(X), R) \xrightarrow{\operatorname{Hom}(\Omega,R)} C^\bullet(X \times X;R) \xrightarrow{\delta^\#} C^\bullet(X;R)$$

and

$$C^\bullet(X;R) \otimes C^\bullet(X;R) \xrightarrow{\mu} \operatorname{Hom}(C_\bullet(X) \otimes C_\bullet(X), R) \xrightarrow{\operatorname{Hom}(\text{twist},R)} \operatorname{Hom}(C_\bullet(X) \otimes C_\bullet(X), R) \xrightarrow{\operatorname{Hom}(\Omega,R)} C^\bullet(X \times X;R) \xrightarrow{\delta^\#} C^\bullet(X;R)$$

induce the same map in cohomology. Taking $\Omega = \Omega_{\text{AW}}$ and applying Proposition 31.14, we see that the first composition induces

$$\langle \alpha \rangle \otimes \langle \beta \rangle \mapsto \langle \alpha \rangle \smile \langle \beta \rangle$$

in cohomology, whereas using both Proposition 31.14 and Lemma 31.15 shows that for $\Omega = \Omega_{\text{AW}}$ the second composition induces

$$\langle \alpha \rangle \otimes \langle \beta \rangle \mapsto (-1)^{nm}\langle \beta \rangle \smile \langle \alpha \rangle, \qquad \forall\, \langle \alpha \rangle \in H^n(X;R),\ \langle \beta \rangle \in H^m(X;R)$$

in cohomology. This completes the proof of Theorem 31.1. ■

LECTURE 32

The Alexander-Whitney Formula

We begin this lecture by writing down an explicit formula for an Eilenberg-Zilber morphism, which we call the **Alexander-Whitney choice of Eilenberg-Zilber morphism**, using the front and back face maps from the Lecture 30. In the following, given a map $\sigma\colon \Delta^n \to X \times Y$, we denote by $\pi'\colon X \times Y \to X$ and $\pi''\colon X \times Y \to Y$ the two projections, and set $\sigma' = \pi' \circ \sigma$ and $\sigma'' = \pi'' \circ \sigma$, so that $\sigma = (\sigma', \sigma'')$.

Theorem 32.1 (Alexander-Whitney Formula). *The following formula defines an Eilenberg-Zilber morphism:*

$$\Omega_{\mathrm{AW}}(\sigma) := \sum_{i+j=n} (\sigma' \circ F_i) \otimes (\sigma'' \circ B_j),$$

for $\sigma = (\sigma', \sigma'')\colon \Delta^n \to X \times Y$.

Note that this makes sense: if σ is a singular zero-simplex in $X \times Y$, then $\sigma = (x, y)$ for some $(x, y) \in X \times Y$, and $\Omega_{\mathrm{AW}}(x, y) = x \otimes y$ as it should do.

Proof. We will prove the result in three steps. Throughout this proof we will write Ω instead of Ω_{AW}, since we only consider one Eilenberg-Zilber morphism.

1. As explained above in Remark 31.7, all we need to do is show that this formula defines a natural chain map. Then the Augmented Acyclic Models Theorem will immediately tell us it is a natural chain equivalence $C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$, and hence an Eilenberg-Zilber morphism.

Next, proving naturality is easy: if $f\colon X \to X'$ and $g\colon Y \to Y'$ then

$$\Omega \circ (f, g)_\#(\sigma) = \sum_{i+j=n} (f \circ \sigma' \circ F_i) \otimes (g \circ \sigma'' \circ B_j) = (f_\# \otimes g_\#) \circ \Omega(\sigma).$$

We can then use naturality to reduce the problem to proving Ω is a chain map for $X = Y = \Delta^n$. This is a similar argument to how we proved that barycentric subdivision was a chain map in Lecture 13. Indeed, suppose we already know that $\Omega\colon C_\bullet(\Delta^n \times \Delta^n) \to C_\bullet(\Delta^n) \otimes C_\bullet(\Delta^n)$ is a chain map. Let us denote by $\delta\colon \Delta^n \to \Delta^n \times \Delta^n$ the diagonal. We can also regard as a singular n-simplex in $\Delta^n \times \Delta^n$. For notational clarity, we will write δ_n (instead of just δ) to indicate the singular n-simplex, so that $\delta_n \in C_n(\Delta^n \times \Delta^n)$. Now if $\sigma\colon \Delta^n \to X \times Y$ is any singular n-simplex, we can write

$$\sigma = (\sigma', \sigma'') \circ \delta,$$

or alternatively

$$\sigma = (\sigma', \sigma'')_\#(\delta_n). \tag{32.1}$$

In the following we add subscripts to each operator to show what they are defined on: thus $\Delta_{X,Y}$ denote the boundary operator[1] in $C_\bullet(X) \otimes C_\bullet(Y)$, $\partial_{X\times Y}$ the boundary operator on $C_\bullet(X \times Y)$, $\Omega_{X,Y}$ is the map defined on $X \times Y$, etc. Assume we prove that

$$\Delta_{\Delta^n,\Delta^n} \circ \Omega_{\Delta^n,\Delta^n}(\delta_n) = \Omega_{\Delta^n,\Delta^n} \circ \partial_{\Delta^n,\Delta^n}(\delta_n) \tag{32.2}$$

Then we have the following horrendous looking computation:

$$\begin{aligned}
\Delta_{X,Y} \circ \Omega_{X,Y}(\sigma) &= \Delta_{X,Y} \circ \Omega_{X,Y} \circ (\sigma', \sigma'')_\#(\delta_n), && \text{by (32.1)},\\
&= \Delta_{X,Y} \circ (\sigma'_\# \otimes \sigma''_\#) \circ \Omega_{\Delta^n,\Delta^n}(\delta_n), && \text{by naturality},\\
&= (\sigma'_\# \otimes \sigma''_\#) \circ \Delta_{\Delta^n,\Delta^n} \circ \Omega_{\Delta^n,\Delta^n}(\delta_n), && \text{since } (\sigma'_\# \otimes \sigma''_\#) \text{ is a chain map},\\
&= (\sigma'_\# \otimes \sigma''_\#) \circ \Omega_{\Delta^n,\Delta^n} \circ \partial_{\Delta^n,\Delta^n}(\delta_n), && \text{by (32.2)},\\
&= \Omega_{X,Y} \circ (\sigma', \sigma'')_\# \circ \partial_{\Delta^n,\Delta^n}(\delta_n), && \text{by naturality},\\
&= \Omega_{X,Y} \circ \partial_{X,Y} \circ (\sigma', \sigma'')_\#(\delta_n), && \text{since } (\sigma', \sigma'')_\# \text{ is a chain map},\\
&= \Omega_{X,Y} \circ \partial_{X,Y}(\sigma), && \text{by (32.1)},
\end{aligned}$$

which—provided we can verify (32.2)—shows the chain map property:

$$\Delta_{X,Y} \circ \Omega_{X,Y} = \Omega_{X,Y} \circ \partial_{X,Y}.$$

2. We now verify that (32.2). This is a messy, but straightforward. Since everything now happens on the standard n-simplex, we will ditch the subscripts. Note that δ_n is an affine singular n-simplex in the sense of Definition 13.5. We will use the notation[2] $[e_{k_0}, \dots, e_{k_i}]$ for the affine singular i-simplex $\Delta^i \to \Delta^n$ that sends e_k to e_{k_i} for $k = 0, \dots, i$. Thus $F_i = [e_0, \dots, e_i]$ and $B_{n-i} = [e_j, \dots, e_n]$. Then, recalling we use a circumflex ˆ to mean "delete" (cf. Definition 7.7), we have:

$$\begin{aligned}
\Delta \circ \Omega(\delta_n) &= \Delta\left(\sum_{i=0}^{n} [e_0, \dots, e_i] \otimes [e_i, \dots, e_n]\right)\\
&= \sum_{i=0}^{n} \big(\partial[e_0, \dots, e_i] \otimes [e_i, \dots, e_n] + (-1)^i [e_0, \dots, e_i] \otimes \partial[e_i, \dots, e_n]\big)\\
&= \sum_{i=0}^{n} \sum_{j \le i} (-1)^j [e_0, \dots, \hat{e}_j, \dots, e_i] \otimes [e_i, \dots, e_n]\\
&\quad + \sum_{i=0}^{n} \sum_{k=0}^{n-i} (-1)^i \cdot (-1)^k [e_0, \dots, e_i] \otimes [e_i, \dots, \hat{e}_{i+k}, \dots, e_n]\\
&= \sum_{j<i} (-1)^j [e_0, \dots, \hat{e}_j, \dots, e_i] \otimes [e_i, \dots, e_n]\\
&\quad + \sum_{j>i} (-1)^j [e_0, \dots, e_i] \otimes [e_i, \dots, \hat{e}_j, \dots, e_n],
\end{aligned}$$

[1] Here this notation is slightly awkward, since we will be using $X = Y = \Delta^n$, and thus we will be considering the boundary operator $\Delta_{\Delta^n,\Delta^n}$. Oh well, too late to change it now...

[2] This notation was also used in the solution to Problem G.1.

where in the last equation we observed that the the terms $j = i$ in the first sum, which are:

$$\sum_{i=1}^{n}(-1)^i[e_0, \dots, e_{i-1}] \otimes [e_i, \dots, e_n]$$

cancelled with the term $j = i$ in the second sum, which are:

$$\sum_{i=0}^{n-1}(-1)^i[e_0, \dots, e_i] \otimes [e_{i+1}, \dots, e_n]$$

3. Going the other way, we have

$$\Omega \circ \Delta(\delta_n) = \sum_{i=0}^{n-1}((\partial\delta_n)' \circ F_i) \otimes ((\partial\delta_n)'' \circ B_{n-i-1}).$$

But $(\partial\delta_n)' = \pi'_\# \circ \partial\delta_n$, where $\pi' \colon \Delta^n \times \Delta^n \to \Delta^n$ is projection onto the first factor. Since $\pi_\#$ is a chain map, $\pi'_\# \circ \partial\delta_n = \partial\pi'_\#\delta_n = \partial\ell_n$, where[3] $\ell_n \colon \Delta^n \to \Delta^n$ is the identity map (regarded as a singular n-simplex in Δ^n). Similarly $(\partial\delta_n)'' = \partial\ell_n$. Now by definition

$$\partial\ell_n = \sum_{j=0}^{n}(-1)^j\varepsilon_j,$$

where $\varepsilon_j \colon \Delta^{n-1} \to \Delta^n$ is the jth face. Thus

$$\Omega \circ \Delta(\delta_n) = \sum_j\sum_i(-1)^j(\varepsilon'_j \circ F_i) \otimes (\varepsilon''_j \circ B_{n-1-i}).$$

Clearly $\varepsilon'_j = \varepsilon''_j = \varepsilon_j$, and hence we can write this as

$$\begin{aligned}\Omega \circ \Delta(\delta_n) &= \sum_i\sum_j(-1)^j(\varepsilon_j \circ F_i) \otimes (\varepsilon_j \circ B_{n-i-1})\\ &= \sum_{i>j}(-1)^j(F_{i+1} \circ \varepsilon_j) \otimes B_{n-i} + \sum_{i<j}(-1)^jF_i \otimes (B_{n-i} \circ \varepsilon_{j-i-1}),\end{aligned}$$

where we used part (3) of Lemma 30.12. But unravelling our notation, we have

$$(F_{i+1} \circ \varepsilon_j) \otimes B_{n-i} = [e_0, \dots, \hat{e}_j, \dots, e_i] \otimes [e_i, \dots, e_n],$$

and

$$F_i \otimes (B_{n-i} \circ \varepsilon_{j-i-1}) = [e_0, \dots, e_i] \otimes [e_i, \dots, \hat{e}_j, \dots, e_n],$$

so that these two terms agree with the formula at the end of Step 2. This completes the proof. ■

Let us now prove Proposition 31.14 from Lecture 31, which we restate here:

[3] As in Lecture 13, we use the symbol ℓ_n to denote the identity map, when thought of as a singular n-simplex in Δ^n.

Proposition 32.2. *For $\Omega = \Omega_{\mathrm{AW}}$, the composite*

$$C^\bullet(X;R)\otimes C^\bullet(X;R) \xrightarrow{\mu} \mathrm{Hom}(C_\bullet(X)\otimes C_\bullet(X),R) \xrightarrow{\mathrm{Hom}(\Omega_{\mathrm{AW}},R)} C^\bullet(X\times X;R) \xrightarrow{\delta^\#} C^\bullet(X;R)$$

is precisely the cup product. Thus if $\alpha \in C^n(X;R)$ and $\beta \in C^m(X;R)$ then

$$\delta^\#(\alpha \times_{\Omega_{\mathrm{AW}}} \beta) = \alpha \smile \beta.$$

Proof. Take $\alpha \in C^n(X;R)$, $\beta \in C^m(X;R)$, and let $\sigma\colon \Delta^{n+m} \to X$ be a singular $(n+m)$-simplex. Then

$$\delta^\# \circ \mathrm{Hom}(\Omega_{\mathrm{AW}},R)(\alpha\otimes\beta)(\sigma) = (\alpha\otimes\beta)(\Omega_{\mathrm{AW}} \circ \delta_\#(\sigma)) = (\alpha\otimes\beta)(\Omega_{\mathrm{AW}}(\delta\circ\sigma)).$$

Now using the Alexander-Whitney Formula,

$$\Omega_{\mathrm{AW}}(\delta\circ\sigma) = \sum_{i=0}^{n+m} ((\delta\sigma)'\circ F_i)\otimes((\delta\sigma)''\circ B_{n+m-i}).$$

But both $\pi'\circ\delta$ and $\pi''\circ\delta$ are the identity on X, so this becomes

$$\Omega_{\mathrm{AW}}(\delta\circ\sigma) = \sum_{i=0}^{n+m} (\sigma\circ F_i)\otimes(\sigma\circ B_{n+m-i}).$$

Since $\alpha\otimes\beta$ vanishes off $C_n(X)\otimes C_m(X)$, we have

$$\begin{aligned}(\alpha\otimes\beta)(\Omega_{\mathrm{AW}}(\delta\circ\sigma)) &= (\alpha\otimes\beta)(\sigma\circ F_n)\otimes(\sigma\circ B_m)\\ &= \alpha(\sigma\circ F_n)\cdot\beta(\sigma\circ B_m)\\ &= (\alpha\smile\beta)(\sigma).\end{aligned}$$

■

Remark 32.3. Eilenberg-Zilber morphisms are associative. This means that for any triple of spaces X,Y,Z, the following diagram commutes *up to natural chain homotopy*:

$$\begin{array}{ccc} C_\bullet(X\times Y\times Z) & \xrightarrow{\Omega_{X,Y\times Z}} & C_\bullet(X)\otimes C_\bullet(Y\times Z) \\ \downarrow{\scriptstyle \Omega_{X\times Y,Z}} & & \downarrow{\scriptstyle \mathrm{id}_X\otimes\Omega_{Y,Z}} \\ C_\bullet(X\times Y)\otimes C_\bullet(Z) & \xrightarrow[\Omega_{X,Y}\otimes\mathrm{id}_Z]{} & C_\bullet(X)\otimes C_\bullet(Y)\otimes C_\bullet(Z) \end{array}$$

This follows from the uniqueness part of Augmented Acyclic Models Theorem. From this one can deduce that the cross product is always associative on the level of cohomology (although this already follows from the fact that the cup product is associative.)

If we take the Alexander-Whitney choice of Eilenberg-Zilber morphisms, then a stronger statement is true: the diagram *genuinely* commutes (i.e. not just up to natural chain homotopy). This follows directly from the formula defining the Alexander-Whitney choice of the Eilenberg-Zilber morphism, as I invite you to check.

Remark 32.4. Eilenberg-Zilber morphisms are also commutative. This means that for any pair of spaces X and Y the following diagram commutes up to natural chain homotopy:

$$\begin{array}{ccc} C_\bullet(X\times Y) & \xrightarrow{\Omega_{X,Y}} & C_\bullet(X)\otimes C_\bullet(Y) \\ {\scriptstyle t^{\#}_{X,Y}}\downarrow & & \downarrow{\scriptstyle \mathrm{twist}_{X,Y}} \\ C_\bullet(Y\times X) & \xrightarrow[\Omega_{Y,X}]{} & C_\bullet(Y)\otimes C_\bullet(X) \end{array}$$

Here $t_{X,Y}\colon X\times Y\to Y\times X$ is the map $(x,y)\mapsto(y,x)$, and $\mathrm{twist}_{X,Y}$ is the natural chain equivalence from Lemma 31.15. This can again be proved via an application of the Augmented Acyclic Models Theorem.

However there is an important difference compared to the associativity diagram in the previous remark: Even if one chooses the Alexander-Whitney choice of Eilenberg-Zilber morphism, this diagram does *not* have to commute on the cochain level[4]. This can again be verified directly from the formula defining the Alexander-Whitney choice of the Eilenberg-Zilber morphism.

The aim of the rest of this lecture is to show that the cross product is a homomorphism of graded rings. We begin with the following trivial result, whose proof I will leave as an exercise.

Lemma 32.5. *Let R and S be rings. Then $R\otimes S$ also carries a ring structure, where the multiplication is defined by*

$$(r\otimes s)\cdot(r'\otimes s') := rr'\otimes ss', \qquad r,r'\in R,\ ,s'\in S.$$

If instead R and S are graded rings, then $R\otimes S$ can be given the structure of a graded ring, where

$$(R\otimes S)^n := \sum_{i+j=n} R^i\otimes S^j,$$

and the multiplication is defined by

$$(r\otimes s)\cdot(r'\otimes s') := (-1)^{mn} rr'\otimes ss', \qquad r\in R,\ r'\in R^n,\ s\in S^m,\ s'\in S.$$

Remark 32.6. The operation $(R,S)\mapsto R\otimes S$ is the coproduct (cf. Definition 16.8) in the category of commutative rings.

Remark 32.7. If R and S are graded rings, then in fact we should use different notation to denote the two tensor products: $R\otimes_{\text{ungraded}} S$ and $R\otimes_{\text{graded}} S$. Indeed, these two rings are *not* the same. For example, if $R=\mathbb{Z}[P]$ and $S=\mathbb{Z}[Q]$ then $R\otimes_{\text{ungraded}} S\cong\mathbb{Z}[P,Q]$ (i.e. polynomials over $\mathbb{Z}$ with two commuating variables P and Q), but $R\otimes_{\text{graded}} S$ consists of all polynomials over $\mathbb{Z}$ in two variables P and Q for which $PQ=-QP$, as you can easily check. From now on, if R and S are graded we always implicitly take $R\otimes S$ to mean $R\otimes_{\text{graded}} S$.

[4] Thank you "asdf" for correcting me here! This implies that Theorem 32.8 does *not* hold on the cochain level.

Theorem 32.8. *Let X and Y be topological spaces and let R be a commutative ring. Then the cross product $H^\star(X;R)\otimes_{\text{graded}} H^\star(Y;R)\to H^\star(X\times Y;R)$ is a homomorphism of graded rings.*

Proof. Take cocycles $\alpha\in Z^n(X;R)$, $\alpha'\in Z^m(X;R)$, $\beta\in Z^p(Y;R)$ and $\beta'\in Z^q(Y;R)$. To simplify the notation we write $a:=\langle\alpha\rangle$, $b=\langle\beta\rangle$ and define a',b' similarly. We want to prove that:

$$(a\times b)\smile_{X\times Y}\left(a'\times b'\right)=(-1)^{mp}\left(a\smile_X a'\right)\times\left(b\smile_Y b'\right),$$

where we added subscripts to the cup products to indicate ring the multiplication was occurring.

Consider the following commutative diagram, where $\delta_X\colon X\to X\times X$ is the diagonal map (and similarly for δ_Y and $\delta_{X\times Y}$), and $t_{X,Y}$ was defined in Remark 32.4.

$$\begin{array}{ccc} X\times Y & \xrightarrow{(\delta_X,\delta_Y)} & X\times(X\times Y)\times Y \\ & \searrow^{\delta_{X\times Y}} & \big\downarrow{\scriptstyle (\mathrm{id}_X,t_{X,Y},\mathrm{id}_Y)} \\ & & X\times(Y\times X)\times Y, \end{array}$$

We then compute:

$$\begin{aligned}(a\times b)\smile_{X\times Y}\left(a'\times b'\right)&\overset{(\dagger)}{=}H^\star(\delta_{X\times Y})\left(a\times b\times a'\times b'\right)\\&=H^\star(\delta_X,\delta_Y)\circ H^\star(\mathrm{id}_X\times t_{X,Y}\times\mathrm{id}_Y)\left(a\times b\times a'\times b'\right)\\&\overset{(\heartsuit)}{=}H^\star(\delta_X,\delta_Y)\circ\left(a\times H^\star(\mathrm{twist}_{X,Y})(b\times a')\times b'\right)\\&\overset{(\spadesuit)}{=}(-1)^{mp}H^\star(\delta_X,\delta_Y)\left(a\times a'\times b\times b'\right)\\&\overset{(\dagger)}{=}(-1)^{mp}H^\star(\delta_X)\left(a\times a'\right)\times H^\star(\delta_Y)\left(b\times b'\right)\\&=(-1)^{mp}\left(a\smile_X a'\right)\times\left(b\smile_Y b'\right),\end{aligned}$$

where both equations labelled $(\dagger)$ used associativity of the cross product (Remark 32.4), $(\heartsuit)$ used commutativity of the cross product (Remark 32.4), and $(\spadesuit)$ used Lemma 31.15. ∎

Corollary 32.9. *Let X and Y are topological spaces of finite type, and assume that $H_n(Y)$ is free abelian for all $n\geq 0$. Then the cross product $H^\star(X)\otimes_{\text{graded}} H^\star(Y)\to H^\star(X\times Y)$ is a isomorphism of graded rings.*

Proof. We have already noted in Remark 31.13 that for $R=\mathbb{Z}$, the cross product is the map η from the Künneth Formula in Cohomology (Theorem 29.16):

$$0\to\bigoplus_{i+j=n}H^i(X)\otimes H^j(Y)\xrightarrow{\eta=\times}H^n(X\times Y)\to\bigoplus_{k+l=n+1}\mathrm{Tor}\big(H^k(X),H^l(Y)\big)\to 0.$$

The assumption that $H_n(Y)$ is free abelian means that $H^n(Y)$ is also free abelian (Corollary 29.9), and hence each Tor group is zero on the right-hand side by part (1) of Theorem 25.6. ∎

Remark 32.10. Using Theorem 32.8, one can give an easier definition of the cross product that *starts* from the cup product. Namely, if $\pi' \colon X \times Y \to X$ and $\pi'' \colon X \times Y \to Y$ are the two projections, then for $\langle \alpha \rangle \in H^\star(X; R)$ and $\langle \beta \rangle \in H^\star(Y; R)$, set

$$\langle \alpha \rangle \times \langle \beta \rangle := H^\star(\pi')\langle \alpha \rangle \smile H^\star(\pi'')\langle \beta \rangle, \tag{32.3}$$

where the cup product on the right-hand side takes place in the ring $H^\star(X \times Y; R)$. To see this, we first claim that

$$H^\star(\pi'')\langle \beta \rangle = \langle \nu_X \rangle \times \langle \beta \rangle, \tag{32.4}$$

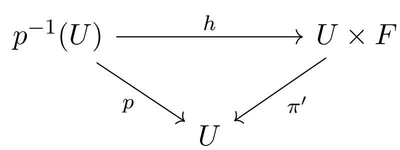
where $\nu_X \in C^0(X; R)$ is the unit (cf. (30.2)). To prove (32.4), first note that since $\nu_X \in H^0(X; R)$ is the image of $\nu_x \in H^0(x; R)$ under the projection $X \to \{x\}$, it suffices to consider the case where X is a point. The case when X is a point is left as an exercise. Anyway, with (32.4) in hand, (32.3) follows from Theorem 32.8.

LECTURE 33

Fibre bundles

In this lecture we define fibre bundles. Next lecture we will state and prove the *Leray-Hirsch Theorem*, which gives conditions on when the cohomology of a fibre bundle looks like the cohomology of a trivial (product) bundle.

DEFINITION 33.1. Let $p\colon E \to X$ be a continuous map between two topological spaces E and X, and let F be a non-empty topological space. We say that $p\colon E \to X$ (or sometimes just "p") is a **fibre bundle** over X with **fibre** F if for every $x \in X$, there exists a neighbourhood $U \subset X$ of x and a homeomorphism $h\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:



where $\pi'\colon U \times F \to U$ denotes the projection onto the first factor. It is often convenient to denote by $E_x := p^{-1}(x)$. Thus $E_x \cong F$ for all $x \in X$. We call X the **base space** and E is called the **total space**. One calls h a **local trivialisation** of the fibre bundle p. We often write $F \to E \xrightarrow{p} X$ to indicate that p is a fibration with fibre F.

One can alternatively define fibre bundles via open covers: a map $p\colon E \to X$ is a fibre bundle with fibre F if and only if there exists an open covering $\{U_\lambda \mid \lambda \in \Lambda\}$ such that for each $\lambda \in \Lambda$ there is a homeomorphism $h_\lambda\colon p^{-1}(U_\lambda) \to U_\lambda \times F$ such that $\pi' \circ h_\lambda = p|_{p^{-1}(U_\lambda)}$. We call $\{U_\lambda\}$ a **trivialising cover** of X.

A fibre bundle $p\colon E \to X$ is necessarily an open map (since projections of products are open maps), and thus X carries the quotient topology determined by p.

EXAMPLE 33.2. It F and X are any two topological spaces then $\pi'\colon X \times F \to X$ is obviously a fibre bundle (where π' denotes the first projection). We call $X \times F$ a **trivial bundle**. Similarly if E is any fibre bundle such that $E \cong X \times F$ (that is, the homeomorphism h from Definition 33.1 can be chosen to be defined on the entire space E) then we say that E is a trivial bundle. Thus for instance $F \times X \to X$ is also a trivial bundle.

Trivial bundles may seem pointless, but they are more common than one would think. We shall see next lecture that *any* fibre bundle over a contractible cell complex is automatically a trivial bundle.

EXAMPLE 33.3. The map $\exp\colon \mathbb{R} \to S^1$ by

$$\exp(s) := e^{2\pi i s}$$

from Definition 5.1 is a fibre bundle over S^1 with fibre $\mathbb{Z}$.

Generalising this, we have the following definition, which you may have seen previously in your point-set topology class.

DEFINITION 33.4. Let $p\colon E \to X$ be a fibre bundle such that F carries the discrete topology. Then we call p a **covering space**.

REMARK 33.5. In fact, covering spaces can be defined slightly more generally. Let us a say a surjective continuous map $p\colon Z \to X$ is a **covering space** if for every $x \in X$, there exists an open neighbourhood $U \subset X$ of x such that $p^{-1}(U)$ is a union of disjoint open sets in Z, each of which is mapped homeomorphically onto U by p. With this definition, a covering space is *not* necessarily a fibre bundle, since if X is not path-connected then the sets $p^{-1}(x)$ and $p^{-1}(y)$ need not be homeomorphic if x and y belong to different path components. However if $p\colon Z \to X$ is a covering space and X' is any path component of X, then setting $E := p^{-1}(X')$ one has that $p|_E\colon E \to X'$ is a fibre bundle.

Here are some other examples of fibre bundles.

DEFINITION 33.6. Let $\mathbb{R}^n \to E \xrightarrow{p} X$ be a fibre bundle. We say that p is a (real) **vector bundle** if the following two additional conditions hold:

1. Each fibre E_x carries the structure of an n-dimensional real vector space.
2. There is an trivialising cover $\{U_\lambda\}$ of X such that the corresponding local trivialisations $h_\lambda\colon p^{-1}(U_\lambda) \to U_\lambda \times \mathbb{R}^n$ have the property that $h_\lambda|_{E_x}\colon E_x \to \{x\} \times \mathbb{R}^n$ is a vector space isomorphism[1] for each $x \in U_\lambda$.

As an example (for those of you familiar with differential geometry), the tangent bundle of any smooth manifold is a real vector bundle. The notion of a complex vector bundle can be defined analogously. Starting from vector bundles, one can define **topological K-theory**, which is a generalised cohomology theory (cf. Remark 7 after Definition 21.9), but that's a different course...

Continuing our list of examples, we now consider a famous class of fibre bundles over a sphere. Let us temporarily denote by $\mathbb{F}$ either the real numbers $\mathbb{R}$, or the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$. For $m = 1, 2$ or 4, we can view the sphere $S^{m(n+1)-1}$ as a subset of $\mathbb{F}^{n+1}$:

$$S^{m(n+1)-1} = \left\{ (x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{F} \text{ and } \sum_{i=0}^{n} |x_i|^2 = 1 \right\}.$$

We set $\mathbb{F}P^n = \big(\mathbb{F}^{n+1} \setminus \{0\}\big)/\sim$, where

$$(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n) \qquad \Leftrightarrow \qquad \exists\, \mu \in \mathbb{F}\backslash\{0\} \text{ such that } x_i = \mu y_i,\ \forall\, i = 0, \dots, n.$$

[1]That is, an isomorphism in the category of vector spaces.

There is a continuous function $p\colon S^{m(n+1)-1} \to \mathbb{F}P^n$ that sends a tuple (x_i) to its equivalence class (for $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ this agrees with our normal definition, cf Example 18.8 and Example 18.9.) On Problem Sheet O, you will prove:

PROPOSITION 33.7. *For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$ and $m = 1, 2$ or 4 respectively,*

$$S^{m-1} \to S^{m(n+1)-1} \xrightarrow{p} \mathbb{F}P^n$$

is a fibre bundle.

We call these fibre bundles **Hopf fibrations**.

DEFINITION 33.8. Let $p\colon E \to X$ be a continuous surjective map (not necessarily a fibre bundle). Let $U \subset X$ be open. A **local section** $s\colon U \to E$ is a continuous function such that $p \circ s = \iota$, where $\iota\colon U \hookrightarrow X$ is the inclusion. If $x \in X$ has a neighbourhood U and a local section s of p on U, we say that s is a **local section at** x.

If $p\colon E \to X$ is a fibre bundle then for any $x \in X$, a local section exists. Indeed, if $U \subset X$ is an open set containing x and $h\colon p^{-1}(U) \to U \times F$ is a local trivialisation, then if $z \in F$ is any point,

$$U \xrightarrow{\iota_z} U \times F \xrightarrow{h^{-1}} p^{-1}(U)$$

is a local section defined on U, where $i_z(x) := (x, z)$.

DEFINITION 33.9. Let G be a topological group (i.e. a topological space with a group structure, such that the multiplication and inverse maps are continuous). Let $H \subseteq G$ be a closed subgroup, and consider the quotient space[2] G/H (whose elements are the left cosets gH for $g \in H$), equipped with the quotient topology $p\colon G \to G/H$. We call G/H a **homogeneous space**.

There is a left action of G on G/H given by $g \cdot (g'H) := (gg')H$, and the map p is **equivariant** in the sense that

$$p(gg') = g \cdot p(g'), \qquad \forall\, g, g' \in G. \tag{33.1}$$

We denote by $e \in G/H$ the coset H.

LEMMA 33.10. *Suppose $p\colon G \to G/H$ is a homogeneous space. If p has local section s at e, then p has a local section at every point of G/H.*

Proof. Let $U \subseteq G/H$ be an open set containing e and $s\colon U \to G$ a local section on U. Given $x := gH \in G/H$, the set $g \cdot U$ is open in G/H and contains x, and the function $s'\colon g \cdot U \to G$ defined by

$$s'(gg'H) := gs(g'H), \qquad g'H \in U$$

is a continuous map and by (33.1) one has

$$p \circ s(gg'H) = p(gs(g'H)) = gp(s(g'H)) = gg'H.$$

■

[2]In general this is not a group—H need not be a normal subgroup.

We can use this as a way to construct fibre bundles.

Proposition 33.11. *Let G be a topological group at let $K \subseteq H \subseteq G$ be closed subgroups. Let $p\colon G \to G/H$ and $p'\colon G/K \to G/H$ denote the two quotient maps. Assume that $p\colon G \to G/H$ has a local section at e. Then*

$$H/K \to G/K \xrightarrow{p'} G/H$$

is a fibre bundle.

Proof. Let $x \in G/H$. By Lemma 33.10, there exists an open set $U \subseteq G/H$ containing x and a local section $s\colon U \to G$ of p. Define

$$f\colon U \times H/K \to G/K, \qquad f(gH, hK) := s(gH)hK.$$

for $g \in G$, $h \in H$ and $gH \in U$. Then $p'f(gH, hK) = gH$ and so $f(gH, hK) \in (p')^{-1}(U)$. Now define

$$\eta\colon (p')^{-1}(U) \to U \times H/K, \qquad \eta(gK) := \big(gH, s(gH)^{-1}gK\big).$$

Both f and η are continuous, and one easily checks they invert each other. Thus in particular η is a homeomorphism. Since x was arbitrary, this completes the proof. ∎

On Problem Sheet O there are several applications of Proposition 33.11 for you to investigate.

Let us now make fibre bundles into a category.

Definition 33.12. Let $F \to E \xrightarrow{p} X$ and $F' \to E' \xrightarrow{p'} X'$ be fibre bundles. A **morphism** from p to p' is a pair (f, φ) of continuous maps, where $f\colon X \to X'$ and $\varphi\colon E \to E'$ are such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ X & \xrightarrow[f]{} & X' \end{array}$$

Thus φ maps fibres of p to fibres of p', i.e. for any $x \in X$, φ induces a map $F \cong p^{-1}(x) \to F' \cong (p')^{-1}(f(x))$.

It is easy to check this defines the category Bun of fibre bundles. If we fix a topological space X then there is a full subcateogory Bun_X of fibre bundles over X whose objects are bundles over X and whose morphisms are pairs (id_X, φ).

The following definition is arguably more important (as far as algebraic topology is concerned). We will only briefly cover it here—the study of fibrations will be taken up again in Lecture 45.

Definition 33.13. Let $p\colon E \to X$ be a continuous map between two topological spaces. Let W be a topological space. We say that p has the **homotopy lifting property** with respect to W if for any homotopy $f_t\colon W \to X$ (for $t \in [0,1]$) and any continuous map $g_0\colon W \to E$ such that $p \circ g_0 = f_0$, there exists a homotopy $g_t\colon W \to E$ such that $p \circ g_t = f_t$:

The homotopy g_t is called a **covering homotopy** of f_t.

EXAMPLE 33.14. It follows from Proposition 5.2 that $\exp\colon \mathbb{R} \to S^1$ has the covering homotopy property with respect to any compact convex subset of $X \subset \mathbb{R}^n$. Indeed, if $X \subset \mathbb{R}^n$ is compact convex, then so is $X \times I$. Now given $f_t\colon X \to S^1$ and $g_0\colon X \to \mathbb{R}$ such that $\exp \circ g_0 = f_0$, Proposition 5.2 gives the existence of a unique continuous map $g\colon X \times I \to \mathbb{R}$ such that $\exp \circ g(x,t) = f_t(x)$. Setting $g_t(x) := g(x,t)$ gives the desired homotopy.

DEFINITION 33.15. If $p\colon E \to X$ has the homotopy lifting property with respect to any topological space W, then p is called a **fibration**. If p has the homotopy lifting property with respect to any cell complex W then p is called a **weak fibration**.

DEFINITION 33.16. Let X be a topological space and let $\{U_\lambda \mid \lambda \in \Lambda\}$ be an open cover of X. We say that $\{U_\lambda\}$ is **locally finite** if for any $x \in X$ there exists an open neighbourhood V such that

$$\{\lambda \in \Lambda \mid U_\lambda \cap V \neq \emptyset\}$$

is finite. We say that a topological space X is **paracompact** if every open cover admits a locally finite refinement.

Any cell complex is paracompact. The next important theorem relates (weak) fibrations to fibre bundles. We will prove (half of) it when we study fibrations in Lecture 45.

THEOREM 33.17. *Let $F \to E \xrightarrow{p} X$ be a fibre bundle. Then p is a weak fibration. If X is paracompact then p is a fibration.*

REMARK 33.18. A fibre bundle with discrete fibre (i.e. a covering space) satisfies a stronger version of the homotopy lifting property: given $f_t\colon W \to X$ and a lift g_0 of f_0, there exists a *unique* lift g_t of f_t. This uniqueness is the main difference between the theory of covering spaces and the general theory of fibre bundles.

We need a bit more algebra before we can get state the main result on the cohomology of fibre bundles (the Leray-Hirsch Theorem), which will take us the rest of the lecture.

DEFINITION 33.19. Let R be a ring. A **left R-module** M is an additive abelian group having a *scalar multiplication* $R \times M \to M$, denoted by $(r,m) \mapsto rm$ such that for all $x, y \in M$ and $r, s \in R$, one has:

1. $r(x+y) = rx + ry$.
2. $(r+s)x = rx + sx$.
3. $rs(x) = r(sx)$.
4. $1_R x = x$.

If M and N are left R-modules, a R-module homomorphism $f\colon M \to N$ is a function such that for all $r \in R$ and $x, y \in M$, one has:

1. $f(x+y) = f(x) + f(y)$,
2. $f(rx) = rf(x)$.

In this way modules over R form a category ${}_R\mathsf{Mod}$.

EXAMPLE 33.20. If $R = \mathbb{Z}$ then ${}_{\mathbb{Z}}\mathsf{Mod}$ is just the familiar category Ab.

There is an analogous notion of a free module.

DEFINITION 33.21. Let R be a ring and let M be a left R-module. A set $B \subset M$ is called a **generating set** if every element of M can be written as a finite sum of elements of B multiplied by elements of R. A subset B is called **linearly independent** if for any given distinct elements $b_1, b_2, \ldots, b_n$ of B, the only solution to

$$r_1 b_1 + r_2 b_2 + \cdots + r_n b_n = 0_M$$

is $r_1 = r_2 = \cdots = r_n = 0_R$. A subset B of M is called a **basis** if it is both a generating set and linearly independent. We say that M is **free** if it admits a basis.

EXAMPLE 33.22. If R is a field then any R-vector space V is a free left R-module.

In an analogous way one can speak of *right* R-modules:

DEFINITION 33.23. Let R be a ring. A **right R-module** M is an additive abelian group having a *scalar multiplication* $R \times M \to M$, denoted by $(r, m) \mapsto mr$ such that for all $x, y \in M$ and $r, s \in R$, one has:

1. $(x+y)r = xr + yr$.
2. $x(r+s) = xr + xs$.
3. $x(rs) = (xr)s$.
4. $x 1_R = x$.

If M and N are right R-modules, a R-module homomorphism $f\colon M \to N$ is a function such that for all $r \in R$ and $x, y \in M$, one has:

1. $f(x+y) = f(x) + f(y)$,
2. $f(xr) = f(x)r$.

In this way right R-modules form a category Mod_R.

DEFINITION 33.24. Let R, S be rings. An (R,S)**-bimodule** M over R is a module which is simultaneously a left R-module and a right S-module and for which the two scalar multiplications satisfy

$$r(xs) = (rx)s, \qquad \forall\, x \in M,\ r \in R,\ s \in S.$$

Thus for an (R,S)-bimodule one can unambiguously write rxs without parentheses.

DEFINITION 33.25. Let R be a ring, and let M be a right R-module and let N be a left R-module. We define the **tensor product over** R of M and N, written $M \otimes_R N$ to be the quotient of $M \otimes N$ by all relations of the form $(xr, y) = (x, ry)$ for $r \in R$, $x \in M$ and $y \in N$.

If M is a right R-module then one can show that $M \otimes_R \square \colon {}_R\mathsf{Mod} \to \mathsf{Ab}$ is a functor, and similarly $\square \otimes_R N \colon \mathsf{Mod}_R \to \mathsf{Ab}$ is a functor for each fixed left R-module N. If M is an (R,S)-bimodule then $M \otimes_S \square$ is actually a functor $\mathsf{Mod}_S \to {}_R\mathsf{Mod}$.

If $R = \mathbb{Z}$ then $M \otimes_{\mathbb{Z}} N$ is just the normal tensor product $M \otimes N$. But in general $M \otimes_R N$ really is quotient of $M \otimes N$. For example, if $R = \mathbb{Q}(\sqrt{2})$ then $R \otimes_R R = R$ but $R \otimes R$ is a four-dimensional vector space over $\mathbb{Q}$.

Suppose now for simplicity that R is commutative. Then left R-modules and right R-modules and (R,R)-bimodules are the same thing, and we just call them all "R-modules". From now on we will only talk about the commutative case, since we are only interested in the cohomology ring of a topological space, and as we have seen, this is most interesting when the coefficient ring is commutative, since then the cohomology ring is a commutative graded ring.

Indeed, if R is a commutative ring and X is a topological space, then we can view the cohomology ring $H^\star(X;R)$ as an R-module. This means that given two topological spaces, we can take the tensor product $H^\star(X;R) \otimes_R H^\star(Y;R)$, which in general is *not* the same as the normal tensor product $H^\star(X;R) \otimes H^\star(Y;R)$. Nevertheless, the analogue of Theorem 32.8 holds: the cross product defines a (graded) R-module homomorphism

$$H^\star(X;R) \otimes_R H^\star(Y;R) \xrightarrow{\times} H^\star(X \times Y;R). \tag{33.2}$$

One can then repeat the arguments from the last lecture to show that (33.2) is a homomorphism of graded rings.

The aim of the rest of this lecture is to prove a result analogous to Corollary 32.9. Unfortunately this is quite a bit harder than the (two line) proof of Corollary 32.9—the reason being that the proof we gave of Corollary 32.9 used the Künneth Theorem for Cohomology (Theorem 29.16), and that theorem was proved only for $R = \mathbb{Z}$.

THEOREM 33.26. *Let X and Y be topological spaces and let R be a commutative ring. Assume that $H^n(Y;R)$ is a finitely generated free R-module for all $n \geq 0$. Then the cross product (33.2) is an isomorphism of graded rings.*

We will eventually give three (!) different proofs of Theorem 33.26 (two this lecture, and a third next lecture, cf. Corollary 34.13). The first method only allows us to prove a weak version of Theorem 33.26, but it has the advantage of being closer to methods we understand.

Definition 33.27. A **principal ideal domain** R is a non-zero commutative ring with the following two additional properties:

1. If r, s are two non-zero elements of R then the product rs is also non-zero.
2. If $I \subset R$ is an ideal then there exists a single element $x \in I$ that generates I in the sense that $I = R \cdot x$.

Principal ideal domains are a useful class of commutative rings. They share many properties with the integers: for the instance, the analgoue of the fundamental theorem for finitely generated abelian groups holds for finitely generated modules over a principal ideal domain. We however will only need one fact: if R is a principal ideal domain and M is a free module, then any submodule of M is also free (this should be compared to the fact that any subgroup of a free abelian group is itself free.)

Proposition 33.28. *Let R be a principal ideal domain and let M be any R-module. Then there exists a short exact sequence $0 \to K \to F \to M \to 0$ where K and F are free R-modules.*

Proof. Exactly the same as Proposition 24.12 ■

For principal ideal domains the definition of Tor is almost the same:

Definition 33.29. Suppose R is a commutative ring and M and N are R-modules. There is a version of Tor, which we denote by $\operatorname{Tor}_1^R(M, N)$, which is defined as

$$\operatorname{Tor}_1^R(M, N) = \ker(f \otimes_R \mathrm{id}_N),$$

where $0 \to K \xrightarrow{f} F \to M \to 0$ is a short exact sequence of R-modules and K, F are free R-modules.

We won't go into the properties of Tor_1^R, save to say that if M is a free R-module then $\operatorname{Tor}_1^R(M, N) = \operatorname{Tor}_1^R(N, M) = 0$ for any N.

Remark 33.30. The reason for the subscript 1 in Tor_1^R is because for rings that are not principal ideal domains, the definition of Tor^R is more complicated, and there end up being higher Tor-groups Tor_n^R for $n \geq 2$.

For principal ideal domains, the proof we gave of the Künneth Formula for Cohomology (Theorem 29.16) goes through with almost no changes. This gives us:

Theorem 33.31 (The Cohomology Künneth Formula for Principal Ideal Domains). *Let R be a principal ideal domain. Suppose X and Y are any two topological spaces of finite type. Then for every $n \geq 0$, there is a split short exact sequence*

$$0 \to \bigoplus_{i+j=n} H^i(X; R) \otimes_R H^j(Y; R) \xrightarrow{\times} H^n(X \times Y; R) \to \bigoplus_{k+l=n+1} \operatorname{Tor}_1^R\big(H^k(X; R), H^l(Y; R)\big) \to 0.$$

REMARK 33.32. There is a version of the Künneth Formula for general rings, but it is much harder: the presence of higher Tor-groups means that one ends up with a *spectral sequence*. This is beyond the scope of this course.

Theorem 33.31 immediately allows one to prove Theorem 33.26, if we make the additional assumption that R is a principal ideal domain and X and Y are both of finite type. Indeed, the first map in Theorem 33.31 is the cross product (cf. Remark 31.13), and thus if $H^n(Y;R)$ is free for every n implies that all the relevant Tor_1^R groups in the Künneth Formula vanishes, and thus

$$\bigoplus_{i+j=n} H^i(X;R) \otimes_R H^j(Y;R) \xrightarrow{\times} H^n(X \times Y;R).$$

This completes the proof of Theorem 33.26 under the additional assumption that R is a principal ideal domain and X and Y are both of finite type. ∎

The assumption that X and Y are of finite type is harmless, but restricting to principal ideal domains can be annoying, so we will now outline a different argument that avoids this. This argument is quite nice, as it uses uniqueness of Eilenberg-Steenrod generalised (co)homology theories (the cohomological variant of Theorem 21.12.) In the proof, we will need the relative version of the two products. The relative cup product is slightly harder, and I will leave this for you on Problem Sheet O.

PROPOSITION 33.33. *Let X be a topological space and let X', X'' be open subsets of X. Let R be a ring. The cup product induces a relative product*

$$H^\star(X, X';R) \otimes H^\star(X, X'';R) \xrightarrow{\smile} H^\star(X, X' \cup X'';R). \tag{33.3}$$

Moreover if $i \colon (X, \emptyset) \hookrightarrow (X, X')$ is an inclusion then and $\langle \alpha \rangle \in H^m(X;R)$ and $\langle \beta \rangle \in H^m(X, X';R)$ then

$$\langle \alpha \rangle \smile H^m(i)\langle \beta \rangle = H^{n+m}(i)(\langle \alpha \rangle \smile \langle \beta \rangle), \tag{33.4}$$

where the left-hand side is the normal cup product in X, and the right-hand side is the relative cup product $H^\star(X;R) \otimes H^\star(X, X';R) \xrightarrow{\smile} H^\star(X, X';R)$.

The relative cross product is much easier (using the known relation between the cup and cross products, and the fact that we already did[3] the relative cup product.)

DEFINITION 33.34. If $X' \subseteq X$ and $Y' \subseteq Y$ are open subsets then for any ring R there is a **relative cross product**

$$H^\star(X, X';R) \otimes_R H^\star(Y, Y';R) \xrightarrow{\times} H^\star(X \times Y, X' \times Y \cup X \times Y';R).$$

This is defined in exactly the same way as Remark 32.10: if $\langle \alpha \rangle \in H^n(X, X';R)$ and $\langle \beta \rangle \in H^m(Y, Y';R)$, then $H^n(\pi')\langle \alpha \rangle \in H^n(X \times Y, X' \times Y;R)$ and $H^m(\pi'')\langle \beta \rangle \in H^m(X \times Y, X \times Y';R)$, where π' and π'' are the two projections. We then use the relative cup product (33.3) and set

$$\langle \alpha \rangle \times \langle \beta \rangle := H^n(\pi')\langle \alpha \rangle \smile H^m(\pi'')\langle \beta \rangle \in H^{n+m}(X \times Y, X' \times Y \cup X \times Y';R).$$

[3]More precisely: we already left as an exercise.

Proof of Theorem 33.26. We will prove the result in three steps. Let us first assume that Y is a finite cell complex Y (in this case the hypothesis that $H^\bullet(Y;R)$ is finitely generated is automatic.)

1. In this first step, we define two generalised cohomology theories with coefficients in R. We set

$$\mathcal{H}^n(X,X') := \bigoplus_i H^i(X,X';R)\otimes_R H^{n-i}(Y;R)$$

and

$$\mathcal{K}^n(X,X') = H^n(X\times Y, X'\times Y;R).$$

Let us check the Eilenberg-Steenrod axioms (we will construct the connecting homomorphisms when discussing the long exact sequence axiom):

1. *The homotopy axiom:* this is clear for both $\mathcal{H}^\bullet$ and $\mathcal{K}^\bullet$.
2. *The long exact sequence axiom:* This is clear for $\mathcal{K}^\bullet$, but less so for $\mathcal{H}^\bullet$. Here are the details. Begin with the long exact sequence in singular cohomology with coefficients in R:

   $$\cdots\to H^n(X,X';R)\to H^n(X;R)\to H^n(X';R)\xrightarrow{\delta} H^{n+1}(X,X';R)\dots$$

   Now fix $k\geq 0$ and tensor every term with the free R-module $H^k(Y;R)$. The sequence remains exact, since $H^k(Y;R)$ is just a direct sum of copies of R. Now to get the desired long exact sequence for $\mathcal{H}^\bullet$, we simply add various shifted sequences of this form together.
3. *The excision axiom* This is obvious for $\mathcal{H}^\bullet$: namely, if X_1 and X_2 are subspaces of X such that $X = X_1^\circ\cup X_2^\circ$ then $\mathcal{H}^n(X,X_1)\cong\mathcal{H}^n(X_2,X_1\cap X_2)$. For $\mathcal{K}^\bullet$ this is also clear, since $(X_1\times Y)\cup(X_2\times Y) = X\times Y$ and $(X_1\times Y)\cap(X_2\times Y) = (X_1\cap X_2)\times Y$.
4. *The "dimension" axiom:* Both $\mathcal{H}^\bullet$ and $\mathcal{K}^\bullet$ are generalised cohomology theories, and so do not satisfy the dimension axiom, but it is important for us later to check that they both do the same thing to a point. For this note that if $X=\{*\}$ is a one-point space then $\mathcal{H}^n(*) = R\otimes_R H^n(Y;R)$ and $\mathcal{K}^n(*) = H^n(Y;R)$.

2. We now construct a natural transformation $\Phi\colon\mathcal{H}^\bullet\to\mathcal{K}^\bullet$. We take Φ to be the relative cross product. I leave it up to check that the following diagram commutes:

$$\begin{array}{ccc}\mathcal{H}^n(X') & \xrightarrow{\delta} & \mathcal{H}^{n+1}(X,X')\\ \times\downarrow & & \downarrow\times\\ \mathcal{K}^n(X') & \xrightarrow[\delta]{} & \mathcal{K}^{n+1}(X,X')\end{array}$$

and thus Φ is a natural transformation. Moreover $\Phi^n(*)\colon\mathcal{H}^n(*)\to\mathcal{K}^n(*)$ is an isomorphism, since it is just the scalar multiplication map $R\otimes_R H^n(Y;R)\to H^n(Y;R)$. Thus by a slight generalisation[4] of Theorem 21.12, we conclude that Φ is an isomorphism on any pair (X,X'), where X is a finite cell complex and X' is a subcomplex.

[4]The only differences are that we are working in cohomology, and that the map $\Phi^n(*)\colon\mathcal{H}^n(*)\to\mathcal{K}^n(*)$ needs to be explicitly checked to be an isomorphism in all degrees—in the setting of Theorem 21.12 this was automatic, since we were working with actual homology theories.

3. Suppose now that X and Y are infinite dimensional cell complexes. The same strategy of proof works, but now we need an upgraded version of Theorem 21.12 that also works for infinite dimensional cell complexes. We will discuss more a more general version of Theorem 21.12 in Lecture 46, but in this case the only difference in the infinite dimensional case is that we need to check explicitly that both $\mathcal{H}^\bullet$ and $\mathcal{K}^\bullet$ satisfy the additivity axion from Definition 21.9. This is clear for $\mathcal{K}^\bullet$, but less so for $\mathcal{H}^\bullet$, and this is where the assumption that $H^n(Y; R)$ is (free and) finitely generated in each degree is used (see (34.1) in the next lecture for more details.)

4. We now complete the proof. As mentioned in Lectures 18 and 21, one can always "approximate" an arbitrary topological space by a cell complex in such a way that the singular homology does not change. We will state this precisely in Theorem 46.15. Thus it suffices to prove the theorem when X and Y are cell complexes, and we already did this in the last step. This completes the proof. ■

The Leray-Hirsch Theorem

In this lecture we generalise Theorem 33.26 to fibre bundles. This is the famous Leray-Hirsch Theorem. We begin with the following general result about fibre bundles, which we will need in the proof.

DEFINITION 34.1. Let $F \to E \xrightarrow{p} X$ be a fibre bundle and let $f\colon Y \to X$ be a continuous map. Let

$$E_f := \{(y, z) \in Y \times E \mid f(y) = p(z)\}.$$

The map $p_f\colon E_f \to Y$ given by $(y, z) \mapsto y$ makes $F \to E_f \xrightarrow{p_f} Y$ a fibre bundle. It is usually called the **pullback bundle**. The map $\varphi\colon E_f \to E$ given by $\varphi(y, z) = z$ makes the following diagram commute:

$$\begin{array}{ccc} E_f & \xrightarrow{\varphi} & E \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f]{} & X \end{array}$$

THEOREM 34.2. *Let $F \to E \xrightarrow{p} X$ be a fibre bundle. Let Y be a paracompact Hausdorff space, and let $f, g\colon Y \to X$ be two homotopic maps. Then the two fibre bundles p_f and p_g are isomorphic.*

COROLLARY 34.3. *Let $F \to E \xrightarrow{p} X$ be a fibre bundle over a contractible paracompact Hausdorff base space X. Then p is a trivial fibre bundle. That is, there exists a homeomorphism $h\colon E \to X \times F$ such that $p = \pi' \circ h$, where π' is the first projection.*

The proof of Theorem 34.2 starts with the following lemma.

LEMMA 34.4. *Any fibre bundle $F \to E \xrightarrow{p} I$ is trivial.*

Proof. Since p is locally trivial, for N large enough, p is trivial over each interval $I_i := [i/N, (i+1)/N]$. Let $h_i\colon I_i \times F \to p^{-1}(I_i)$ be a trivialisation. Now set

$$\eta_i\colon F \to F, \qquad \eta_i(y) := \pi'' \circ h_i^{-1} \circ h_{i-1}(y, i/n)$$

for $i \geq 1$ (where π'' is the second projection.) Then define

$$\tilde{\eta}_i := \eta_i \circ \cdots \circ \eta_1$$

and $\tilde{\eta}_0 := \mathrm{id}_F$. Finally, define

$$h\colon F \times I \to E, \qquad h(y, t) := h_i(\tilde{\eta}_i(y), t), \qquad t \in I_i.$$

By construction, h is continuous, and is our desired trivialisation. ■

The same proof shows that if we are given a fibre bundle over $U \times I$, then a collection of trivialisations over $U \times I_i$ can be glued together to give a global trivialisation. Now let us sketch the proof of the following result.

PROPOSITION 34.5. *Let X be paracompact and Hausdorff. Let $\imath_0$ and $\imath_1$ denote the two maps $X \hookrightarrow X \times I$ given by $\imath_i(x) = (x, i)$. Suppose $F \to E \xrightarrow{p} X \times I$ is a fibre bundle, and let p_i denote the pullback bundle over X associated to $\imath_i$. Then p_0 and p_1 are isomorphic.*

Proof (Sketch). Let $\{U_\lambda\}$ be locally finite cover and let N be large enough so that p is trivial over $U_\lambda \times I_i$, where as before $I_i = [i/N, (i+1)/N]$. By the paragraph above the statement of the proposition, we can construct trivialisations

$$h_\lambda \colon U_\lambda \times F \times I \to E|_{U_\lambda \times I}.$$

The goal now is to glue all these trivialisations together. The idea is roughly similar to the one used in Lemma 34.4 above, only now one needs to work a little harder to ensure the resulting map really is continuous. The trick is to use a *partition of unity* subordinate to the locally finite cover $\{U_\lambda\}$. This falls somewhat out of the remit of this course, and so I will omit the details. ∎

We can now prove Theorem 34.2. The argument is the same as the one used to prove Theorem 8.9 from Proposition 8.5.

Proof of Theorem 34.2. Let $G \colon f \simeq g$ be a homotopy. Then we have a fibre bundle $F \to E \xrightarrow{p_G} Y \times I$. The fibre bundles p_f and p_g are the pullback bundles of p_G under the two maps $\imath_0$ and $\imath_1$ respectively. Thus the result follows from Proposition 34.5. ∎

Let us now get started on the Leray-Hirsch Theorem. For this we need to introduce some notation. Let $F \to E \xrightarrow{p} X$ denote a fibre bundle, and let R denote a ring. Given $x \in X$ we denote by $\imath_x \colon E_x \hookrightarrow E$ the inclusion of the fibre.

DEFINITION 34.6. A **cohomology extension of the fibre** is a homomorphism $\xi \colon H^\bullet(F; R) \to H^\bullet(E; R)$ such that for every $x \in X$ and $n \geq 0$, the composition

$$H^n(F; R) \xrightarrow{\xi} H^n(E; R) \xrightarrow{H^n(\imath_x)} H^n(E_x; R)$$

is an isomorphism.

THEOREM 34.7 (The Leray-Hirsch Theorem). *Let $F \to E \xrightarrow{p} X$ be a fibre bundle and let R be a commutative ring. Assume that $H^n(F; R)$ is a finitely generated free R-module for all $n \geq 0$, and that a cohomology extension ξ of the fibre exists. Then the map*

$$L \colon H^\star(X; R) \otimes_R H^\star(F; R) \to H^\star(E; R)$$

given by

$$L \colon \langle \alpha \rangle \otimes \langle \beta \rangle \mapsto H^\star(p)\langle \alpha \rangle \smile \xi\langle \beta \rangle$$

is an isomorphism.

Remark 34.8. The assumption that a cohomology extension exists is a necessary one. Indeed, consider the Hopf fibration $S^1 \to S^3 \to S^2$ and take $R = \mathbb{Z}$. Then $H^n(S^1)$ is finitely generated and free for all $n \geq 0$, but $H^\star(S^2) \otimes H^\star(S^1)$ is not isomorphic to $H^\star(S^3)$. Thus by the Leray-Hirsch Theorem, no cohomology extension can exist.

Remark 34.9. Theorem 34.7 is *not* asserting that L is an isomorphism of rings, and indeed this is not always the case.

The Leray-Hirsch Theorem is only useful if a cohomology extension of the fibre exists. In the next lecture we will show that such an extension always exists when p is an *orientable sphere bundle*.

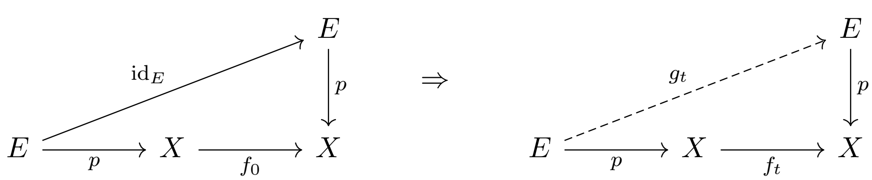
We will need the following preliminary result for the proof of the Leray-Hirsch Theorem.

Lemma 34.10. *Let $p\colon E \to X$ be a fibration. Assume $X' \subset X$ is a subspace which X strongly deformation retracts onto, in the sense that there exists a homotopy $f_t\colon X \to X$ such that*

$$f_0 = \mathrm{id}_X, \qquad f_1(X) \subset X', \qquad f_t(X') \subset X', \qquad \forall\, t \in I.$$

Then the inclusion $\imath\colon p^{-1}(X') \hookrightarrow E$ is a homotopy equivalence.

Proof. By the homotopy lifting property with respect to E, there exists a homotopy $g_t\colon E \to E$ such that $g_0 = \mathrm{id}_E$ and such that $f_t \circ p = p \circ g_t$ for each $t \in [0, 1]$, as the following picture indicates:



We claim that g_1 is a homotopy inverse to $\imath$. Indeed, going in one direction we have $\imath \circ g_1 = g_1 \simeq \mathrm{id}_E$. Going in the other direction, since $g_t(p^{-1}(X')) \subseteq p^{-1}(X')$ for all $t \in I$, we see that $g_1|_{p^{-1}(X')} \simeq \mathrm{id}_{p^{-1}(X')}$ (as maps $p^{-1}(X') \to p^{-1}(X')$), and thus also $g_1 \circ \imath \simeq \mathrm{id}_{p^{-1}(X')}$. ■

We now move onto the proof of Theorem 34.7. We will be completely rigorous for the case when X is a finite dimensional cell complex, and then somewhat sketchier for the general case.

Proof of Theorem 34.7. We begin with more notation. Given $Y \subseteq X$, denote by $E_Y := p^{-1}(Y)$. We denote by ξ_Y the composite

$$H^\bullet(F; R) \xrightarrow{\xi} H^\bullet(E; R) \xrightarrow{H^\bullet(\imath_Y)} H^\bullet(E_Y; R),$$

where $\imath_Y\colon E_Y \hookrightarrow E$ is the inclusion. Abbreviate by M the graded free module $H^\star(F; R)$. Set

$$L_Y\colon H^\star(Y; R) \otimes_R M \to H^\star(E_Y; R)$$

the map defined by

$$L_Y\colon \langle\alpha\rangle \otimes \langle\beta\rangle \mapsto H^\star(p)\langle\alpha\rangle \smile \xi_Y\langle\beta\rangle$$

Thus our goal is to show that for $Y = X$, the map L_X is an isomorphism. We will prove the theorem in four steps.

1. We first prove the result in the special case where X is paracompact and pointed contractible[1]. In this case, by Corollary 34.3, we may assume that $E = X \times F$, and moreover by Lemma 34.10 there exists $x \in X$ such that the inclusion $\imath_x\colon E_x \cong F \hookrightarrow E$ is a homotopy equivalence. Thus $H^\bullet(\imath_x)$ is an isomorphism. Since $H^\bullet(\imath_x) \circ \xi$ is an isomorphism by assumption, it follows that $\xi\colon H^\bullet(F;R) \to H^\bullet(E;R)$ is also an isomorphism. We then have the following commutative diagram:

$$\begin{array}{ccc} H^\star(X;R) \otimes H^\star(F;R) & \xrightarrow{L} & H^\star(E;R) \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ H^\star(F;R) & \xrightarrow[\xi]{} & H^\star(E;R) \end{array}$$

The left-hand vertical map is an isomorphism since $H^\star(X;R) \cong R$ as X is contractible, and so the cup product reduces to scalar multiplication map $R \otimes_R H^n(F;R) \to H^n(F;R)$. Thus L is an isomorphism.

2. We now prove the theorem for X a finite dimensional cell complex, by inducting on their dimension. If X is a point then the claim follows from Step (1). Suppose now the theorem holds for any cell complex of dimension $n-1$, including the $(n-1)$-skeleton X^{n-1}. Let us write $X^n = U \cup V$, where U is obtained from X^n by deleting a single point in each n-cell, and V is the union of the n-cells. We then have a commutative diagram of Mayer-Vietoris sequences:

$$\begin{array}{ccc} H^\star(X^n;R) \otimes_R M & \xrightarrow{L_{X^n}} & H^\star(E_{X^n};R) \\ \big\downarrow & & \big\downarrow \\ (H^\star(U;R) \otimes_R M) \oplus (H^\star(V;R) \otimes_R M) & \xrightarrow{(L_U, L_V)} & H^\star(E_U;R) \oplus H^\star(E_V;R) \\ \big\downarrow & & \big\downarrow \\ H^\star(U \cap V;R) \otimes_R M & \xrightarrow[L_{U\cap V}]{} & H^\star(E_{U\cap V};R) \end{array}$$

The left-hand column is the tensor product of the Mayer-Vietoris sequence for (U,V), tensored with the free module M. It remains exact, since as we have seen, tensoring with a free module is an exact functor. We will prove that in Step (3) below that L_U, L_V and $L_{U\cap V}$ are isomorphisms. Then the Five Lemma (Proposition 11.3) tells us that L_{X^n} is an isomorphism, which thus completes the inductive step.

[1]That is, X can be contracted to a basepoint by a homotopy that fixes the basepoint. Thanks to O. Edtmair for simplfying this argument.

3. Let's start with L_U. We have a commutative diagram:

$$\begin{array}{ccc} H^\star(U;R)\otimes_R M & \xrightarrow{L_U} & H^\star(E_U;R) \\ \downarrow & & \downarrow \\ H^\star(X^{n-1};R)\otimes_R M & \xrightarrow[L_{X^{n-1}}]{} & H^\star(E_{X^{n-1}};R) \end{array}$$

The left-hand vertical map is an isomorphism, since X^{n-1} is a strong deformation retract of U (this is a special case of Proposition 19.8). By Lemma 34.10 (which is applicable due to Theorem 33.17), E_U is homotopy equivalent to E_{X^n}, and hence the right-hand vertical map is also an isomorphism. By induction $L_{X^{n-1}}$ is an isomorphism, and hence so is L_U.

Now let's do the case of L_V. By assumption, $V = \bigsqcup E^n_\lambda$ for some n-cells E^n_λ (possibly uncountably many). We obtain a commutative diagram

$$\begin{array}{ccc} H^\star(V;R)\otimes_R M & \xrightarrow{L_V} & H^\star(E_V;R) \\ \downarrow & & \downarrow \\ \left(\prod H^\star(E^n_\lambda;R)\right)\otimes_R M & & \prod H^\star(E_{E^n_\lambda};R) \\ \downarrow & & \downarrow \\ \left(\prod H^\star(E^n_\lambda;R)\otimes_R M\right) & \xrightarrow[\prod L_{E^n_\lambda}]{} & \prod H^\star(E_{E^n_\lambda};R) \end{array}$$

All the vertical maps are isomorphisms: the two top are isomorphisms due to the additivity of cohomology. The bottom left is an isomorphism due to the fact that if Q is a finitely generated free R-module and N_λ are an arbitrary collection of R-modules then

$$\left(\prod N_\lambda\right)\otimes_R Q \cong \prod (N_\lambda \otimes_R Q) \tag{34.1}$$

It thus suffices to show that $L_{E^n_\lambda}$ is an isomorphism, but this follows from Step (1).

Finally, consider $L_{U\cap V}$. By arguing as above, this reduces to considering $L_{U\cap E^n_\lambda}$ for a given n-cell E^n_λ. Such a set $U\cap E^n_\lambda$ has as a strong deformation retract a sphere S^{n-1}_λ:

$$\begin{array}{ccc} H^\star(U\cap E^n_\lambda;R)\otimes_R M & \xrightarrow{L_{U\cap E^n_\lambda}} & H^\star(E_{U\cap E^n_\lambda};R) \\ \downarrow & & \downarrow \\ H^\star(S^{n-1}_\lambda;R)\otimes_R M & \longrightarrow & H^\star(E_{S^{n-1}_\lambda};R) \end{array}$$

As in Step (2), the two vertical maps are isomorphisms (using Lemma 34.10 again). But S^{n-1}_λ is a cell complex of dimension $n-1$, and hence the bottom horizontal map is an isomorphism by induction.

4. We have now proved the result for all finite-dimensional cell complexes. Let us briefly outline how to prove it for an infinite-dimensional cell complex.

Let X be a cell complex with skeleton filtration (X^n). Then a cohomological version of Proposition 20.3 tells us that the inclusion $X^n \hookrightarrow X$ induces an isomorphism $H^i(X;R) \to H^i(X^n;R)$ for $i \leq n$. Similarly if $F \to E \xrightarrow{p} X$ is a fibre bundle then $H^i(E;R) \cong H^i(E_{X^n};R)$ for $i \leq n$. This isn't hard to prove, but it uses terminology we haven't introduced yet, and hence for now we will take it on faith and come back to justify it when we start homotopy theory towards the end of the course.

From this is is easy to deduce Theorem 34.7 for X an infinite dimensional cell complex. Fix $n \geq 0$, and let X^n denote the n-skeleton of X. Then we have a commutative diagram

$$\begin{array}{ccc} H^\star(X;R) \otimes_R H^\star(F;R) & \longrightarrow & H^\star(X^n;R) \otimes_R H^\star(F;R) \\ \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L_{X^n}} \\ H^\star(E;R) & \longrightarrow & H^\star(E_{X^n};R) \end{array}$$

By assumption the horizontal maps are isomorphisms in degree $i \leq n$. The right-hand vertical map is an isomorphism by Step (3). Thus the left-hand map is an isomorphism in degree $i \leq n$. Since n was arbitrary, the theorem is proved for infinite dimensional cell complexes.

5. Let us sketch how to extend the proof to arbitrary topological spaces X. Just like Theorem 33.26 last lecture, this requires us to use a cellular approximation to X. So assume there exists a cell complex Y and a continuous map $f \colon Y \to X$ such that f induces an isomorphism on all (co)homology groups (we will construct such a pair (Y, f) at the end of the course). Let $p_f \colon E_f \to Y$ denote the pullback bundle from Definition 34.1. Define $\varphi \colon E_f \to E$ by $\varphi(y, z) = z$, so that the following diagram commutes:

$$\begin{array}{ccc} E_f & \xrightarrow{\varphi} & E \\ \downarrow{\scriptstyle p_f} & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f]{} & X \end{array}$$

The fact that f induces an isomorphism on all cohomology groups implies that φ does too—this is an immediate corollary of the long exact sequence of homotopy groups associated to a fibration (Corollary 45.18.) It is easy to see that the cohomology extension ξ induces one in E_f, and thus $F \to E_f \to Y$ satisfies the hypotheses of the theorem. By Step (4), we obtain an isomorphism

$$L \colon H^\star(Y;R) \otimes_R H^\star(F;R) \to H^\star(E_f;R)$$

A naturality argument gives commutativity of the following diagram:

$$\begin{array}{ccc} H^\star(X;R) \otimes_R H^\star(F;R) & \longrightarrow & H^\star(Y;R) \otimes_R H^\star(F;R) \\ \downarrow{\scriptstyle L} & & \downarrow{\scriptstyle L} \\ H^\star(E;R) & \longrightarrow & H^\star(E_f;R) \end{array}$$

The horizontal maps are isomorphisms by the discussion above. Thus the left-hand L is also an isomorphism. This completes the proof. ■

Remark 34.11. We proved in Theorem 16.22 that homology commutes with colimits, and we used this in Proposition 18.17 to note that the homology of a cell complex X is the colimit of the homology of its skeleta X^n. Let us briefly say a few words about the corresponding situation in cohomology. (This is purely for interest—it has no relevance to the rest of this lecture!)

When working with cohomology, instead of colimits, we need to take *limits*[2]. Limits are defined analogously to how we defined colimits in Lecture 16, by appropriately reversing the arrows. I invite you to guess the correct definition (We will do this formally in Lecture 39.) This gives us a group $\varprojlim H^\bullet(X^n;R)$ (note the arrow is pointing to the left). One might hope that the $H^i(X;R) = \varprojlim H^i(X^n;R)$, but unfortunately in general this isn't true. This is because limits are less well behaved than colimits, and are not exact functors from diagrams of abelian groups to diagrams of abelian groups. Consequently one needs to worry about the first *derived* functor of $\varprojlim$, denoted by $R^1\varprojlim$. Here is a correct statement:

Proposition 34.12. *Suppose we are given a family $i_n\colon X_n \to X_{n+1}$ for $n \in \mathbb{N}$ of closed inclusions. Let A be an abelian group. Assume in addition that for each n the space X_n is weakly Hausdorff. Let $X = \bigcup_n X_n$, endowed with the colimit topology. Then there is a natural short exact sequence*

$$0 \to R^1 \varprojlim_n \left(H^{i-1}(X_n;A)\right) \to H^i(X;A) \to \varprojlim_n \left(H^i(X^n;A)\right) \to 0.$$

Sadly we don't have time to define the derived functor $R^1\varprojlim$ properly, or to prove Proposition 34.12.

One can view Theorem 33.26 as a special case of the Leray-Hirsch Theorem. Let us recall the statement:

Corollary 34.13. *Let X and Y be topological spaces and let R be a commutative ring. Assume that $H^n(Y;R)$ is a finitely generated free R-module for all $n \geq 0$. Then the cross product (33.2) is an isomorphism of graded rings.*

Proof. We view $X \times Y \to X$ as a fibre bundle via the first projection. The second projection $X \times Y \to Y$ gives us a cohomology extension of the fibre. Theorem 34.7 is thus applicable, and the result immediately follows. ∎

Let us now discuss a stronger "relative" version of Theorem 34.7, which will also be useful. For this we need another definition.

Definition 34.14. A **fibre-bundle pair**, written $(F,F') \to (E,E') \xrightarrow{p} X$, consists of a continuous map $p\colon E \to X$ with the property that there exists an open cover $\{U_\lambda\}$ of X and homeomorphisms of pairs

$$h_\lambda\colon \left(U_\lambda \times F, U_\lambda \times F'\right) \to \left(p^{-1}(U_\lambda), p^{-1}(U_\lambda) \cap E'\right)$$

such that $p|_{p^{-1}(U_\lambda)} \circ h_\lambda = \pi'$ for each λ, where as usual π' is the projection onto the first factor.

[2] Yes, this is one example where the terminology is somehow "backwards"... Don't blame me!

Equivalently, a fibre bundle pair $(F, F') \to (E, E') \xrightarrow{p} X$ consists of a fibre bundle $F \to E \xrightarrow{p} X$ together with a subspace $E' \subseteq E$ and a subspace $F' \subseteq F$ such that $F' \to E' \xrightarrow{p|_{E'}} X$ is itself a fibre bundle, and moreover such that local trivialisations of $p|_{E'}$ can be obtained by restricting local trivialisations of p.

Example 34.15. A **disk bundle** $B^n \to E \xrightarrow{p} X$ is a fibre bundle with fibre a ball B^n. Let E' denote the union of the boundary spheres of the fibres and let $p' = p|_{E'}$. Then $S^{n-1} \to E' \xrightarrow{p'} X$ is a **sphere bundle** (i.e. a fibre bundle with fibre a sphere), and moreover (E, E') is a fibre bundle pair since local trivialisations of E restrict to local trivialisations of E' (this is because a homeomorphism $B^n \to B^n$ restricts to define a homeomorphism $S^{n-1} \to S^{n-1}$.). We call p' the **boundary sphere bundle** of the disk bundle p.

Remark 34.16. Let $S^n \to E \xrightarrow{p} X$ denote a sphere bundle. Let

$$Z := \Big((E \times I) \sqcup X \Big) \Big/ \sim$$

where we identify $(y, 0) \in E \times I$ with $p(y)$. There is a natural map $f\colon Z \to X$ whose fibre over $x \in X$ is $Z_x = (E_x \times I)/(E_x \times \{0\})$. This is a cone over a sphere, and hence $Z_x \cong B^{n+1}$. Thus $B^{n+1} \to Z \xrightarrow{f} X$ is a disk bundle. Moreover, identifying $E_x \cong E_x \times \{1\}$, we see that $(B^{n+1}, S^n) \to (Z, E) \xrightarrow{f} X$ is a fibre bundle pair with $f|_E = p$. Thus any sphere bundle can be seen as the boundary sphere bundle of a disk bundle.

Definition 34.17. Let $(F, F') \to (E, E') \xrightarrow{p} X$ denote a fibre bundle pair and let R denote a ring. A **cohomology extension of the fibre** is again a homomorphism $\xi\colon H^\bullet(F, F'; R) \to H^\bullet(E, E'; R)$ such that for every $x \in X$ and $n \geq 0$, the composition

$$H^n(F, F'; R) \xrightarrow{\xi} H^n(E, E'; R) \xrightarrow{H^n(\imath_x)} H^n(E_x, E'_x; R)$$

is an isomorphism, where $E'_x = p^{-1}(x) \cap E'$.

Here is the Relative Leray-Hirsh Theorem. It includes the (absolute) Leray-Hirsch Theorem 34.7 as a special case (take $E' = F' = \emptyset$.) The proof is almost exactly the same as the absolute case, and I omit the details.

Theorem 34.18 (The Relative Leray-Hirsch Theorem). *Let $(F, F') \to (E, E') \xrightarrow{p} X$ be a fibre bundle pair and let R be a commutative ring. Assume that $H^n(F, F'; R)$ is a finitely generated free R-module for all $n \geq 0$, and that a cohomology extension ξ of the fibre exists. Then the map*

$$L\colon H^\star(X; R) \otimes_R H^\star(F, F'; R) \to H^\star(E, E'; R)$$

given by

$$L\colon \langle\alpha\rangle \otimes \langle\beta\rangle \mapsto H^\star(p)\langle\alpha\rangle \smile \xi\langle\beta\rangle$$

is an isomorphism.

LECTURE 35

The Thom Isomorphism Theorem

There are many important applications of the Leray-Hirsch Theorem 34.7 and its relative version Theorem 34.18. However since we only have finitely many lectures left, we will immediately simplify the picture and restrict our attention to disk bundles.

Let $(B^n, S^{n-1}) \to (E, E') \xrightarrow{p} X$ be a fibre bundle pair. Since

$$H^i(B^n, S^{n-1}; R) \cong \begin{cases} R, & i = n, \\ 0, & i \neq n, \end{cases}$$

we see that a cohomology extension ξ of the fibre is the same thing as specifying a *single* class $t \in H^n(E, E'; R)$:

$$t := \xi(r^*),$$

where $r^* \in R \cong H^n(B^n, S^{n-1}; R)$ is any generator of R (i.e. any element with a multiplicative inverse). If $R = \mathbb{Z}_2$, then there is only one such element r^*, but for other rings there may be more (for instance, if $R = \mathbb{Z}$ then both 1 and -1 work.) The class t then has the property that $H^n(\imath_x)(t)$ is a generator $H^n(E_x, E'_x; R)$ for every $x \in X$. Such a class t has a special name (which historically predates the notion of a cohomology extension of the fibre).

Definition 35.1. Let $(B^n, S^{n-1}) \to (E, E') \xrightarrow{p} X$ be a fibre bundle pair and let R be a commutative ring. A **Thom class** for p is a cohomology class $t \in H^n(E, E'; R)$ with the property that $H^n(\imath_x)(t)$ is a generator of $H^n(E_x, E'_x; R)$ for every $x \in X$.

We then have:

Lemma 35.2. *Let $(B^n, S^{n-1}) \to (E, E') \xrightarrow{p} X$ be a fibre bundle pair. A Thom class exists if and only if there exists a cohomology extension of the fibre.*

Proof. We have just seen that a cohomology extension of the fibre determines a Thom class. Conversely, a choice of Thom class allows us to define a cohomology extension of the fibre by sending a generator of $H^n(B^n, S^{n-1}; R)$ to t and then extending by linearity (and setting ξ to be zero in all other degrees.) ■

The next result is just a rephrasing of the Relative Leray-Hirsch Theorem 34.18 in this special case.

Theorem 35.3 (The Thom Isomorphism Theorem). *Let $(B^n, S^{n-1}) \to (E, E') \xrightarrow{p} X$ be a fibre bundle pair and let R be a commutative ring. Suppose a Thom class t exists. Then the map $L\colon H^i(X; R) \to H^{i+n}(E, E'; R)$ given by*

$$L\colon \langle \alpha \rangle \mapsto H^i(p)\langle \alpha \rangle \smile t$$

is an isomorphism for all $i \geq 0$, and $H^i(E, E'; R) = 0$ for $i < n$.

The Thom Isomorphism Theorem 35.3 is completely useless unless one knows that a Thom class exists. We will now prove that for $R = \mathbb{Z}_2$, a Thom class always exists. Afterwards we discuss the case $R = \mathbb{Z}$, which is a bit more subtle.

Theorem 35.4. *Let $(B^n, S^{n-1}) \to (E, E') \xrightarrow{p} X$ be a fibre bundle pair. Then for $R = \mathbb{Z}_2$, a Thom class t always exists. Moreover t is unique.*

In the course of the proof, we will use the following result about cell complexes. The proof is not too hard (it is a variation of the argument used in Lecture 19 to show that cell complexes and their subcomplexes are "nice" pairs). Nevertheless, the ideas are somewhat tangential to the discussion at hand, and hence we will skip it.

Lemma 35.5. *Let X be a path connected cell complex of dimension n. Then X can be covered by $(n+1)$ open paracompact subsets that are contractible in X.*

Proof of Theorem 35.4. Just as with our proof of the Leray-Hirsch Theorem 34.7 last lecture, we will prove the result first when X is contractible, then for bundles where X is a finite-dimensional cell complex, then when X is an arbitrary cell complex, and finally the general case. This time round though we will be very sketchy in the infinite-dimensional case[1]. It suffices to prove the result when X is path-connected.

1. Suppose E is trivial (which by Corollary 34.3 is the case if X is paracompact and contractible). For $R = \mathbb{Z}_2$, there is a *unique* generator $\langle \gamma \rangle \in H^n(B^n, S^{n-1}; \mathbb{Z}_2)$. Consider now the map of pairs

$$(B^n, S^{n-1}) \xrightarrow{\imath_x} X \times (B^n, S^{n-1}) \xrightarrow{\pi''} (B^n, S^{n-1})$$

where $\imath_x(y) = (x, y)$ and π'' is the second projection. If $t := H^n(\pi'')\langle \gamma \rangle$ then it is clear that t is a Thom class, which moreover is unique (t is the image of $\nu_X \times \langle \gamma \rangle$ under Theorem 33.26, where $\nu_X \in H^0(X; \mathbb{Z}_2)$ is the unit class (30.2).) Let us also note that by Theorem 33.26 we have that

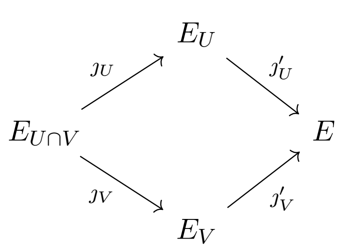
$$H^i(E, E'; \mathbb{Z}_2) = \bigoplus_{k+l=i} \left(H^k(X; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^l(B^n, S^{n-1}; \mathbb{Z}_2) \right),$$

and thus $H^i(E, E'; \mathbb{Z}_2) = 0$ for $i < n$.

2. Now suppose X is the union of two open sets U and V such that all three of $E_U = p^{-1}(U)$ and $E_V = p^{-1}(V)$ and $E_{U\cap V} = p^{-1}(U \cap V)$ admit a unique Thom class t_U, t_V and $t_{U\cap V}$ respectively and satisfy

$$H^i(E_U, E'_U; \mathbb{Z}_2) = H^i(E_V, E'_V; \mathbb{Z}_2) = H^i(E_{U\cap V}, E'_{U\cap V}; \mathbb{Z}_2) = 0, \qquad \text{for } i < n.$$

Consider the inclusions:



[1] In the exam, I will only expect you to know how to prove this in the finite-dimensional case!

Then we have a Mayer-Vietoris sequence

$$H^{i-1}(E_{U\cap V}, E'_{U\cap V}; \mathbb{Z}_2) \to H^i(E, E'; \mathbb{Z}_2) \to H^i(E_U, E'_U; \mathbb{Z}_2) \oplus H^i(E_V, E'_V; \mathbb{Z}_2) \to H^i(E_{U\cap V}, E'_{U\cap V}; \mathbb{Z}_2)$$

For $i < n$ the first and third groups are zero, and thus so is the second group: $H^i(E, E'; \mathbb{Z}_2) = 0$ for $i < n$. For $i = n$ the sequence is:

$$0 \to H^n(E, E'; \mathbb{Z}_2) \to H^n(E_U, E'_U; \mathbb{Z}_2) \oplus H^n(E_V, E'_V; \mathbb{Z}_2) \xrightarrow{h} H^n(E_{U\cap V}, E'_{U\cap V}; \mathbb{Z}_2)$$

Uniqueness of the Thom classes imply that

$$H^n(\jmath_U)(t_U) = H^n(\jmath_V)(t_V) = t_{U\cap V}. \tag{35.1}$$

The map h is defined by

$$h(\langle\alpha\rangle, \langle\beta\rangle) = H^n(\jmath_U)\langle\alpha\rangle - H^n(\jmath_V)\langle\beta\rangle,$$

and thus by (35.1) one has $h(t_U, t_V) = 0$. Thus by exactness, there is a unique element $t \in H^n(E, E'; \mathbb{Z}_2)$ such that

$$H^n(\jmath'_U)(t) = t_U, \qquad H^n(\jmath'_V)(t) = t_V.$$

Since t_U and t_V are Thom classes for E_U and E_V it follows that t is a Thom class for X.

3. It thus follows by by induction on m that if E can be written as the union of m trivial bundles then E itself has a unique Thom class, and moreover $H^i(E, E'; \mathbb{Z}_2) = 0$ for $i < n$. Indeed, Step (1) did the case $m = 1$, and if we assume that the result is true for $\leq m - 1$ and that $X = U_1 \cup \cdots \cup U_m$ where $E|_{U_i}$ is trivial for $i = 1, \ldots, m$, then we can apply the inductive hypothesis to all three of U_1 and $U_2 \cup \cdots \cup U_m$ and $(U_1 \cap U_2) \cup \cdots \cup (U_1 \cap U_m)$. Then Step (2) shows that a Thom class exists for E and that $H^i(E, E'; \mathbb{Z}_2) = 0$ for $i < n$.

This implies the theorem is true for X a finite-dimensional cell complex. Indeed, by Lemma 35.5, if X has dimension n, then we can write X as the union of $n + 1$ contractible open sets, and thus Corollary 34.3 implies that we can write E as the union of $n + 1$ trivial bundles.

4. Now let us sketch how this extends to an infinite dimensional cell complex X. If X^k is the kth skeleta, then $E_k := E_{X^k}$ admits a Thom class t_k. Now consider

$$(t_0, t_1, t_2, \ldots) \in \prod_k H^n(E_k, E'_k; \mathbb{Z}_2).$$

This gives rise to an element $t \in \varprojlim_k H^n(E_k, E'_k; \mathbb{Z}_2)$. Moreover since $H^{n-1}(E_k, E'_k; \mathbb{Z}_2) = 0$ by the previous step, one also has $R^1 \varprojlim_k H^{n-1}(E_k, E'_k; \mathbb{Z}_2) = 0$, and thus from Proposition 34.12 we have $H^n(E, E'; \mathbb{Z}_2) = \varprojlim_k H^n(E_k, E'_k; \mathbb{Z}_2)$. Thus t is an element of $H^n(E, E'; \mathbb{Z}_2)$, and this is our desired Thom class.

Finally, using our by now standard (but as yet, still unjustified!) step of approximating an arbitrary space by a cell complex, the general case follows, just as in the last step of Theorem 34.7 last lecture. ■

Now let us consider the case $R = \mathbb{Z}$. The previous proof goes wrong in exactly one place—namely, right at the beginning. There is no longer a *unique* choice of generator of $H^n(B^n, S^{n-1})$ (there are two possible choices). Since the argument repeatedly used uniqueness, the entire proof breaks down. This leads us to the concept of *orientability*.

For this, rather than disk bundles, it is slightly more convenient to talk about vector bundles. This is for two reasons: firstly, you are all already familiar with an orientation of a vector space, and secondly, this will be more helpful when we discuss topological manifolds next lecture.

Suppose $\mathbb{R}^n \to E \xrightarrow{p} X$ is a vector bundle (cf. Definition 33.6). Let

$$E^\times = \bigcup_x E_x \setminus 0_x,$$

where $0_x \in E_x$ is the zero element of the vector space E_x. Since one also has

$$H^i(\mathbb{R}^n, \mathbb{R}^n \setminus 0; R) \cong \begin{cases} R, & i = n, \\ 0, & i \neq n \end{cases}$$

(as $\mathbb{R}^n \setminus B^n \hookrightarrow \mathbb{R}^n \setminus 0$ is a homotopy equivalence), one can define a Thom class in exactly the same way.

Definition 35.6. Let $\mathbb{R}^n \to E \xrightarrow{p} X$ be a vector bundle, and let R be a commutative ring. A **Thom class** for p is a cohomology class $t \in H^n(E, E^\times; R)$ with the property that $H^n(\imath_x)(t)$ is a generator of $H^n(E_x, E_x \setminus 0_x; R)$ for every $x \in X$.

Theorem 35.7 (The Thom Isomorphism Theorem—vector bundle version). *Let $\mathbb{R}^n \to E \xrightarrow{p} X$ be a vector bundle, and let R be a commutative ring. Suppose a Thom class t exists. Then the map $L\colon H^i(X; R) \to H^{i+n}(E, E^\times; R)$ given by*

$$L\colon \langle \alpha \rangle \mapsto H^i(p)\langle \alpha \rangle \smile t$$

is an isomorphism for all $i \geq 0$, and $H^i(E, E^\times; R) = 0$ for $i < n$.

The proof of Theorem 35.7 proceeds in exactly the same way as Theorem 35.3, and we will refer to both results as the "Thom Isomorphism Theorem". Moreover the same argument as in Theorem 35.4 gives:

Theorem 35.8. *Let $\mathbb{R}^n \to E \xrightarrow{p} X$ be a vector bundle. Then for $R = \mathbb{Z}_2$, a Thom class $t \in H^n(E, E^\times; \mathbb{Z}_2)$ always exists. Moreover t is unique.*

Now let us discuss orientability. We begin by choosing a canonical generator of $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$.

Definition 35.9. In Lecture 7 we defined (cf Definition 7.8) e_i to be the standard basis vector of $\mathbb{R}^{n+1}$ with a 1 in the $(i+1)$st position and zeroes in all other coordinates. This convention was convenient for simplices, but is rather awkward for linear algebra. Thus we introduce more notation: let $q_i \in \mathbb{R}^n$ be the standard basis vector with a 1 in the ith coordinate and zeros in the other coordinates.

The following lemma is an easy computation whose proof is on Problem Sheet O.

LEMMA 35.10. *Let $n \geq 1$. Let $\sigma_n \colon \Delta^n \to \mathbb{R}^n$ denote the unique affine map such that*

$$\sigma_n(e_0) = -\sum_{i=1}^{n} q_i, \qquad \text{and} \qquad \sigma_n(e_i) = q_i, \qquad i = 1, \dots, n.$$

Then σ_n determines a generator $\langle \sigma_n \rangle$ of $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{Z}$.

DEFINITION 35.11. Let R be an ring. Let $\gamma_n \in \operatorname{Hom}(H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0), R)$ denote the class determined by $\gamma_n(\langle \sigma_n \rangle) = 1$. Then from the Dual Universal Coefficients Theorem 29.5, γ_n determines a generator $\langle \gamma_n \rangle$ of $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0; R) \cong R$. We call it the **canonical generator**.

DEFINITION 35.12. Let V be real vector space of dimension $n \geq 1$. An **orientation** on V is an equivalence class of an ordered bases, where two bases being equivalent if the transformation matrix taking one to the other has positive determinant.

There are thus two orientations on V, and a choice of orientation allows us to speak of a **positively oriented basis** (i.e. one in the chosen equivalence class) and a **negatively oriented basis**. We use the convention that $(q_1, \dots, q_n)$ is a positively oriented basis of $\mathbb{R}^n$. If V and W are n-dimensional vector spaces equipped with orientations, then a linear isomorphism $f \colon V \to W$ is **orientation preserving** if it sends a positively oriented basis to a positively oriented basis.

Here is a cohomological version of the definition.

DEFINITION 35.13. Fix an ordered basis $B = (v_1, v_2, \dots, v_n)$ of V. Define a linear map $f_B \colon V \to \mathbb{R}^n$ by setting $f(v_i) = q_i$. Two bases B and B' are equivalent in the sense of Definition 35.12 if and only if[2] the maps f_B and $f_{B'}$ are homotopic through maps of pairs $(V, V \setminus 0) \to (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$. The map f_B determines an isomorphism

$$H^n(f_B) \colon H^n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \to H^n(V, V \setminus 0).$$

Set $\langle \gamma_B \rangle := H^n(f_B)\langle \gamma_n \rangle$, where $\langle \gamma_n \rangle$ is the canonical generator from Definition 35.11 for $A = \mathbb{Z}$. We call $\langle \gamma_B \rangle$ a **cohomological orientation** of the vector space V.

It follows immediately that B and B' are equivalent in the sense of Definition 35.12 if and only if $\langle \gamma_B \rangle = \langle \gamma_{B'} \rangle$. (If they are not equivalent, then $\langle \gamma_B \rangle = -\langle \gamma_{B'} \rangle$.) Thus a choice of cohomological orientation is the same thing as the choice of a (normal) orientation.

Now let us extend this to vector bundles. Suppose $\mathbb{R}^n \to E \xrightarrow{p} X$ is a vector bundle and $h \colon p^{-1}(U) \to U \times \mathbb{R}^n$ is a vector bundle trivialisation (as in Definition 33.6). We define a cohomology class

$$\langle \gamma_U \rangle := H^n(\pi'' \circ h)\langle \gamma_n \rangle \in H^n(E_U, E_U^{\times}),$$

where as always, $\pi'' \colon U \times \mathbb{R}^n \to \mathbb{R}^n$ is the second projection, and $E_U^{\times} = \bigcup_{x \in U} E_x \setminus 0_x$. Next, set $h_x := h|_{E_x} \colon E_x \times \{x\} \cong E_x \to \mathbb{R}^n$. If we set

$$\langle \gamma_x \rangle := H^n(h_x)\langle \gamma_n \rangle,$$

[2] Exercise: Why?

then we have

$$H^n(\imath_x)\langle\gamma_U\rangle = \langle\gamma_x\rangle, \qquad \forall\, x \in U,$$

where $\imath_x\colon E_x \hookrightarrow E$ is the inclusion of the fibre. This follows from the obvious commutativity of the following diagram:

$$\begin{array}{ccccc} E_x & \xrightarrow{\imath_x} & p^{-1}(U) & \xrightarrow{h} & U \times \mathbb{R}^n \\ {\scriptstyle h_x}\downarrow & & & & \downarrow{\scriptstyle \pi''} \\ \mathbb{R}^n & & \xrightarrow[\mathrm{id}]{\hspace{4em}} & & \mathbb{R}^n \end{array}$$

Thus via h we obtain a cohomological orientation $\langle\gamma_x\rangle$ of each vector space E_x for $x \in U$. We call this the **cohomological orientation induced by** h.

So far this is just notation (we have not yet imposed any conditions on the vector bundle p). But now suppose $k\colon p^{-1}(V) \to V \times \mathbb{R}^n$ is another vector bundle trivialisation, with $U \cap V \neq \emptyset$. This gives us a class

$$\langle\gamma_V\rangle := H^n(\pi'' \circ k)\langle\gamma_n\rangle \in H^n(E_V, E_V^{\times})$$

and

$$\langle\gamma'_x\rangle := H^n(k_x)\langle\gamma_n\rangle \in H^n(E_x, E_x \setminus 0_x).$$

We would like these classes to be compatible in the sense that the cohomological orientation induced by h should agree with the cohomological orientation induced by k on $U \cap V$:

$$\langle\gamma_x\rangle = \langle\gamma'_x\rangle, \qquad \forall\, x \in U \cap V.$$

If this is the case we say that the two trivialisations h and k are **compatible**.

Definition 35.14. An **orienting atlas** of a vector bundle $\mathbb{R}^n \to E \xrightarrow{p} X$ consists of an open cover $\{U_\lambda\}$ of X and vector bundle trivialisations $h_\lambda\colon p^{-1}(U_\lambda) \to U_\lambda \times \mathbb{R}^n$ such that h_λ is compatible with $h_{\lambda'}$ whenever $U_\lambda \cap U_{\lambda'} \neq \emptyset$. If an orienting atlas exists, we say the vector bundle p is **orientable**. An orienting atlas defines a cohomological orientation $\langle\gamma_x\rangle \in H^n(E_x, E_x \setminus 0_x)$ for every $x \in X$. We call such a choice an **orientation** of p. Thus if p is orientable and X is path-connected, there are precisely two orientations of p.

Not all vector bundles are orientable. Here is an example of one that isn't.

Example 35.15. Let E be the quotient space of $I \times \mathbb{R}$ where we identify

$$(0, s) \sim (1, 1 - s), \qquad \forall s \in \mathbb{R}.$$

The projection $I \times \mathbb{R} \to I$ factors through $\sim$ to define a map $p\colon E \to S^1$. Then $\mathbb{R} \to E \xrightarrow{p} S^1$ is a vector bundle called the **Möbius bundle**. I invite you to prove that p is not orientable[3].

We conclude this lecture by using orientability to give a version of Theorem 35.4 for $R = \mathbb{Z}$.

[3] *Hint*: the name should give you a clue!

Theorem 35.16. *Let* $\mathbb{R}^n \to E \xrightarrow{p} X$ *be an oriented vector bundle over a path connected space* X*. Then for* $R = \mathbb{Z}$*, a Thom class* $t \in H^n(E, E^\times)$ *always exists.*

Proof. As already mentioned, the only place where the proof of Theorem 35.4 goes wrong is in Step (1). But the choice of orientation rectifies this: if we are given an orienting atlas $h_\lambda \colon p^{-1}(U_\lambda) \to U_\lambda \times \mathbb{R}^n$ then we can define Thom classes $t_{U_\lambda} := \langle \gamma_{U_\lambda} \rangle \in H^n(E_{U_\lambda}, E_{U_\lambda}^\times)$, and the compatibility condition plays the role that uniqueness did in the proof of Theorem 35.4. ∎

LECTURE 36

Topological manifolds and presheaves

In this lecture we define *topological manifolds* and *pre(co)sheaves*.

DEFINITION 36.1. A topological space X is said to be **n-dimensional locally Euclidean** if every point $x \in X$ has an open neighbourhood U which is homeomorphic to an open subset V of $\mathbb{R}^n$. A homeomorphism $\varphi\colon U \to V$ is called a **chart** about x. We call n the **dimension** of X.

Since $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$ (this is the Invariance of Domain Theorem, cf. Problem I.2), the dimension is well defined. It is often convenient to assume that a chart $\varphi\colon U \to V$ at x has the property that $0 \in V$ and satisfies $\varphi(x) = 0$. In this case we say that φ is **centred** at x. We can of course also assume $V = \mathbb{R}^n$ when needed.

DEFINITION 36.2. A topological space M is called an **n-dimensional topological manifold** if:

1. M is Hausdorff.
2. M is n-dimensional locally Euclidean.
3. M is paracompact (cf. Definition 33.16.)
4. M has at most countably many connected components.

REMARK 36.3. If a topological manifold M is compact we will—somewhat illogically—say that M is a **closed** topological manifold.

The notion of *boundary* for a manifold is slightly confusing, because there are two different concepts which usually do not coincide: the *topological boundary* and the *manifold boundary*. In this course we will only ever use manifolds as defined above, where (by definition) the manifold boundary is always empty. (So the rest of this remark is for interest only!)

If $M \subset \mathbb{R}^m$ is a topological n-dimensional manifold[1] then $\partial^{\mathrm{top}} M$ (the topological boundary of M as a subset of $\mathbb{R}^m$, i.e. $\overline{M} \setminus M^\circ$) either satisfies $M \cap \partial M = \emptyset$ (when $n = m$) or $M = \partial^{\mathrm{top}} M$ (when $n < m$). This is because the "locally Euclidean" condition is an open condition.

There is a more general notion of an **n-dimensional topological manifold with boundary**, which is a Hausdorff paracompact topological space with countably many connected components such that every point is either homeomorphic to an open

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[1]Any n-dimensional topological manifold can be embedded in $\mathbb{R}^{2n+1}$. Any n-dimensional *smooth* topological manifold can be embedded in $\mathbb{R}^{2n}$. These (difficult) results are usually known as the *Whitney Embedding Theorems*.

subset in $\mathbb{R}^n$ or an open subset in a half-plane $\mathbb{R}^n_+ := \{(q_1, \dots, , q_n) \mid q_1 \geq 0\}$, where we endow $\mathbb{R}^n_+$ with the subspace topology on $\mathbb{R}^n$.

In this case one can define the *interior* of M, written $\operatorname{int} M$ to be those points in M which have neighbourhoods homeomorphic to an open subset of $\mathbb{R}^n$. The manifold boundary $\partial^{\text{man}} M$ is then $M \setminus \operatorname{int} M$.

If M is an n-dimensional topological manifold with boundary of dimension n, then $\operatorname{int} M$ is a manifold (without boundary) of dimension n and $\partial^{\text{man}} M$ is a manifold (without boundary) of dimension $n-1$.

If $M \subset \mathbb{R}^m$ is a manifold with boundary and $n = m$ then the two concepts coincide: $\partial^{\text{top}} M = \partial^{\text{man}} M$. But if $n < m$ then they do not need to: for example, the sphere S^2 with a small open disk E removed is a two-dimensional manifold with boundary, and $\partial^{\text{man}}(S^2 \setminus E)$ is the topological boundary of E in S^2 (which is a circle S^1). But if we sit S^2 inside $\mathbb{R}^3$ in the standard way the topological boundary of $S^2 \setminus E$ in $\mathbb{R}^3$ is all of $S^2 \setminus E$.

REMARK 36.4. We won't prove (or use) the following facts, but they are good to know:

- Any closed topological manifold is homotopy equivalent to a cell complex.
- Any closed topological manifold of dimension $n \neq 4$ is homeomorphic to a cell complex.
- Any closed smooth manifold actually *is* a cell complex.

The following lemma gives a homological way of defining the dimension of a topological manifold.

LEMMA 36.5. *Let M be an n-dimensional topological manifold and let A be an abelian group. For any point $x \in M$, one has*

$$H_i(M, M \setminus x; A) \cong \begin{cases} A, & i = n, \\ 0, & i \neq n, \end{cases}$$

Proof. Choose a chart $\varphi \colon U \to \mathbb{R}^n$ centred at x. Then for any $i \geq 0$, one has

$$\begin{aligned} H_i(M, M \setminus x; A) &\cong H_i(U, U \setminus x; A), && \text{by excision,} \\ &\cong H_i(\mathbb{R}^n, \mathbb{R}^n \setminus 0; A), && \text{as } \varphi \text{ is a homeomorphism.} \end{aligned}$$

■

PROPOSITION 36.6. *Let M be an n-dimensional topological manifold and let A be an abelian group. Let*

$$\mathcal{O}(M; A) := \bigsqcup_{x \in M} H_n(M, M \setminus x; A)$$

and let $p \colon \mathcal{O}(M; A) \to M$ denote the map that sends any element of $H_n(M, M \setminus x; A)$ to x. Then $A \to \mathcal{O}(M; A) \xrightarrow{p} M$ is a fibre bundle, where we give A the discrete topology (thus p is a covering space, cf. Definition 33.4.)

The proof of Proposition 36.6 requires an auxiliary lemma, which in turns needs the following definition.

Definition 36.7. Let M be an n-dimensional topological manifold and let A be an abelian group. Given subsets $X \subseteq Y \subseteq M$ we denote by

$$\rho_X^Y \colon H_n(M, M \setminus Y; A) \to H_n(M, M \setminus X; A)$$

the homomorphism induced by the inclusion $(M, M \setminus Y) \hookrightarrow (M, M \setminus X)$.

Definition 36.8. Let us say a closed set $K \subset M$ is a **convex** set if there exists a chart $\varphi \colon U \to \mathbb{R}^n$ with $K \subset U$ such that $\varphi(K)$ is a compact convex subset of $\mathbb{R}^n$.

Lemma 36.9.

1. *If $K \subset M$ is a compact convex set then ρ_x^K is an isomorphism for all $x \in K$.*
2. *Let $x \in M$ and let W be a neighbourhood of x. Then there exists a neighbourhood $U \subset W$ of x such that for all $y \in U$, the homomorphism ρ_y^U is an isomorphism.*

Proof. Let K be a compact convex set. We may assume that there is a chart $\varphi \colon U \to \mathbb{R}^n$ with $K \subset U$ such that $\varphi(K) = D$, where D is a compact convex subset of $\mathbb{R}^n$ such that $0 \in D \subset B^n$. Consider the following commutative diagram, where we omit the coefficient group A for simplicity:

$$\begin{array}{ccccccc}
H_n(B^n, S^{n-1}) & \longrightarrow & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus D) & \xleftarrow{H_n(\varphi)} & H_n(U, U \setminus K) & \longrightarrow & H_n(M, M \setminus K) \\
{\scriptstyle =}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \rho_x^K} \\
H_n(B^n, S^{n-1}) & \longrightarrow & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) & \xleftarrow[H_n(\varphi)]{} & H_n(U, U \setminus x) & \longrightarrow & H_n(M, M \setminus x)
\end{array}$$

All the horizontal maps are isomorphisms: the left-hand ones are because the inclusion $S^{n-1} \hookrightarrow \mathbb{R}^n \setminus D$ is a homotopy equivalence. The middle two are because φ is a homeomorphism, and the right-hand two are excision isomorphisms.

The second statement is an immediate corollary of the first, since given any neighbourhood W of a point x we can always find a compact convex set K such that $x \in K \subset W$. Then any neighbourhood $U \subset K$ of x works. ∎

We can now prove Proposition 36.6.

Proof of Proposition 36.6. Fix $x \in M$, and let U be a neighbourhood of x such that ρ_y^U is an isomorphism for all $y \in U$. Define

$$k \colon U \times H_n(M, M \setminus x; A) \to p^{-1}(U)$$

by $k(y, \langle c \rangle) := \rho_y^U \circ (\rho_x^U)^{-1} \langle c \rangle$. Define a topology on $\mathcal{O}(M; A)$ by declaring k to be a homeomorphism. We need to check that if k and k' are two such maps defined on overlapping neighbourhoods U and U' of x and x' respectively, then the composition

$$(k')^{-1} \circ k \colon (U \cap U') \times H_n(M, M \setminus x; A) \to (U \cap U') \times H_n(M, M \setminus x'; A)$$

is continuous. Fix $y \in U \cap U'$ and choose a neighbourhood V of y such that ρ_z^V is an isomorphism for all $z \in V$. Now consider the following commutative diagram, where we omit the coefficient group A from the notation:

$$\begin{array}{ccccccccc} & & & & H_n(M, M \setminus z) & & & & \\ & & & \overset{\rho_z^U}{\nearrow} & \uparrow & \overset{\rho_z^{U'}}{\nwarrow} & & & \\ H_n(M, M \setminus x) & \overset{\rho_x^U}{\longleftarrow} & H_n(M, M \setminus U) & & \rho_z^V & & H_n(M, M \setminus U') & \overset{\rho_{x'}^{U'}}{\longrightarrow} & H_n(M, M \setminus x') \\ & & & \underset{\rho_V^U}{\searrow} & & \underset{\rho_V^{U'}}{\swarrow} & & & \\ & & & & H_n(M, M \setminus V) & & & & \end{array}$$

The map $(k')^{-1} \circ k$ is thus given by

$$(k')^{-1} \circ k(z, \langle c \rangle) = \big(z, \rho_{x'}^{U'} \circ (\rho_V^{U'})^{-1} \circ \rho_V^U \circ (\rho_x^U)^{-1} \langle c \rangle\big).$$

In particular, the second component of $(k')^{-1} \circ k$ does not depend on the choice of $z \in V$, which implies that $(k')^{-1} \circ k$ is continuous.

To complete the proof, we set $h := k^{-1} \colon p^{-1}(U) \to U \times H_n(M, M \setminus x; A)$, which gives us a local trivialisation of $\mathcal{O}(M; A)$. ■

Let us now put this in an nice categorical setting.

Definition 36.10. Let X be an topological space. Given two open subsets $U \subseteq V \subseteq X$, let $\imath_U^V \colon U \hookrightarrow V$ denote the inclusion. Let $\mathsf{Op}(X)$ denote the category whose objects are the open sets of X, and whose morphism sets $\mathrm{Hom}(U, V)$ are empty if $U \not\subseteq V$ and have precisely one element if $U \subseteq V$:

$$\mathrm{Hom}(U, V) = \begin{cases} \{e_U^V\}, & U \subseteq V, \\ \emptyset, & U \not\subseteq V, \end{cases}$$

Definition 36.11. Let C be a category. A **presheaf over X with values in C** is a *contravariant* functor $T \colon \mathsf{Op}(X) \to \mathsf{C}$. A **pre-cosheaf** over X with values in C is[2] a *covariant* functor $\mathsf{Op}(X) \to \mathsf{C}$.

Presheaves are more common than pre-cosheaves in mathematics; however in this course we will mainly work with pre-cosheaves. Here is the standard example of a presheaf.

Example 36.12. Let X be a topological space. Given an open set U of X, let $C(U)$ denote the set of continuous functions $f \colon U \to \mathbb{R}$. We can give $C(U)$ the structure of a commutative ring under pointwise operations. If $U \subset V$ and $\imath_U^V \colon U \hookrightarrow V$ is the inclusion then if $f \in C(V)$ one has $f \circ \imath_U^V \in C(U)$. Thus we have the **presheaf of continuous functions**

$$C \colon \mathsf{Op}(X) \to \mathsf{ComRings}, \qquad U \mapsto C(U),$$

and

$$C(e_U^V) \colon C(V) \to C(U), \qquad f \mapsto f \circ \imath_U^V.$$

One normally writes $f|_U$ instead of $f \circ \imath_U^V$.

[2]Yes, I agree this terminology seems backwards... But that's just life.

Remark 36.13. In fact, this is an example of a **sheaf** (not just a presheaf). However we won't define sheaves (or cosheaves) in this course, since we do not need them, and thus we will not discuss this.

Example 36.14. Let $F \to E \xrightarrow{p} X$ be a fibre bundle. Given an open subset $U \subset X$, let $\Gamma(U; E)$ denote the space of continuous sections $s \colon U \to E$ of p, i.e. continuous maps $s \colon U \to E$ such that $s(x) \in E_x$ for all $x \in U$. If $U \subset V$ and $s \in \Gamma(V; E)$ then $s \circ \imath_U^V \in \Gamma(U; E)$, where again $\imath_U^V \colon U \hookrightarrow V$ is the inclusion. Thus we have another presheaf

$$\Gamma \colon \mathsf{Op}(X) \to \mathsf{Sets}.$$

Here is an example of a pre-cosheaf:

Example 36.15. Let M be an n-dimensional topological manifold and let A be an abelian group. Consider the functor

$$H_n(M, \square; A) \colon \mathsf{Op}(M) \to \mathsf{Ab}$$

which sends an open set U to the homology group $H_n(M, U; A)$ and sends an inclusion $U \hookrightarrow V$ to the induced map $\rho_{M \setminus V}^{M \setminus U} \colon H_n(M, U; A) \to H_n(M, V; A)$

Now let us specialise Example 36.14 to the situation at hand.

Example 36.16. Let M be an n-dimensional topological manifold and let A be an abelian group. Given a closed subset $K \subset M$, let $\Gamma(K; A)$ denote the space of continuous sections $s \colon K \to \mathcal{O}(M; A)$ of p. Since A has the discrete topology, a section is continuous if and only if its expression in a local trivialisation is locally constant. This means that if $x \in K$ and U is a path-connected neighbourhood of x such that $\mathcal{O}(M; A)$ is trivial over U via $h \colon p^{-1}(U) \to U \times H_n(M, M \setminus x; A)$ then the function $\pi'' \circ h \circ s \colon U \to H_n(M, M \setminus x; A)$ should be a constant function.

Let $\Gamma_c(K; A) \subset \Gamma(K; A)$ denote those sections with *compact support*, i.e. for which there exists a compact set $C \subseteq K$ such that $s(x) = 0$ for all $x \in K \setminus C$.

Since A is a group, and not just a topological space, if $s, t \colon K \to \mathcal{O}(M; A)$ are two sections, we can add them together to form a new section $s + t$. Thus $\Gamma(K; A)$ and $\Gamma_c(K; A)$ are also abelian groups.

So far, this is not quite the same as in Example 36.14 since here we have defined $\Gamma_c(K; A)$ for K a *closed* set, not an open set. To this end we consider the functor

$$\Gamma_c(M \setminus \square; A) \colon \mathsf{Op}(M) \to \mathsf{Ab}$$

and if $V \subseteq U$ then

$$\Gamma_c(e_V^U; A) \colon \Gamma_c(M \setminus V; A) \to \Gamma_c(M \setminus U; A), \qquad s \mapsto s|_{M \setminus U}.$$

Note that $\Gamma_c(M \setminus \square; A)$ is actually a pre-cosheaf (not a presheaf), since the operation $U \mapsto M \setminus U$ is contravariant (i.e. if $U \subset V$ then $X \setminus V \subseteq X \setminus U$.)

In conclusion, we have two pre-cosheaves $\mathsf{Op}(M) \to \mathsf{Ab}$:

$$H_n(M, \square; A) \qquad \text{and} \qquad \Gamma_c(M \setminus \square; A).$$

Our aim now is to construct a natural transformation between them. This requires the following lemma.

Lemma 36.17. *Let $U \subset M$ be open. If $\langle c \rangle \in H_n(M, U; A)$ then $\phi_{\langle c \rangle}(y) := \rho_y^{M \setminus U} \langle c \rangle$ belongs to $\Gamma_c(M \setminus U; A)$.*

Proof. Fix $x \in M \setminus U$ and let W be a small neighbourhood of x with $W \subset M \setminus U$. Using a local trivialisation $h \colon p^{-1}(W) \to W \times H_n(M, M \setminus x; A)$ as in the proof of Proposition 36.6, the section $\phi_{\langle c \rangle}$ becomes $y \mapsto (y, \rho_x^W \circ \rho_W^{M \setminus U} \langle c \rangle)$ on $p^{-1}(W)$, which is continuous by the definition of the topology on $\mathcal{O}(M; A)$.

It remains to show $s \in \Gamma_c(M \setminus U; A)$. Choose a cycle $c = \sum_i a_i \otimes \sigma_i \in Z_n(M; A)$ that represents $\langle c \rangle$. There exists a compact set $C \subset M$ such that each singular simplex $\sigma_i \colon \Delta^n \to M$ in c has $\sigma_i(\Delta^n) \subset C$ (cf. Lemma 17.5). Now suppose $y \in M \setminus C$. Then the image of c under the composition

$$C_n(C; A) \to C_n(M; A) \to C_n(M, C; A) \to C_n(M, M \setminus y; A)$$

is zero. But this image is a representative of $\phi_{\langle c \rangle}(y)$. Thus the support of $\phi_{\langle c \rangle}$ is contained in the compact set $(M \setminus U) \cap C$. ■

Lemma 36.18. *There is a natural transformation $\Phi \colon H_n(M, \square; A) \to \Gamma_c(M \setminus \square; A)$ given by setting*

$$\Phi(U)\langle c \rangle = \phi_{\langle c \rangle},$$

where $\phi_{\langle c \rangle}$ was defined in Lemma 36.17.

Proof. One needs only check that for closed sets $K \subset L$ the following diagram commutes:

$$\begin{array}{ccc} H_n(M, M \setminus L; A) & \xrightarrow{\rho_K^L} & H_n(M, M \setminus K; A) \\ {\scriptstyle \Phi(L)} \downarrow & & \downarrow {\scriptstyle \Phi(K)} \\ \Gamma_c(L; A) & \xrightarrow[s \mapsto s|_K]{} & \Gamma_c(K; A) \end{array}$$

This however is immediate from the definition. ■

What is much less obvious is that Φ is actually an natural isomorphism. We will prove the following theorem next lecture.

Theorem 36.19. *Let M be an n-dimensional topological manifold. Let $K \subset M$ be closed. Let A be an abelian group. Then $\Phi(M \setminus K) \colon H_n(M, M \setminus K; A) \to \Gamma_c(K; A)$ is an isomorphism of abelian groups. Moreover*

$$H_i(M, M \setminus K; A) = 0, \qquad \forall\, i > n.$$

Theorem 36.19 is the first of two major theorems about the homology of manifolds (the second is Poincaré Duality, which we will cover in Lecture 39. The reason is it useful is that the group $\Gamma_c(K; A)$ is much easier to compute. For instance, if we take $A = \mathbb{Z}_2$ then it follows immediately[3] from Theorem 36.19 that if M is a closed connected n-dimensional topological manifold then $H_i(M; \mathbb{Z}_2) = 0$ for $i > n$ and $H_n(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$.

[3] Exercise: Why? (If you can't do the exercise, fear not: we will go over this in detail next lecture.)

LECTURE 37

Orientability and the cap product

We begin this lecture by proving Theorem 36.19 from the last lecture. Let us recall the statement.

THEOREM 37.1. *Let M be an n-dimensional topological manifold. Let $K \subset M$ be closed. Let A be an abelian group. Then $\Phi(M \setminus K) \colon H_n(M, M \setminus K; A) \to \Gamma_c(K; A)$ is an isomorphism of abelian groups. Moreover*

$$H_i(M, M \setminus K; A) = 0, \qquad \forall\, i > n. \tag{37.1}$$

To simplify the notation during the proof we will write $\Phi(K)$ instead of $\Phi(M \setminus K)$.

Proof. The proof is quite complicated, and we will induct over the "complexity" of K (the precise meaning of this will become clear during the proof.) Let $(\heartsuit K)$ be short for the statement that the theorem is true for the closed set K. So our goal is to prove $(\heartsuit K)$ for every closed set K. Throughout the proof we will suppress the coefficient group A from all our notation, since it plays no role. We will prove the result in five steps.

1. Assume that $(\heartsuit K)$, $(\heartsuit L)$, and $(\heartsuit K \cap L)$ hold. We prove that $(\heartsuit K \cup L)$ holds. For this consider the Mayer-Vietoris sequence associated to $M \setminus (K \cap L)$, $M \setminus K$ and $M \setminus L$. We obtain the following commutative diagram, where we are using the fact that Φ is a natural transformation.

$$\begin{array}{ccc}
H_{n+1}(M, M \setminus (K \cap L)) & \xrightarrow{\cong} & 0 \\
\downarrow & & \downarrow \\
H_n(M, M \setminus (K \cup L)) & \xrightarrow{\Phi(K \cup L)} & \Gamma_c(K \cup L) \\
\downarrow & & \downarrow \\
H_n(M, M \setminus K) \oplus H_n(M, M \setminus L) & \xrightarrow{(\Phi(K), \Phi(L))} & \Gamma_c(K) \oplus \Gamma_c(L) \\
\downarrow & & \downarrow \\
H_n(M, M \setminus (K \cap L)) & \xrightarrow[\Phi(K \cap L)]{} & \Gamma_c(K \cap L)
\end{array}$$

The Five Lemma tells us that $\Phi(K \cup L)$ is an isomorphism, and the fact that $H_i(M, M \setminus (K \cup L)) = 0$ for $i > n$ follows directly from the Mayer-Vietoris sequence. Thus $(\heartsuit K \cup L)$ holds.

2. We now prove that $(\heartsuit K)$ holds for any compact convex set K. Indeed, for such K we apply the commutative diagram from the proof of part (1) of Lemma 36.9 but with H_i instead of H_n. If $i > n$ then the left-hand groups are all zero, and hence

$H_i(M, M\setminus K) = 0$ for $i > n$. Moreover from the same lemma we have that ρ_x^K is an isomorphism for each $x \in K$. This implies that $\Phi(K)$ is an isomorphism, since a section over a connected set is determined by a single value.

Now by induction on $m \geq 1$, we see that $(\heartsuit K_1 \cup \cdots \cup K_m)$ holds if $K_1 \cup \cdots \cup K_m$ is contained in the domain of a chart $\varphi\colon U \to \mathbb{R}^n$ and each K_i is compact convex. Indeed, we just did the case $m = 1$ and Step (1) gives the inductive step.

3. Now suppose K is any closed compact set contained in the domain of a chart $\varphi\colon U \to \mathbb{R}^n$. Let $W \subset U$ be any neighbourhood of K. Then there exists a closed set L which is a finite union of compact convex sets such that $K \subseteq L \subset W$. The previous step tells us $\Phi(L)$ is an isomorphism. We now take the filtered colimit over all such L to obtain

$$H_i(M, M\setminus K) = \underrightarrow{\operatorname{colim}}_L H_i(M, M\setminus L) = 0, \qquad \forall\, i > n,$$

and for $i = n$ we get a commutative diagram:

$$\begin{array}{ccc} \underrightarrow{\operatorname{colim}}_L H_n(M, M\setminus L) & \longrightarrow & H_n(M, M\setminus K) \\ \downarrow & & \downarrow{\scriptstyle \Phi(K)} \\ \underrightarrow{\operatorname{colim}}_L \Gamma_c(L) & \longrightarrow & \Gamma_c(K) \end{array}$$

The top horizontal map is an isomorphism by a relative version of Corollary 17.6 (we used this in the proof of Theorem 17.9.) The left-hand vertical map is an isomorphism, since each individual $\Phi(L)$ is an isomorphism.

Let us prove that the bottom horizontal map is an isomorphism. Indeed, if s, s' are two sections in $\Gamma_c(L)$ for some L such that $s|_K = s'|_K$ then since sections are locally constant, there is another such $K \subseteq L' \subseteq L$ such that $s|_{L'} = s'|_{L'}$, and thus s and s' give the same element in the filtered colimit. This shows that bottom horizontal map is injective. To show it is surjective, we must show that any $s \in \Gamma_c(K)$ extends to some L containing K. Again by locally constancy we can do this locally, so if $x \in K$ then there is an open set $U(x)$ such that s extends to a section s_x defined on $U(x) \cap K$. Since K is compact we can cover K by finitely many such $U(x)$. Now set

$$U := \left\{ z \in \bigcup_i U(x_i) \mid s_{x_i}(z) = s_{x_j}(z) \text{ if } z \in U(x_i) \cap U(x_j) \right\}.$$

This is open and contains K, and by construction s extends to U. Now choose any L of the desired form such that $K \subseteq L \subset U$. Then s extends to L, and hence the map is surjective.

An arbitrary compact set can be written as a finite union of compact sets contained in charts, so by induction and Step (1) again we obtain the result in the compact case

4. Now suppose K is a closed set with the property that we can write $K = \bigcup_{i=1}^{\infty} L_i$ where each L_i is compact and has a neighbourhood U_i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. We show that $(\heartsuit K)$ holds. Indeed, by Step (3) $(\heartsuit L_i)$ holds for all i. Moreover since the L_i have disjoint open neighbourhoods , using additivity of both homology groups and sections we see that $\Phi(K) = \sum_i \Phi(L_i)$, and hence $(\heartsuit K)$ holds too.

5. We now complete the proof for an arbitrary closed set K. Since M is paracompact and has at most countably many connected components, we can write M as a union $\bigcup_{i=1}^{\infty} C_i$ where $C_i \subset C_{i+1}^{\circ}$ are compact sets. Set $C_0 = \emptyset$ and set

$$L_i = K \cap \left(C_i \setminus C_{i-1}^{\circ}\right).$$

Set $K' = \bigcup_{i \text{ is odd}} L_i$ and $K'' = \bigcup_{i \text{ is even}} L_i$. Now K', K'' and $K \cap K''$ satisfy the hypotheses[1] of Step (4), and hence $(\heartsuit K')$, $(\heartsuit K'')$ and $(\heartsuit K' \cap K'')$ hold. Since $K = K' \cup K''$, by Step (1) we also have that $(\heartsuit K)$ holds. ∎

We now specialise our discussion of the fibre bundle $\mathcal{O}(M; A)$ to the case where A is a commutative ring, and using this to define what it means for a topological manifold to be *orientable*.

Definition 37.2. Let M be an n-dimensional topological manifold and let R be a commutative ring. Then for any $x \in M$, the group $H_n(M, M \setminus x; R)$ is a free R-module of rank 1. A generator of $H_n(M, M \setminus x; R)$ (which corresponds to an element of R with a multiplicative inverse) is called a **local R-orientation** of M at x.

Definition 37.3. Let M be an n-dimensional topological manifold and let R be a commutative ring. Let $K \subset M$ be a closed subset. An **R-orientation of M along K** is a continuous section $s \in \Gamma(K; R)$ such that $s(x)$ is a generator of $H_n(M, M \setminus x; R)$ for all $x \in K$. Thus an R-orientation along K is a continuous choice of local R-orientation at every point $x \in K$. We say M is **R-orientable along K** if an R-orientation along K exists. For $K = M$, we simply say M is **R-orientable**. If we consider M equipped with such an R-orientation s then we say than M is **R-oriented**.

Remark 37.4. Note that if $K \subset L \subset M$ and M is R-orientable along L then M is also R-orientable along K, since if $s \in \Gamma_c(L; R)$ is an R-orientation of M along L then $s|_K \in \Gamma_c(K; R)$ is an R-orientation of M along K.

Definition 37.5. Given a commutative ring R, let $U(R)$ denote elements of R with a multiplicative inverse (the "units"). Then $U(R)$ becomes an abelian group under multiplication.

Definition 37.6. Let $\mathrm{Ori}(M; R) \subset \mathcal{O}(M; R)$ denote the subbundle whose fibre over $x \in M$ is $U(R) \subset R \cong H_n(M, M \setminus x; R)$. We call $\mathrm{Ori}(M; R)$ the **R-orientation covering** of M. Then $U(R) \to \mathrm{Ori}(M; R) \to M$ is another fibre bundle with fibre $U(R)$, and $(R, U(R)) \to (\mathcal{O}(M; R), \mathrm{Ori}(M; R)) \to M$ is a fibre bundle pair.

Thus an R-orientation of M along $K \subset M$ is the same thing as a section of $s\colon K \to \mathrm{Ori}(M; R)$. For $R = \mathbb{Z}$, we drop the R from the notation and the terminology. Since $U(\mathbb{Z}) = \{\pm 1\} \cong \mathbb{Z}_2$, $\mathrm{Ori}(M)$ is a **double covering**, which we call the **orientation covering**.

Proposition 37.7. *Let M be an n-dimensional topological manifold. The following are equivalent:*

[1]Here we are using the point-set topological fact that manifolds are always *normal* topological spaces.

1. *M is orientable.*
2. *M is orientable along any compact subset.*
3. *$\operatorname{Ori}(M) \to M$ is a trivial fibre bundle.*
4. *$\mathcal{O}(M) \to M$ is a trivial fibre bundle.*

Proof.

- (1) $\Rightarrow$ (2) is a special case.
- (2) $\Rightarrow$ (3): We may assume M is connected, since a fibre bundle is trivial if and only if it is trivial over every connected component of the base. Since $\operatorname{Ori}(M)$ is a double cover (i.e. the fibre is $\mathbb{Z}_2$), either $\operatorname{Ori}(M)$ is connected or it is trivial. If $\operatorname{Ori}(M)$ is connected, then there exists a path in $\operatorname{Ori}(M)$ between any two points of a given fibre. The image of such a path is a compact connected subset K of M, and by assumption $\operatorname{Ori}(M)|_K$ is non-trivial, since we can connect two points of a fibre in it. But by (2) there exists a section of $\operatorname{Ori}(M)|_K$, and hence $\operatorname{Ori}(M)|_K$ is trivial. Contradiction. Thus $\operatorname{Ori}(M)$ is not connected, and hence it is trivial.
- (3) $\Rightarrow$ (4): The bundle $\operatorname{Ori}(M)$ is trivial if and only if it has a section. If s is a section of $\operatorname{Ori}(M)$ then

$$M \times \mathbb{Z} \to \mathcal{O}(M), \qquad (x, n) \mapsto ns(x)$$

  is a trivialisation of $\mathcal{O}(M)$.
- (4) $\Rightarrow$ (1): If $\mathcal{O}(M)$ is trivial then in particular it has a section with values in the set of generators.

■

So far we have defined orientability in terms of the existence of sections of $\mathcal{O}(M; R)$. Using Theorem 36.19 however we can transfer to a statement about homology classes. (This is the main point of Theorem 36.19!)

Theorem 37.8. *Let M be an n-dimensional topological manifold and let R be a commutative ring. Let $K \subseteq M$ be a closed connected subset.*

1. *If K is not compact then $H_n(M, M \setminus K; R) = 0$.*
2. *If K is compact and M is R-orientable along K then $H_n(M, M \setminus K; R) \cong R$. There exists a homology class $\langle o_K \rangle \in H_n(M, M \setminus K; R)$ such that $\rho^K_x \langle o_K \rangle$ is a generator of $H_n(M, M \setminus x; R)$ for all $x \in K$.*
3. *If K is compact and M is not R-orientable along K then one has $H_n(M, M \setminus K; R) = \{r \in R \mid 2r = 0\}$.*

Proof. Since K is connected, a section of $\Gamma(K; R)$ is determined by its value at a single point. If this value is non-zero, then the section is non-zero everywhere. Thus there do not exist non-zero sections with compact support if K is non-compact: $\Gamma_c(K; R) = 0$. Thus Theorem 36.19 tells us that if K is connected and non-compact then $H_n(M, M \setminus K; R) = 0$. (This part does not need R to a ring.)

If K is compact then we have a commutative diagram:

$$\begin{array}{ccc} H_n(M, M \setminus K; R) & \xrightarrow{\Phi(K)} & \Gamma(K; R) \\ \rho_x^K \downarrow & & \downarrow s \mapsto s(x) \\ H_n(M, M \setminus x; R) & \xrightarrow[\Phi(\{x\})]{} & \Gamma(\{x\}; R). \end{array}$$

If M is orientable along K then we can choose s such that $s(x)$ is a generator. The desired class is then given by $\langle o_K \rangle = \Phi(K)^{-1}(s)$.

The third part is on Problem Sheet P. ■

In the case $R = \mathbb{Z}$, one can also obtain information about the torsion subgroup of $H_{n-1}(M, M \setminus K)$. This is the content of the following result, whose proof is again on Problem Sheet P.

Proposition 37.9. *Let M be an n-dimensional topological manifold. Let $K \subseteq M$ be a closed connected subset.*

1. *If K is non-compact then the torsion subgroup of $H_{n-1}(M, M \setminus K)$ is zero.*
2. *If K is compact and M is orientable along K then the torsion subgroup of $H_{n-1}(M, M \setminus K)$ is also zero.*
3. *If K is compact and M is not orientable along K then the torsion subgroup of $H_{n-1}(M, M \setminus K)$ is isomorphic to $\mathbb{Z}_2$.*

Remark 37.10. Let us rewrite Remark 37.4 in terms of homology classes. Suppose $K \subset L$ are compact subsets of M and M is orientable along L with corresponding class $\langle o_L \rangle \in H_n(M, M \setminus L; R)$. Then the class $\langle o_K \rangle := \rho_K^L \langle o_L \rangle$ has the property that $\rho_x^K \langle o_K \rangle$ is a generator of $H_n(M, M \setminus x; R)$ for all $x \in K$. Thus an R-orientation of M is equivalent to a collection

$$\{\langle o_K \rangle \mid K \subset M \text{ is compact}\}$$

of homology classes $\langle o_K \rangle \in H_n(M, M \setminus K; R)$ that satisfy the following two compatibility conditions:

1. For every $x \in K$, the class $\rho_x^K \langle o_K \rangle$ is a generator of $H_n(M, M \setminus x; R)$.
2. If $K \subset L$ then $\rho_K^L \langle o_L \rangle = \langle o_K \rangle$.

Let us now assume that M itself is compact and connected. Then Theorem 37.8 can be improved:

Corollary 37.11. *Let M be a compact connected n-dimensional topological manifold. If M is non-orientable then $H_n(M) = 0$. If M is orientable then $H_n(M) \cong \mathbb{Z}$, and for each $x \in M$ the restriction $H_n(M) \to H_n(M, M \setminus x)$ is an isomorphism.*

Definition 37.12. Let M be an orientable compact connected n-dimensional topological manifold. A generator of $H_n(M)$ is called a **fundamental class** of M. As above, we use the notation $\langle o_M \rangle$ to denote a fundamental class. An oriented[2] closed manifold is thus a manifold together with a choice of fundamental class.

[2] "Orientable" means: a fundamental class exists. "Oriented" means: we have chosen a fundamental class.

Remark 37.13. Fundamental classes can be used to extend the definition of degree (cf. Definition 15.2) to orientable compact connected manifolds.

Remark 37.14. A vector space of dimension n is trivially an n-dimensional topological manifold. As a check to make sure you have understood the definitions, convince yourself that Definition 37.3 is consistent with Definition 35.12 from the two lectures ago.

Now let us take a step back and define a new product, called the *cap* product. This definition works on an arbitrary topological space, although it is most useful on manifolds due to the duality theorems we will prove in two lectures time.

Definition 37.15. Let X be a topological space and let X', X'' be open sets, let R be a commutative ring. The **cap product** consists of a family of R-linear maps

$$H^n(X, X'; R) \otimes H_{n+m}(X, X' \cup X''; R) \to H_m(X, X''; R), \qquad \langle \alpha \rangle \otimes \langle c \rangle \mapsto \langle \alpha \rangle \frown \langle c \rangle.$$

As with the cup product, we will give the definition first in the absolute case (where $X' = X'' = \emptyset$) and then discuss the relative case. For this we begin on the chain level and define an operation

$$C^n(X; R) \otimes C_{n+m}(X; R) \to C_m(X; R), \qquad \alpha \otimes c \mapsto \alpha \frown c. \tag{37.2}$$

If $\sigma\colon \Delta^{n+m} \to X$, we set

$$\alpha \frown \sigma := \alpha(\sigma \circ F_n) \otimes (\sigma \circ B_m)$$

This is indeed an element of $C_m(X; R)$, since $\alpha(\sigma \circ F_n)$ is an element of R, and $\sigma \circ B_m\colon \Delta^m \to X$ is a singular m-simplex in X. Now extend this by linearity to define the operation (37.2).

The following lemma is a proved in the same way as we dealt with similar statements for the cup product in Lecture 30, using the face relations from Lemma 30.12. We omit most of the details.

Lemma 37.16. *Let X, Y be topological spaces. Let $f\colon X \to Y$ be continuous. Let $\alpha \in C^n(X; R)$, $\beta \in C^p(X; R)$, $c \in C_{n+m}(X; R)$ and $\gamma \in C^n(Y; R)$. Then:*

1. *$\partial(\alpha \frown c) = (-1)^n(\alpha \frown \partial c - d\alpha \frown c)$.*
2. *If $p = m$ then $\beta(\alpha \frown c) = (\alpha \smile \beta)(c)$.*
3. *If $p \le m$ then $(\alpha \smile \beta) \frown c = \beta \frown (\alpha \frown c)$.*
4. *$\nu_X \frown c = c$, where $\nu_X \in C^0(X; R)$ was defined in (30.2).*
5. *$f_\#(f^\# \gamma \frown c) = \gamma \frown f_\# c$.*

Proof. The proof of (1) is on Problem Sheet P.

Let us prove (2). Assume that $\sigma\colon \Delta^{n+m} \to X$ is a singular n-simplex. Then $\beta(\alpha \frown \sigma) = \alpha(\sigma \circ F_n) \cdot \beta(\sigma \circ B_m)$, which is exactly the formula for $(\alpha \smile \beta)(\sigma)$.

The proof of (3) is similar and I omit this, and the proof of (4) is immediate. So let us skip to the proof of the naturality statement (5). For this we have

$$\begin{aligned} \gamma \frown f_{\#}\sigma &= \gamma(f \circ \sigma \circ F_n) \otimes (f \circ \sigma \circ B_m) \\ &= f^{\#}\gamma(\sigma \circ F_n) \otimes (f \circ \sigma \circ B_m) \\ &= f_{\#}(f^{\#}\gamma \frown \sigma), \end{aligned}$$

as required. ■

It follows from statement (1) of Lemma 37.16 that the cap product operation (37.2) is well defined on the level of (co)homology. This is the same argument as in the proof of Theorem 30.20: indeed, if $d\alpha = \partial c = 0$ then $\partial(\alpha \frown c) = 0$, and if either α or c is a (co)boundary then $\alpha \frown c$ is a boundary.

To extend the definition to relative groups, suppose $\alpha \in C^n(X, X'; R) \subset C^n(X; R)$ and $c \in C_{m+n}(X'; R) + C_{m+n}(X''; R)$. Then $\alpha \frown c \in C_m(X''; R)$ from the definition. Thus we have an induced cap product:

$$C^n(X, X'; R) \otimes \frac{C_{m+n}(X; R)}{C_{m+n}(X'; R) + C_{m+n}(X''; R)} \to \frac{C_m(X; R)}{C_m(X''; R)}.$$

Using the chain equivalence $C_\bullet(X'; R) + C_\bullet(X''; R) \to C_\bullet(X' \cup X''; R)$ (cf. the proof of Theorem 14.8), we obtain the cap product as in Definition 37.15.

Remark 37.17. An inspection of the definition of (37.2) shows that we can be a little more precise with the choice of coefficients. Indeed, suppose M and N are R-modules. Then the cap product in Definition 37.15 can actually be realised as

$$H^n(X, A; M) \otimes H_{n+m}(X, A \cup B; N) \to H_m(X, B; M \otimes_R N), \qquad \langle\alpha\rangle \otimes \langle c\rangle \mapsto \langle\alpha\rangle \frown \langle c\rangle.$$

LECTURE 38

Čech cohomology

We now introduce a different type of cohomology which will be useful in our study of duality. This is called **Čech cohomology**.

REMARK 38.1. Some of you may have seen the definition of Čech cohomology before– for example, in an Algebraic Geometry course—and it looked terribly complicated. Luckily we will give the definition only in a special case (namely: compact subsets of a Euclidean neighbourhood retract), for which one can give a much easier definition.

DEFINITION 38.2. A topological space X is called a **Euclidean neighbourhood retract** if there exists an embedding $i\colon X \to \mathbb{R}^n$ and a neighbourhood U of $i(X)$ that retracts onto $i(X)$. By identifying X with $i(X)$, we can think of a Euclidean neighbourhood retract as a subset X of $\mathbb{R}^n$ with the property that there exists an open set U containing X and a continuous map $r\colon U \to X$ such that $\imath \circ r = \mathrm{id}_X$, where $\imath\colon X \hookrightarrow U$ is the inclusion.

Let us now begin by stating without proof two pieces of "point-set topology" about Euclidean neighbourhood retracts. They show in particular that the property of being a Euclidean neighbourhood retract is a topological invariant. The proofs are not *that* difficult, but we don't have time to give them here, and they don't involve any ideas relevant to the course. Nevertheless, I invite you to try them as exercises if you are bored and in the mood for some point-set topology.

The first result characterises Euclidean neighbourhood retracts.

PROPOSITION 38.3. *(Properties of Euclidean neighbourhood retracts)*

1. *Let X be a Hausdorff space. Assume X can be covered by finitely many locally compact*[1] *open sets X_i. Assume each X_i is a Euclidean neighbourhood retract. Then X is an Euclidean neighbourhood retract.*
2. *A set $X \subset \mathbb{R}^n$ is a Euclidean neighbourhood retract if and only if X is locally compact and locally contractible*[2].

The second result explains why they are useful. Again, we emphasise we are only introducing the special case we are going to use (for instance, in the next result one could replace "compact" with "locally compact".)

PROPOSITION 38.4. *(Extending homotopies in Euclidean neighbourhood retracts) Let X and Y be Euclidean neighbourhood retracts, and let $K \subset X$ be compact.*

[1]A topological space Y is locally compact if each point has a compact neighbourhood.

[2]A topological space Y is locally contractible if each point has a local basis of contractible neighbourhoods.

1. *If $f\colon K \to Y$ is continuous then there exists a neighbourhood U of K in X and a continuous map $F\colon U \to Y$ such that $F|_K = f$.*
2. *If $f, g\colon X \to Y$ are continuous maps and $h_t\colon K \to Y$ is a homotopy from $f|_K$ to $g|_K$ then there exists a neighbourhood U of K in X and a homotopy $\tilde{h}_t\colon U \to Y$ such that $\tilde{h}_0 = f|_U$, $\tilde{h}_1 = g|_U$ and $\tilde{h}_t|_K = h_t$ for all $t \in I$.*

A corollary of these results is the following statement, which explains why they fit into the framework we are interested in.

COROLLARY 38.5.

1. *If M is a topological manifold and $K \subset M$ is a compact set, then there exists an Euclidean neighbourhood retract $X \subset M$ such that $K \subset X$.*
2. *If M is a closed topological manifold then M is a Euclidean neighbourhood retract.*
3. *If X is a finite cell complex then X is a Euclidean neighbourhood retract.*

Now let us begin the construction.

DEFINITION 38.6. Let K denote the category[3] whose objects are pairs (L, K), where $K \subseteq L$ are compact subsets of some Euclidean neighbourhood retract, and whose morphisms are just the usual continuous maps of pairs:

$$\mathrm{Hom}_{\mathsf{K}}((L,K),(L',K')) = \left\{ f\colon (L,K) \to (L',K'), f \text{ continuous} \right\},$$

with composition just the normal composition of functions. Thus K is a full subcateogory of Top^2.

Suppose $(L, K) \in \mathrm{obj}(\mathsf{K})$. Then by assumption there exists a Euclidean neighbourhood retract X such that $K \subset L \subset X$. Suppose we are given two pairs $U \subset V \subset X$ and $U' \subset V' \subset X$ of open sets, such that $K \subset U \subset U'$, $L \subset V \subset V'$. Then we have a commutative diagram of inclusions:

$$\begin{array}{ccc} (L,K) & \hookrightarrow & (V,U) \\ & \searrow & \downarrow \\ & & (V',U'). \end{array}$$

Now fix $k \geq 0$ and pass to singular cohomology with coefficients in A:

$$\begin{array}{ccc} H^k(L,K;A) & \longleftarrow & H^k(V,U;A) \\ & \nwarrow & \uparrow \\ & & H^k(V',U';A) \end{array} \tag{38.1}$$

[3] I did initially try and call this category by the more “logical” name $\mathsf{CompSubsetsENR}^2$ but then decided that was stupid and just went with K...

Let $\mathsf{Op}^2_{(L,K)}(X)$ denote the subcategory of $\mathsf{Op}^2(X)$ of pairs of open sets (V,U) such that $(L,K) \subset (V,U)$, and whose morphisms are again given by inclusions $(V,U) \hookrightarrow (V',U')$. Now let $\mathcal{U}_X(L,K)$ denote the *opposite* category (cf Definition 28.1), i.e. with the same objects but "reverse" inclusions as morphisms. Thus if (V,U) and (V',U') belong to $\mathcal{U}_X(L,K)$ then:

$$\mathrm{Hom}((V',U'),(V,U)) = \begin{cases} \{e^{V',U'}_{V,U}\}, & U \subseteq U',\ V \subseteq V', \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then from (38.1), we have a filtered diagram:

$$H^k \colon \mathcal{U}_X(L,K) \to \mathsf{Ab},$$

$$(V,U) \mapsto H^k(V,U;A), \qquad e^{V',U'}_{V,U} \mapsto H^k(\imath^{V',U'}_{V,U}) \colon H^k(V',U';A) \to H^k(V,U;A),$$

where $\imath^{V',U'}_{V,U} \colon (V,U) \hookrightarrow (V',U')$ is the inclusion. This is indeed a filtered diagram, i.e. a *covariant* functor, since we already passed to opposite category $\mathcal{U}_X(L,K)$ to filter over, which "cancels out" the contravariance of H^k.

Thus we can take the colimit (remember we already know that colimits always exist in Ab, cf. Example 16.19), which gives us our desired construction.

DEFINITION 38.7. Let $K \subset L \subset X$ be compact subspaces of a Euclidean neighbourhood retract, and let A be an abelian group. We define the **Čech cohomology** of (L,K) with coefficients in A to be the abelian group

$$\check{H}^k(L,K;A) := \varinjlim H^k(V,U;A)$$

We use the notation $\check{H}^k(K;A)$ for $\check{H}^k(K,\emptyset;A)$.

You may wonder why we have omitted the ambient Euclidean neighbourhood retract X in our notation $\check{H}^k(L,K)$. In fact, the Čech cohomology groups do not depend on X, see Remark 38.13 below.

Before going any further, we will need the following technical statement about filtered colimits.

DEFINITION 38.8. Let $(\Lambda, \preceq)$ and $(\Lambda', \preceq')$ be two directed sets (cf. Example 16.12). A map $f \colon \Lambda \to \Lambda'$ is **order preserving** if $\lambda \preceq \mu$ implies that $f(\lambda) \preceq' f(\mu)$. We say that f is **cofinal** if for all $\lambda' \in \Lambda'$ there exists $\lambda \in \Lambda$ such that $\lambda' \preceq' f(\lambda)$. If Λ is a directed subset of Λ' and f is the inclusion $\Lambda \hookrightarrow \Lambda'$ then we simply say that Λ is **cofinal** in Λ'.

Recall we denote by $\mathsf{J}(\Lambda, \preceq)$ the filtered category associated to the directed set $(\Lambda, \preceq)$. An order preserving map $f \colon \Lambda \to \Lambda'$ induces a functor P_f from $\mathsf{J}(\Lambda, \preceq)$ to $\mathsf{J}(\Lambda', \preceq')$. Explicitly, on objects the functor P_f is defined by

$$P_f(\lambda) := f(\lambda), \qquad \lambda \in \Lambda,$$

On morphisms, if $\lambda \preceq \mu$ and $i_{\lambda,\mu} \in \mathrm{Hom}_{\mathsf{J}(\Lambda,\preceq)}(\lambda,\mu)$ denotes the (unique) morphism from λ to μ in $\mathsf{J}(\Lambda,\preceq)$ then P_f is defined as

$$P_f(i_{\lambda,\mu}) := i_{f(\lambda),f(\mu)},$$

where $i_{f(\lambda),f(\mu)} \in \mathrm{Hom}_{\mathsf{J}(\Lambda',\preceq')}(f(\lambda), f(\mu))$ is the unique morphism from $f(\lambda)$ to $f(\mu)$ in $\mathsf{J}(\Lambda',\preceq')$.

Suppose C is a category and $T' \colon \mathsf{J}(\Lambda',\preceq') \to \mathsf{C}$ is a filtered diagram (i.e. a functor). If $f \colon \Lambda \to \Lambda'$ is an order preserving map then f induces a filtered diagram

$$T = T' \circ P_f \colon \mathsf{J}(\Lambda,\preceq) \to \mathsf{C}.$$

The following result is left as an exercise to check how much you remember of colimits from last semester.

Lemma 38.9. *Suppose $f \colon \Lambda \to \Lambda'$ is a cofinal map between directed sets $(\Lambda,\preceq)$ and $(\Lambda',\preceq')$. Suppose C is a category and $T' \colon \mathsf{J}(\Lambda',\preceq') \to \mathsf{C}$ is a filtered diagram. Then let $T := T' \circ P_f \colon \mathsf{J}(\Lambda,\preceq) \to \mathsf{C}$. Then $\varinjlim T$ exists if and only if $\varinjlim T'$ exists, and if so, then*

$$\varinjlim T \cong \varinjlim T'$$

Lemma 38.9 implies that if $K \subset L \subset X$ are compact subspaces and $W \subset X$ is any open subset containing L then it suffices to take the filtered colimits over those neighbourhoods $(V,U) \in \mathcal{U}_X(L,K)$ with $V \subset W$. In the following, we will repeatedly use Lemma 38.9 without further comment.

Everything we have said so far did not actually use the assumption that both K and L were subsets of an Euclidean neighbourhood retract. In order to make $\check{H}^k$ into a contravariant functor, we need to associate a homomorphism $\check{H}^k(f) \colon \check{H}^k(L',K';A) \to \check{H}^k(L,K;A)$ to any morphism $f \colon (L,K) \to (L',K')$ in K. This is where the Euclidean neighbourhood retract assumption comes in handy.

Definition 38.10. Assume that $K \subset L \subset X$ and $K' \subset L' \subset X'$ and $f \colon (L,K) \to (L',K')$ is a continuous map of pairs. We use part (1) of Proposition 38.4 to find an open set $W \subset X$ containing L and a continuous map $F \colon W \to X'$ such that $F|_L = f$. Now if $(V',U') \in \mathcal{U}_{X'}(L',K')$ then $(F^{-1}(V'), F^{-1}(U'))$ belongs to $\mathcal{U}_X(L,K)$. Thus we can consider the composition:

$$H^k(V',U';A) \xrightarrow{H^k(F))} H^k(F^{-1}(V'),F^{-1}(U');A) \to \check{H}^k(L,K;A),$$

where the right-hand map is the map induced from the colimit. But now we get such a map for any pair (V',U'). Since $H^k(F)$ commutes with (reverse) inclusions, this shows that $\check{H}^k(L,K;A)$ is a solution of the filtered diagram $H^k \colon \mathcal{U}_{X'}(L',K') \to \mathsf{Ab}$. Thus by definition of the colimit we get a map

$$\check{H}^k(L',K';A) \to \check{H}^k(L,K;A).$$

Let us call this map $\check{H}^k(f)$. In order for this to be a functor, we need to know that $\tilde{H}^k(f)$ is independent of the choice of extension F. This is a nice exercise using part (2) of Proposition 38.4, and I am generously leaving it for you on Problem Sheet P.

THEOREM 38.11. *Čech cohomology defines contravariant functors*

$$\check{H}^k\colon \mathsf{K} \to \mathsf{Ab}, \qquad \forall\, k \geq 0.$$

The aim of the rest of this lecture is to show that Čech cohomology with coefficients in A is an Eilenberg-Steenrod cohomology theory with coefficients in A, at least when (L, K) are themselves Euclidean neighbourhood retracts (by Proposition 38.3 this includes cell complexes.) For less well behaved spaces Čech cohomology can differ from singular homology (and hence is *not* a homology theory.)

The homotopy axiom follows from the same argument that is used to show that map $\check{H}^k(f)$ is well defined, and this is also on Problem Sheet P.

COROLLARY 38.12. *Let $K \subset L \subset X$ and $K' \subset L' \subset X'$ be compact pairs in two Euclidean neighbourhood retracts. Suppose $f\colon (L, K) \to (L', K')$ is a homotopy equivalence. Then $\check{H}^k(f)\colon \check{H}^k(L', K') \to \check{H}^k(L, K)$ is an isomorphism for all $k \geq 0$.*

Proof. Since $\check{H}^k$ satisfies the homotopy axiom (Problem P.4), this follows from a general fact about functors (Problem A.2.) ■

REMARK 38.13. It follows from Corollary 38.12 that the Čech cohomology groups $\check{H}^k(L, K)$ do not depend on the choice of ambient Euclidean neighbourhood retract X.

We will now verify the remaining axioms.

PROPOSITION 38.14 (Long exact sequence). *If $(L, K) \in \mathsf{K}$ then there is a long exact sequence*

$$\cdots \to \check{H}^k(L; A) \to \check{H}^k(K; A) \xrightarrow{\check{\delta}} \check{H}^{k+1}(L, K; A) \to \check{H}^{k+1}(L; A) \to \dots \qquad (38.2)$$

Proof. The construction of $\check{\delta}$ is similar to how we constructed $\check{H}^k(f)$. Let X be a Euclidean neighbourhood retract containing L. Choose open $U \subset V \subset X$ such that $K \subset U$ and $L \subset V$. Then we have a map

$$\delta\colon H^k(U; A) \to H^{k+1}(V, U; A)$$

coming from the long exact sequence of the pair (V, U) in normal singular cohomology. Composing this with the map $H^{k+1}(V, U; A) \to \check{H}^{k+1}(L, K; A)$ coming from the colimit, we then have maps

$$H^k(U; A) \to \check{H}^{k+1}(L, K; A)$$

for each such pair (V, U). It follows that $\check{H}^l(L, K; A)$ is a solution of the filtered diagram $H^k\colon \mathcal{U}_X(K, \emptyset) \to \mathsf{Ab}$. Thus by the universal property of the colimit again, this gives us an induced map $\check{\delta}\colon \check{H}^k(K; A) \to \check{H}^{k+1}(L, K; A)$.

It remains to check that the sequence (38.2) is exact. This is a general fact about filtered colimits, but rather than prove the general statement let us simply check exactness in our special case.

Suppose $\check{\delta}\langle\alpha\rangle = 0$. Let (V, U) be as above, and suppose $\langle\alpha\rangle$ is represented by $\langle\alpha'\rangle \in H^k(U; A)$. Then $\delta\langle\alpha'\rangle \in H^{k+1}(V, U; A)$ represents $\check{\delta}\langle\alpha\rangle$. Since $\check{\delta}\langle\alpha\rangle = 0$, $\delta\langle\alpha'\rangle$ is contained in the kernel of some restriction $H^{k+1}(V, U; A) \to H^{k+1}(V', U'; A)$ (for some $(V', U') \subset (V, U)$).

Fix such a pair (V', U'). Let $\langle\alpha''\rangle \in H^k(U'; A)$ denote the restriction of $\langle\alpha'\rangle$. Then $\delta\langle\alpha''\rangle = 0$. By exactness there exists a primitive $\langle\beta'\rangle \in H^k(V'; A)$ that maps onto $\langle\alpha''\rangle$ under the map $H^k(V'; A) \to H^k(U'; A)$. This represents a class $\langle\beta\rangle \in \check{H}^k(L; A)$, and this class maps onto $\langle\alpha\rangle$ under the map $\check{H}^k(L; A) \to \check{H}^k(K; A)$. This proves one of the six conditions one needs to check exactness, and I leave the other five to you as a wholesome exercise. ■

Before going any further, let us build a natural transformation $\Phi\colon \check{H}^\bullet \to H^\bullet$ (normal singular cohomology) which commutes with the boundary operators. Indeed, if $(V, U) \in \mathcal{U}_X(L, K)$ then the inclusion $(L, K) \to (V, U)$ induces a map $H^k(V, U; A) \to H^k(L, K; A)$. Since H^k is a contravariant functor, this shows that $H^k(L, K; A)$ is a solution of the diagram $H^k\colon \mathcal{U}_X(L, K) \to \mathsf{Ab}$, and hence there is a map $\Phi(L, K)\colon \check{H}^k(L, K; A) \to H^k(L, K; A)$. An argument similar to those above shows that Φ is natural and commutes with the long exact sequence maps. In general Φ is *not* a natural isomorphism (it need not be injective or surjective), but we do have the following result:

PROPOSITION 38.15. *Suppose $(L, K) \in \mathsf{K}$ are both Euclidean neighbourhood retracts themselves. Then $\Phi(L, K)\colon \check{H}^k(L, K; A) \to H^k(L, K; A)$ is an isomorphism.*

Proof. Suppose K is an Euclidean neighbourhood retract. Then $\mathcal{U}_K(K, \emptyset)$ has a "maximal element", namely $(K, \emptyset)$ itself. (We will define this more formally in Lecture 41 and call it a *terminal object*, cf. Definition 41.2.) Thus $\check{H}^k(K; A) = H^k(K; A)$, and so $\Phi(K, \emptyset)$ is an isomorphism. Similarly if L is an Euclidean neighbourhood retract we have $\check{H}^k(L; A) \cong H^k(L; A)$. Finally to deduce the relative case we use the fact we already know that Φ is natural, and apply the Five Lemma. ■

PROPOSITION 38.16 (Excision). *Let M, N be compact subsets of an Euclidean neighbourhood retract X. Then the inclusion $(M, M \cap N) \hookrightarrow (M \cup N, N)$ induces an isomorphism*

$$\check{H}^k(M \cup N, N; A) \cong \check{H}^k(M, M \cap N; A), \qquad \forall\, k \geq 0.$$

Proof. This follows from the corresponding isomorphism $H^k(U \cup V, V; A) \cong H^k(U, U \cap V; A)$ in singular cohomology. ■

PROPOSITION 38.17 (Mayer-Vietoris). *For each pair M, N of compact subsets of a Euclidean neighbourhood retract there is a long exact Mayer-Vietoris sequence:*

$$\dots \check{H}^k(M \cup N; A) \to \check{H}^k(M; A) \oplus \check{H}^k(N; A) \to \check{H}^k(M \cap N; A) \to \check{H}^{k+1}(M \cup N; A) \to \dots$$

Proof. The usual formal consequence of excision. ■

Remark 38.18. Everything we did prior to Proposition 38.16 works in the more general setting where we only assume that L and K are locally compact. However excision does *not* hold for an arbitrary pair of locally compact subsets of a Euclidean neighbourhood retract—more hypotheses are needed. Since we will only ever use Čech cohomology for compact subsets, I didn't think it worth exploring this here.

In summary, we have proved:

Theorem 38.19. *Let* $\mathsf{CompENR}^2$ *denote the category of compact pairs* (L, K) *where* $K \subseteq L$ *are both Euclidean neighbourhood retracts, with morphisms given by continuous maps of pairs. Then Čech cohomology is a cohomology theory in the sense of Eilenberg-Steenrod on* $\mathsf{CompENR}^2$.

We conclude this lecture by emphasising once again: for an arbitrary pair $(L, K) \in \mathsf{K}$, it is *not* true that the Čech cohomology of (L, K) agrees with the singular homology. This is only true if K and L are themselves Euclidean neighbourhood retracts. Note this is consistent with the cohomology version of Theorem 21.12—the natural transformation Φ induces an isomorphism on a point, and hence is an isomorphism on any pair (L, K) of compact cell complexes. But compact cell complexes are Euclidean neighbourhood retracts (Proposition 38.3), so this doesn't tell us anything we don't already know.

LECTURE 39

The Duality Theorem

In this lecture we prove one of the truly key results in algebraic topology, which we call simply the **Duality Theorem**. At the end of the lecture we deduce the important special case of **Poincaré Duality**.

Let M be an n-dimensional topological manifold. Throughout this lecture we fix a commutative ring R, which we will omit from the notation wherever possible.

Given any pair $K \subset L$ of compact subsets of M, we can always find a Euclidean neighbourhood retract X such that $L \subset X \subset M$. This follows from Proposition 38.3. In particular, the Čech cohomology $\check{H}^k(L, K)$ is defined for any pair of compact subsets of M. Assume now that M is oriented, so that for each compact subset $K \subset M$, we have a class $\langle o_K \rangle \in H_n(M, M \setminus K)$.

Notation: In this lecture $K \subset L$ are always compact subsets of M and U and V are open subsets $U \subset V$ with

$$K \subset U \qquad \text{and} \qquad L \subset V.$$

(We are not necessarily assuming that $U \subset L$ though). We denote by

$$i_{KL}^{UV} \colon (V \setminus K, U \setminus L) \hookrightarrow (V, U)$$

the inclusion.

We use the relative cap product from Definition 37.15 with

$$X := V \setminus K, \qquad X' := U \setminus K, \qquad X'' := V \setminus L.$$

This gives us a map

$$H^k(V \setminus K, U \setminus K) \otimes H_{k+m}(V \setminus K, (U \setminus K) \cup (V \setminus L)) \xrightarrow{\frown} H_m(V \setminus K, V \setminus L).$$

Now consider the composition

$$H_{k+m}(M, M \setminus L) \to H_{k+m}(M, (M \setminus L) \cup U) \xrightarrow{\cong} H_{k+m}(V \setminus K, (U \setminus K) \cup (V \setminus L)),$$

where in the second map we excised the closed set $M \setminus (V \setminus K)$ contained in the open set $(M \setminus L) \cup U$.

If $\langle c \rangle \in H_{k+m}(M, M \setminus L)$ then we denote by $\langle c \rangle_{KL}^{UV}$ the image of $\langle c \rangle$ in $H_{k+m}(V \setminus K, (U \setminus K) \cup (V \setminus L))$.

In particular, for $m = n - k$ and $\langle c \rangle = \langle o_L \rangle$ we get a map

$$d_{KL}^{UV} \colon H^k(V \setminus K, U \setminus K) \to H_{n-k}(V \setminus K, V \setminus L)$$

given by

$$d^{UV}_{KL}\langle\gamma\rangle := \langle\gamma\rangle \frown \langle o_L\rangle^{UV}_{KL}.$$

By excision again, this last group is isomorphic to $H_{n-k}(M \setminus K, M \setminus L)$. Hence putting this altogether, we have a map that we denote by

$$D^{UV}_{KL} \colon H^k(V,U) \to H_{n-k}(M \setminus K, M \setminus L)$$

and makes the following diagram commute:

$$\begin{array}{ccc} H^k(V,U) & \xrightarrow{H^k(\imath^{UV}_{KK})} & H^k(V \setminus K, U \setminus K) \\ {\scriptstyle D^{UV}_{KL}}\downarrow & & \downarrow{\scriptstyle d^{UV}_{KL}} \\ H_{n-k}(M \setminus K, M \setminus L) & \xrightarrow[\cong]{} & H_{n-k}(V \setminus K, V \setminus L) \end{array}$$

Our aim is to show that D^{UV}_{KL} induces a map

$$D_{KL} \colon \check{H}^k(L,K) \to H_{n-k}(M \setminus K, M \setminus L).$$

This follows from the following naturality statement.

LEMMA 39.1. *Let $X \subset M$ be an Euclidean neighbourhood retract containing V. Suppose $U' \subset V'$ is another pair of open sets with $U \subset U'$ and $V \subset V' \subset X$. Then the inclusion $\jmath \colon (V,U) \hookrightarrow (V',U')$ satisfies for every $k \geq 0$:*

$$D^{U'V'}_{KL} = D^{UV}_{KL} \circ H^k(\jmath).$$

The proof of Lemma 39.1 is immediate from naturality of the cap product. It follows that the maps

$$D^{UV}_{KL} \colon H^k(V,U) \to H_{n-k}(M \setminus K, M \setminus L)$$

form a solution to the filtered diagram $H^k \colon \mathcal{U}_X(L,K) \to \mathsf{Ab}$, and hence the universal property of the colimit provides us with the desired map $D_{KL} \colon \check{H}^k(L,K) \to H_{n-k}(M \setminus K, M \setminus L)$.

In fact, the maps D_{KL} are natural transformations, as the following lemma proves.

LEMMA 39.2. *Let $(L,K) \subset (L',K')$ (as pairs). Then for any $k \geq 0$, the following commutes:*

$$\begin{array}{ccc} \check{H}^k(L',K') & \xrightarrow{D_{K'L'}} & H_{n-k}(M \setminus K', M \setminus L') \\ \downarrow & & \downarrow \\ \check{H}^k(L,K) & \xrightarrow[D_{KL}]{} & H_{n-k}(M \setminus K, M \setminus L) \end{array}$$

Proof. Let $K \subset K' \subset U$ and $L \subset L' \subset V$ with $U \subset V$ open. Let $\jmath' \colon (M \setminus K', M \setminus L') \hookrightarrow (M \setminus K, M \setminus L)$ denote the inclusion. Then it follows from the definition that

$$H_{n-k}(\jmath') \circ D^{UV}_{K'L'} = D^{UV}_{KL}.$$

The desired commutativity is now a formal consequence of properties of colimits. I invite you to fill in the details. ■

Here then is the main result of today's lecture. It is one of the most important theorems in all of algebraic topology.

THEOREM 39.3 (The Duality Theorem). *Let M be an n-dimensional oriented topological manifold. Then for every pair $K \subset L$ of compact subsets of M, the duality homomorphism*

$$D_{KL} \colon \check{H}^k(L, K) \to H_{n-k}(M \setminus K, M \setminus L)$$

is an isomorphism.

The proof will take us some time. The key technical result needed is the following commutativity statement.

PROPOSITION 39.4. *For each pair $K \subset L$ of compact sets the following diagram commutes up to a sign:*

$$\begin{array}{ccc} \check{H}^{k-1}(K) & \xrightarrow{D_K} & H_{n-k+1}(M, M \setminus K) \\ \check{\delta}\big\downarrow & & \big\downarrow\delta \\ \check{H}^k(L, K) & \xrightarrow[D_{KL}]{} & H_{n-k}(M \setminus K, M \setminus L) \end{array}$$

REMARK 39.5. The fact that the diagram in Proposition 39.4 only commutes up to a sign is due to our convention on defining the cap product. It is possible (and indeed, a lot of the literature does this) to set things up so that the diagram genuinely commutes. This makes the formulae defining the cap product harder to remember (and harder to lecture!) though, so I didn't elect to do this.

In order to prove Proposition 39.4 we need the following general statement about cap products.

PROPOSITION 39.6. *Let X be an arbitrary topological space. Assume $X = X' \cup X''$ with X', X'' open. Let $Y' \subset X'$ and $Y'' \subset X''$ be open sets, and put $Y := Y' \cup Y''$. Fix a homology class*

$$\langle c \rangle \in H_n(X, Y)$$

and let $\langle c' \rangle$ be the element that $\langle c \rangle$ maps to under the following composition:

$$H_n(X, Y) \to H_n(X, Y' \cup X'') \cong H_n(X', Y' \cup (X' \cap X''))$$

where the $\cong$ is an excision isomorphism. Similarly let $\langle c'' \rangle$ be the element that $\langle c \rangle$ maps to under the following composition:

$$H_n(X, Y) \to H_n(X, X' \cup Y'') \cong H_n(X'', (X' \cap X'') \cup Y'')$$

Then the following hexagon commutes up to a sign:

$$
\begin{array}{ccccc}
 & & H^{k-1}(X',Y') & & \\
 & \swarrow{\scriptstyle \frown\langle c'\rangle} & & \searrow & \\
H_{n-k+1}(X', X'\cap X'') & & & & H^{k-1}(X'\cap X'', Y'\cap X'') \\
\downarrow{\scriptstyle \delta} & & & & \downarrow{\scriptstyle \delta} \\
H_{n-k}(X'\cap X'', X'\cap Y'') & & & & H^{k}(X'', X'\cap X'') \\
 & \searrow & & \swarrow{\scriptstyle \frown\langle c''\rangle} & \\
 & & H_{n-k}(X'',Y'') & &
\end{array}
$$

Proof. As the picture suggests[1], the proof of this is a rather tedious application of the Hexagon Lemma (Problem H.2). I omit the details. ∎

Let us now get started on the proof of Proposition 39.4.

Proof of Proposition 39.4. By passing to the colimit is suffices to show that the following diagram commutes up to a sign:

$$
\begin{array}{ccc}
H^{k-1}(U) & \xrightarrow{D^{\emptyset U}_{\emptyset K}} & H_{n-k+1}(M, M\setminus K) \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta} \\
H^{k}(V,U) & \xrightarrow[D^{UV}_{KL}]{} & H_{n-k}(M\setminus K, M\setminus L)
\end{array}
\tag{39.1}
$$

We apply Proposition 39.6 with:

$$X' = U, \qquad X'' = V\setminus K, \qquad Y' = \emptyset, \qquad Y'' = V\setminus L.$$

Then $H_n(X,Y) = H_n(V, V\setminus K) \cong H_n(M, M\setminus L)$, and we take in Proposition 39.6 our class $\langle c\rangle$ to be $\langle o_L\rangle$. Then using the notation from the statement of Proposition 39.6, the two classes $\langle c'\rangle$ and $\langle c''\rangle$ become $\langle o_L\rangle^{\emptyset U}_{\emptyset K}$ and $\langle o_L\rangle^{UV}_{KL}$ respectively. We squidge the hexagon from Proposition 39.6 to form the middle rectangle in the next diagram. The top and bottom rectangles commute by naturality of the connecting homomorphisms in homology and cohomology:

$$
\begin{array}{ccccc}
H^{k-1}(U) & & \xrightarrow{\qquad\qquad\delta\qquad\qquad} & & H^{k}(V,U) \\
\downarrow{\scriptstyle \cong} & & & & \downarrow \\
H^{k-1}(U) & \longrightarrow & H^{k-1}(U\setminus K) & \xrightarrow{\delta} & H^{k}(V\setminus K, U\setminus K) \\
\downarrow{\scriptstyle \frown\langle o_L\rangle^{\emptyset U}_{\emptyset K}} & & & & \downarrow{\scriptstyle \frown\langle o_L\rangle^{UV}_{KL}} \\
H_{n-k+1}(U, U\setminus K) & \xrightarrow{\delta} & H_{n-k}(U\setminus K, U\setminus L) & \longrightarrow & H_{n-k}(V\setminus K, V\setminus L) \\
\downarrow & & & & \downarrow \\
H_{n-k+1}(M, M\setminus K) & & \xrightarrow[\qquad\qquad\delta\qquad\qquad]{} & & H_{n-k}(M\setminus K, M\setminus L).
\end{array}
$$

[1] Actually the picture is a little misleading, since in order to apply the Hexagon Lemma one first transforms this hexagon into a different hexagon...

The outside square is exactly (39.1) (after reflecting and rotating). This completes the proof. ■

Proposition 39.4 allows us to form the following commutative (up to a sign) diagrams:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \check{H}^k(L) & \longrightarrow & \check{H}^k(K) & \xrightarrow{\check{\delta}} & \check{H}^{k+1}(L,K) & \longrightarrow & \cdots \\ & & {\scriptstyle D_L}\big\downarrow & & \big\downarrow{\scriptstyle D_K} & & \big\downarrow{\scriptstyle D_{KL}} & & \\ \cdots & \longrightarrow & H_{n-k}(M, M \setminus L) & \longrightarrow & H_{n-k}(M, M \setminus K) & \xrightarrow[\delta]{} & H_{n-k-1}(M \setminus K, M \setminus L) & \longrightarrow & \cdots \end{array}$$

The top row is the long exact sequence of the pair (L, K) in Čech cohomology (Proposition 38.14). The bottom row is the long exact sequence of the triple $(M, M \setminus K, M \setminus L)$ (Problem F.4). The fact that the diagram commutes (up to a sign) uses Lemma 39.2 and Proposition 39.4. An application of the Five Lemma therefore tells us that it *suffices to prove the Duality Theorem 39.3 in the absolute case*, that is, that $D_K \colon \check{H}^k(K) \to H_{n-k}(M, M \setminus K)$ is always an isomorphism.

We also need to investigate how Mayer-Vietoris sequences behave. This is the content of the following claim:

PROPOSITION 39.7. *Suppose K and K' are two compact sebsets. Then the Mayer-Vietoris sequence in Čech cohomology and the normal Mayer-Vietoris sequence for their complements commutes up to a sign:*

$$\begin{array}{ccc} \vdots & & \vdots \\ \big\downarrow & & \big\downarrow \\ \check{H}^k(K \cup K') & \xrightarrow{D_{K \cup K'}} & H_{n-k}(M, M \setminus (K \cup K')) \\ \big\downarrow & & \big\downarrow \\ \check{H}^k(K) \oplus \check{H}(K') & \xrightarrow{(D_K, D_{K'})} & H_{n-k}(M, M \setminus K) \oplus H_{n-k}(M, M \setminus K') \\ \big\downarrow & & \big\downarrow \\ \check{H}^k(K \cap K') & \xrightarrow{D_{K \cap K'}} & H_{n-k}(M, M \setminus (K \cap K')) \\ {\scriptstyle \delta}\big\downarrow & & \big\downarrow{\scriptstyle \delta} \\ \check{H}^{k+1}(K \cup K') & \xrightarrow[D_{K \cup K'}]{} & H_{n-k-1}(M, M \setminus (K \cup K')) \\ \big\downarrow & & \big\downarrow \\ \vdots & & \vdots \end{array}$$

One can prove this directly, but it is also a formal consequence of Lemma 39.2 and Proposition 39.4. Indeed, the connecting map in the Mayer-Vietoris sequence is defined from induced maps from inclusions and ordinary (co)boundary operators in long exact sequences—see the proof of Proposition 14.9 and the Barratt-Whitehead Lemma (Proposition 11.4) if you are sceptical.

We are ready to prove the Duality Theorem. We will use an argument similar to the one used in the proof of Theorem 36.19. Let us first formalise this in the following "meta" theorem.

THEOREM 39.8. *Let $(\heartsuit K)$ be a statement about a compact set K in a fixed manifold M. Assume that:*

1. *$(\heartsuit K)$ is true for any convex compact set (cf. Definition 36.8).*
2. *If $(\heartsuit K)$, $(\heartsuit L)$ and $(\heartsuit K \cap L)$ holds then so does $(\heartsuit K \cup L)$.*
3. *If $K_1 \supset K_2 \supset \cdots K := \bigcap_i K_i$ and $(\heartsuit K_i)$ holds for every i, then $(\heartsuit K)$ also holds.*

Then $(\heartsuit K)$ is true for any compact set K in M.

Proof. Since the intersection of convex sets is convex, using induction on properties (1) and (2) tell us that $(\heartsuit K_1 \cup \cdots \cup K_m)$ is true for any finite union of convex sets contained in the same chart domain.

Next, if K is any compact set in a chart domain then K is the intersection of a sequence $K_1 \supset K_2 \supset \cdots$ where each K_i is a finite union of compact convex sets. Thus by property (3) in this case $(\heartsuit K)$ holds.

Finally, any compact set K is a finite union of compact sets in chart domains. Thus by induction we see that $(\heartsuit K)$ is true in this case too. ■

Now let us prove the Duality Theorem.

Proof of the Duality Theorem 39.3. We will apply Theorem 39.8 where $(\heartsuit K)$ is the statement: D_K is an isomorphism. Thus we need only verify the three properties in the hypotheses of Theorem 39.8.

The proof of (2) is immediate from the Mayer-Vietoris sequence (Proposition 39.7) and the Five Lemma.

Now let us prove (1) in increasingly more general cases. If K is empty then all groups are zero. Now suppose $K = \{x\}$ is a single point. If $\langle o_x \rangle$ is our given generator of $H_n(M, M \setminus x)$ then $\square \mapsto \square \frown \langle o_x \rangle$ takes the generator $\nu \in H^0(x) = \check{H}^0(x)$ into $\langle o_x \rangle$ again (this is part (4) of Lemma 37.16). Thus D_x is an isomorphism for $k = 0$. But for $k > 0$, both $\check{H}^k(x)$ and $H_{n-k}(M, M \setminus x)$ are zero (cf. Lemma 36.5.)

Now suppose K is an arbitrary convex set in M. Let $x \in K$. By naturality of the duality morphism (Lemma 39.2), the following commutes:

$$\begin{array}{ccc} \check{H}^k(K) & \xrightarrow{D_K} & H_{n-k}(M, M \setminus K) \\ \downarrow & & \downarrow \\ \check{H}^k(x) & \xrightarrow[D_x]{} & H_{n-k}(M, M \setminus x). \end{array}$$

We claim that the two vertical maps are isomorphisms. The right-hand vertical map is an isomorphism by part (1) of Lemma 36.9. To show that $\check{H}^k(K) \to \check{H}^k(x)$ is an isomorphism, choose a chart $\varphi\colon U \to \mathbb{R}^n$ centred at x with $K \subset U$. The set

$C := \varphi(K)$ is a compact convex subset of $\mathbb{R}^n$ and thus is an Euclidean neighbourhood retract (this follows from Proposition 38.3). Thus we have isomorphisms

$$\check{H}^k(K) \cong \check{H}^k(C) \cong H^k(C) \cong H^k(x) \cong \check{H}^k(x).$$

Thus D_K is indeed an isomorphism, and we have proved that hypothesis (1) holds.

To complete the proof we must show that if $K_1 \supset K_2 \supset \cdots = \bigcap_i K_i$ then the maps

$$\underrightarrow{\operatorname{colim}}_i\, \check{H}^k(K_i) \to \check{H}^k(K), \qquad \text{and} \qquad \underrightarrow{\operatorname{colim}}_i\, H_{n-k}(M, M \setminus K_i) \to H_{n-k}(M, M \setminus K)$$

are both isomorphisms. The first isomorphism is an easy consequence of the definition; let us show surjectivity only. If $\langle c \rangle \in \check{H}^k(K)$ is represented by $\langle c' \rangle \in H^k(U)$ for some open $U \supset K$ then there exists i such that $K_i \subset U$ whence $\langle c' \rangle$ represents an element in $\check{H}^k(K_i)$. Finally for singular homology, the fact that the second colimit is an isomorphism follows as singular homology commutes with colimits—we used this exact statement in (17.1), see also Remark 17.8. ■

Finally, let us conclude this lecture by stating two special cases of the Duality Theorem.

Corollary 39.9 (Poincaré Duality). *Let M be an oriented n-dimensional closed topological manifold with fundamental class $\langle o_M \rangle \in H_n(M)$. Then*

$$H^k(M) \to H_{n-k}(M), \qquad \langle c \rangle \mapsto \langle c \rangle \frown \langle o_M \rangle$$

is an isomorphism.

This is immediate from the Duality Theorem, since in this case $\check{H}^k(M) = H^k(M)$ (see part (2) of Corollary 38.5). Take $L = M$ and $K = \emptyset$, and observe that the duality morphism is simply $\square \mapsto \square \cap \langle o_M \rangle$.

On Problem Sheet P, you are asked to prove **Alexander Duality**, which is another famous duality result that can be seen a special case of the Duality Theorem 39.3.

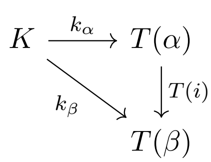
LECTURE 40

Adjoint functors

We now move onto the final section of the course. We will go back to homotopy theory and define the *higher homotopy groups* $\pi_n(X, x)$ for a pointed topological space (X, x).

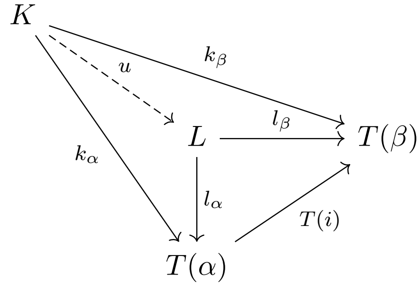
First, however, we will spend two lectures on more abstract nonsense. In this lecture we discuss duality in categories. Recall in Lecture 16 we defined the notion of a *colimit*. We first briefly define the dual version of a colimit, which (unsurprisingly) is called a "limit". In Definition 16.3 we introduced a "solution" to a diagram in a category. As mentioned in a footnote there, our definition of a solution is often called a *co-cone*. This is because there is a dual notion, called a "cone", which is defined in exactly the same way only with the arrows reversed. As with covariant and contravariant functors (which we typically refer to just as "functors"), in general we will simply call both co-cones and cones simply "solutions". Here is the formal definition of a cone.

Definition 40.1. Let C be a category. Let J be an index category and let $T\colon \mathsf{J} \to \mathsf{C}$ be a diagram in C. A **cone** for T is an object K of C together with a family of morphisms $k_\alpha\colon K \to T(\alpha)$ in C for each object $\alpha \in \mathrm{obj}(\mathsf{J})$ such that if $i\colon \alpha \to \beta$ is any morphism in J then the following commutes:



We write $(K, \{k_\alpha\})$ to indicate the solution.

Definition 40.2. Let J be an index category and let $T\colon \mathsf{J} \to \mathsf{C}$ be a diagram in C. A **limit** is a cone $(L, \{l_\alpha\})$ that satisfies the following *universal property*: if $(K, \{k_\alpha\})$ is any other cone then there exists a *unique* morphism $u\colon K \to L$ such that the following diagram commutes for every morphism $i\colon \alpha \to \beta$ in J:



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We write $L = \lim T$. If L exists, it is unique up to isomorphism.

Just as with colimits, some limits have special names. The simplest is the *product*.

Example 40.3. Take J to have exactly two objects and no morphisms (apart from the identity morphisms).

$$\spadesuit \qquad \heartsuit$$

A **product** in C is the limit of a diagram $T\colon \mathsf{J} \to \mathsf{C}$. For example, in Sets the product of X and Y is just the normal cartesian product $X \times Y$. In general, the product (in an arbitrary category) C of two objects A and B is denoted by $A \sqcap B$ (if it exists).

On Problem Sheet Q you will show that commutativity and associativity holds for (co)products in the following sense:

Lemma 40.4. *Let C be a category and $A, B, C \in \mathrm{obj}(\mathsf{C})$. Then:*

1. *If the products $A \sqcap B$ and $B \sqcap A$ exist then they are isomorphic:*
$$A \sqcap B \cong B \sqcap A.$$
2. *If the coproducts $A \sqcup B$ and $B \sqcup A$ exist then they are isomorphic:*
$$A \sqcup B \cong B \sqcup A.$$
3. *If the products $A \sqcap B$, $B \sqcap C$, $(A \sqcap B) \sqcap C$ and $A \sqcap (B \sqcap C)$ all exist then:*
$$(A \sqcap B) \sqcap C \cong A \sqcap (B \sqcap C).$$
4. *If the coproducts $A \sqcup B$, $B \sqcup C$, $(A \sqcup B) \sqcup C$ and $A \sqcup (B \sqcup C)$ all exist then:*
$$(A \sqcup B) \sqcup C \cong A \sqcup (B \sqcup C).$$

Warning: It does *not* follow without a further diagrammatic assumption that the generalised associative law holds from the associative law involving three terms!

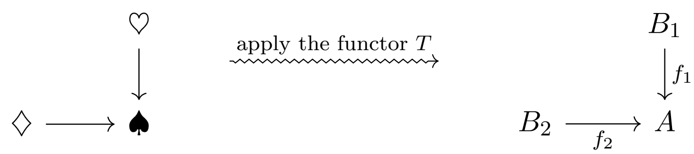
Remark 40.5. In Ab the product and the coproduct (cf. Example 16.8) coincide. In Groups they do not: the product is the direct product and the coproduct is the free product.

The dual notion to a pushout is called a *pullback*.

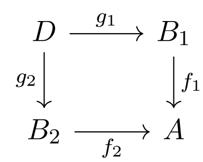
Example 40.6. Let J be a category with exactly three objects, $\{\heartsuit, \spadesuit, \diamondsuit\}$, and assume that there is unique morphism $\heartsuit \to \spadesuit$ and a unique morphism $\diamondsuit \to \spadesuit$, and that the only other morphisms are the identity morphisms (whose existence is forced). We write this pictorially as

$$\begin{array}{ccc} & & \heartsuit \\ & & \downarrow \\ \diamondsuit & \longrightarrow & \spadesuit \end{array}$$

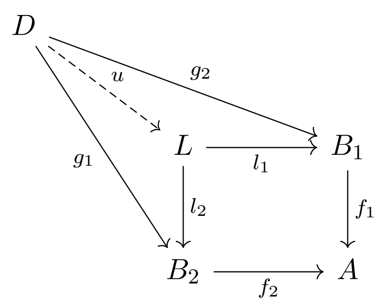
A functor $T\colon \mathsf{J} \to \mathsf{C}$ is the same thing as a triple of objects (A, B_1, B_2) in $\mathrm{obj}(\mathsf{C})$ together with a choice of two morphisms $f_1\colon A \to B_1$ and $f_2\colon A \to B_2$.



A cone corresponds to an object D of C together with two morphisms $g_1\colon D \to B_1$ and $g_2\colon D \to B_2$ such that the following commutes:



A limit of T is a cone (L, l_1, l_2) such that for any other cone (D, g_1, g_2) there is a unique map $u\colon D \to L$ such that the following commutes:



We call L a **pullback**. In the category of sets, if $f\colon X \to Z$ and $g\colon Y \to Z$ then the pullback of f and g is the set

$$X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}\,.$$

I invite you to check what pullbacks are in other familiar categories.

Now let us move onto a *filtered limit*. These are defined in the same way as filtered colimits (i.e. one starts with a filtered category), but since the arrows point the other way one needs to use contravariant functors.

DEFINITION 40.7. Let J be a filtered index category (cf. Definition 16.11). A **filtered limit** is a limit of *contravariant* functor $T\colon \mathsf{J} \to \mathsf{C}$ (or equivalently, a limit of *covariant* functor defined on the opposite category.) We use the notation $\varprojlim T$ to indicate a limit is filtered.

As with filtered colimits, the easiest way to manufacture examples is to begin with a directed set $(\Lambda, \preceq)$, form the corresponding index category $\mathsf{J}(\Lambda, \preceq)$, and then take the opposite category. Forming the opposite category essentially amounts to replacing $\preceq$ with $\succeq$. Let us illustrate this with the simplest possible directed set.

Example 40.8. A **sequential limit** is a filtered limit on $(\mathbb{N}, \geq)$. Explicitly, let C be a category, and assume we are given a sequence

$$f_n \colon C_{n+1} \to C_n, \qquad n \in \mathbb{N},$$

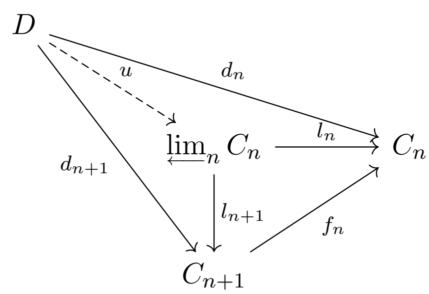
of morphisms in C. The filtered limit of T is an object $\varprojlim T$ (which we will usually write as $\varprojlim_n C_n$ instead) together with a family of morphisms $l_n \colon \varprojlim_n C_n \to C_n$ for $n \in \mathbb{N}$ such that

$$l_n = f_n \circ l_{n+1}, \qquad \forall\, n \in \mathbb{N}.$$

This satisfies the universal property that if $(D, \{d_n\})$ is object of C and a family of morphisms $d_n \colon D \to C_n$ for $n \in \mathbb{N}$ such that

$$d_n = f_n \circ d_{n+1}, \qquad \forall\, n \in \mathbb{N},$$

then there exists a *unique* morphism $u \colon D \to \varprojlim_n C_n$ such that the following diagram commutes:



Remark 40.9. **Warning:** As already mentioned in Remark 34.11, the analogue of Theorem 16.22 for limits is *not* true!

Definition 40.10. Let C and D be two categories. The **product category** $\mathsf{C} \times \mathsf{D}$ is the category whose objects are ordered pairs (C, D) where $C \in \operatorname{obj}(\mathsf{C})$ and $D \in \operatorname{obj}(\mathsf{D})$, and

$$\operatorname{Hom}_{\mathsf{C}\times\mathsf{D}}((C,D),(C',D')) = \big\{(f,g) \mid f \in \operatorname{Hom}_{\mathsf{C}}(C,C')\; g \in \operatorname{Hom}_{\mathsf{D}}(D,D')\big\}\,.$$

The composition $(f,g) \circ_{\mathsf{C}\times\mathsf{D}} (f',g')$ is defined as you expect:

$$(f,g) \circ_{\mathsf{C}\times\mathsf{D}} (f',g') := \big(f \circ_{\mathsf{C}} f'), (g \circ_{\mathsf{D}} g'\big)\,.$$

The identity element $\mathrm{id}_{(C,D)}$ is simply the pair $(\mathrm{id}_C, \mathrm{id}_D)$.

Remark 40.11. As the name indicates, the product category is simply the product (in the sense of Example 40.3 above) in the category Cat of small categories.

Definition 40.12. A **bifunctor** is a functor defined on a product category: $T \colon \mathsf{C} \times \mathsf{D} \to \mathsf{E}$. We will always assume bifunctors are covariant—if we wish to consider a bifunctor which is contravariant in one or both of its variables we simple replace the category with the corresponding opposite category.

Example 40.13. Let C be any category. Then there is a Hom-**bifunctor**

$$\mathrm{Hom}\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Sets}$$

which assigns to an ordered pair (A, B) of objects of C the hom-set $\mathrm{Hom}(A, B)$. This bifunctor is contravariant in the first variable and covariant in the second. Explicitly, if $f\colon A \to B$ and $g\colon C \to D$ are morphism in C then

$$\mathrm{Hom}(f, g)\colon \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, D), \qquad h \mapsto g \circ h \circ f.$$

Now let C and D be categories, and let $S\colon \mathsf{C} \to \mathsf{D}$ and $T\colon \mathsf{D} \to \mathsf{C}$ be covariant functors. Conisder the two bifunctors

$$\mathrm{Hom}(S(\square), \square)\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Sets}, \qquad (C, D) \mapsto \mathrm{Hom}_{\mathsf{D}}(S(C), D),$$

and

$$\mathrm{Hom}(\square, T(\square))\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \to \mathsf{Sets}, \qquad (C, D) \mapsto \mathrm{Hom}_{\mathsf{C}}(C, T(D)).$$

In general there is no reason for these two bifunctors to be related. However, since they are both functors with the same source category ($\mathsf{C}^{\mathrm{op}} \times \mathsf{D}$) and the same target category (Sets), one can ask the question as to when they are naturally isomorphic. This leads to the notion of an *adjoint pair*, which is one of the most important concepts in category theory.

Definition 40.14. Let C and D be categories, and let $S\colon \mathsf{C} \to \mathsf{D}$ and $T\colon \mathsf{D} \to \mathsf{C}$ be covariant functors. We say that (S, T) is an **adjoint pair** if there is a natural isomorphism between the the two bifunctors

$$\Psi\colon \mathrm{Hom}(S(\square), \square) \to \mathrm{Hom}(\square, T(\square)).$$

Explicitly, this means that for every pair of objects (C, D) we are given a bijection

$$\Psi(C, D)\colon \mathrm{Hom}_{\mathsf{D}}(S(C), D) \to \mathrm{Hom}_{\mathsf{C}}(C, T(D)),$$

which satisfies the following naturality condition: if $f\colon C \to C'$ and $g\colon D \to D'$ are morphisms in C and D respectively, then the following diagram should commute:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{D}}(S(C'), D) & \xrightarrow{\Psi(C', D)} & \mathrm{Hom}_{\mathsf{C}}(C', T(D)) \\ \downarrow{\scriptstyle \mathrm{Hom}(S(f), g)} & & \downarrow{\scriptstyle \mathrm{Hom}(f, T(g))} \\ \mathrm{Hom}_{\mathsf{D}}(S(C), D') & \xrightarrow[\Psi(C, D')]{} & \mathrm{Hom}_{\mathsf{C}}(C, T(D')) \end{array} \tag{40.1}$$

One calls S the **left adjoint** functor and T the **right adjoint** functor. The ordering is important: if (S, T) is an adjoint pair then (T, S) may not be.

Here is a trivial example.

Example 40.15. Let Forget: $\mathsf{Ab} \to \mathsf{Sets}$ denote the forgetful functor. Let Free: $\mathsf{Sets} \to \mathsf{Ab}$ denote the **free functor** that sends a set X to the free abelian group $F(X)$ with basis X. Then (Free, Forget) are an adjoint pair. The function

$$\Psi(X, A) \colon \mathrm{Hom}_{\mathsf{Ab}}(F(X), A) \to \mathrm{Hom}_{\mathsf{Sets}}(X, A)$$

is simply given by $f \mapsto f|_X$. The fact that this is a bijection is precisely the statement of Lemma 7.2: any function $f \colon X \to A$ specifies uniquely a homomorphism $\tilde{f} \colon F(X) \to A$.

This also works if we consider Forget as a functor Forget: $\mathsf{Groups} \to \mathsf{Sets}$, but in this case one should define Free: $\mathsf{Sets} \to \mathsf{Groups}$ to be the functor that assigns to a set X the free group (not the free abelian group!) with basis X.

Here is another more interesting example of a forgetful functor forming an adjoint pair.

Example 40.16. Let Met denote the category of metric spaces, and let $\mathsf{CompleteMet}$ denote the full subcategory of complete metric spaces. Then there is a forgetful functor Forget: $\mathsf{CompleteMet} \to \mathsf{Met}$ that forgets a metric space is complete.

Next, as you all (hopefully) remember, to any metric space (X, d) one can construct a complete metric space $(\bar{X}, \bar{d})$, called the **completion** of X, which contains X as a dense subspace. The completion satisfies the following universal property: if Y is any complete metric space and $f \colon X \to Y$ is any uniformly continuous function, then there exists a unique uniformly continuous function $\bar{f} \colon \bar{X} \to Y$ such that $\bar{f}|_X = f$. Like all universal properties, this tells us $\bar{X}$ is unique up to isomorphism in Met (that is, unique up to isometry).

Explicitly, $\bar{X}$ is constructed as follows. First, let $\widehat{X}$ denote the set of all Cauchy sequences (x_n) in X. Define a pseudometric $\widehat{d}$ on $\widehat{X}$ by declaring

$$\widehat{d}((x_n), (x'_n)) := \lim_n d(x_n, x'_n)$$

(this limit exists because $\mathbb{R}$ is a complete metric space!) Then let $\bar{X}$ denote the quotient of $\widehat{X}$ under the equivalence relation

$$(x_n) \sim (x'_n) \qquad \Leftrightarrow \qquad \widehat{d}((x_n), (x'_n)) = 0.$$

Then $\widehat{d}$ factors to define a complete metric on $\bar{X}$. The embedding $X \hookrightarrow \bar{X}$ sends a point x to the equivalence class of the constant sequence $x_n \equiv x$.

With a bit more work, one can show that the operation $X \mapsto \bar{X}$ defines a functor Complete: $\mathsf{Met} \to \mathsf{CompleteMet}$, and (Complete, Forget) forms an adjoint pair.

Here is another algebraic example for you to work through on Problem Sheet Q.

Example 40.17. Let R and R' be rings (not necessarily commutative), and let M be an (R, R')-bimodule. Then $(\square \otimes_R M, \mathrm{Hom}_{R'}(M, \square))$ forms an adjoint pair. Similarly $(M \otimes_{R'} \square, \mathrm{Hom}_R(M, \square))$ forms an adjoint pair.

An adjoint functor pair gives rise to a *unit* and a *counit* as follows.

Definition 40.18. Let C and D be categories, and let $S\colon \mathsf{C} \to \mathsf{D}$ and $T\colon \mathsf{D} \to \mathsf{C}$ be covariant functors. Suppose (S, T) is an adjoint pair. For any $C \in \operatorname{obj}(\mathsf{C})$, taking $D = S(C)$ in the definition of $\Psi(C, D)$ we obtain a natural bijection

$$\Psi(C, S(C))\colon \operatorname{Hom}_{\mathsf{D}}(S(C), S(C)) \to \operatorname{Hom}_{\mathsf{C}}(C, T(S(C)).$$

We define the **unit** of (S, T) to be family of maps $\eta\colon \mathrm{id}_{\mathsf{C}} \to T \circ S$ given by

$$\eta(C) := \Psi(C, S(C))(\mathrm{id}_{S(C)})\colon C \to T(S(C)).$$

If C is any object of C and D is any object of D, then the following **unit equation** holds:

$$\Psi(C, D)(g) = T(g) \circ \eta(C), \qquad \forall\, g\colon S(C) \to D. \tag{40.2}$$

This can be seen by considering the diagram (40.1) as follows:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathsf{D}}(S(C), S(C)) & \xrightarrow{\Psi(C,S(C))} & \operatorname{Hom}_{\mathsf{C}}(C, T(S(C))) \\
{\scriptstyle \operatorname{Hom}(S(\mathrm{id}_C),g)}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Hom}(\mathrm{id}_C,T(g))} \\
\operatorname{Hom}_{\mathsf{D}}(S(C), D) & \xrightarrow[\Psi(C,D)]{} & \operatorname{Hom}_{\mathsf{C}}(C, T(D))
\end{array}$$

Start with $\mathrm{id}_{S(C)}$ in the top left-hand corner. Then going clockwise yields $T(g) \circ \eta(C) \circ \mathrm{id}_C = T(g) \circ \eta(C)$, and going anticlockwise yields $\Psi(C, D)(g)$, which gives the unit equation.

Lemma 40.19. *The unit is a natural transformation* $\eta\colon \mathrm{id}_{\mathsf{C}} \to T \circ S$.

Proof. To see that η is a natural transformation, we need to check that if $f\colon C \to C'$ is any morphism in C then the following diagram commutes:

$$\begin{array}{ccc}
C & \xrightarrow{\eta(C)} & T(S(C)) \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle T(S(f))} \\
C' & \xrightarrow[\eta(C')]{} & T(S(C'))
\end{array}$$

The unit equation (40.2) applied with $g = S(f)$ tells us that

$$\Psi(C, S(C'))(S(f)) = T(S(f)) \circ \eta(C). \tag{40.3}$$

Now apply (40.1) again as follows:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathsf{D}}(S(C'), S(C')) & \xrightarrow{\Psi(C',S(C'))} & \operatorname{Hom}_{\mathsf{C}}(C', T(S(C'))) \\
{\scriptstyle \operatorname{Hom}(S(f),\mathrm{id}_{S(C')}))}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Hom}(f,T(\mathrm{id}_{S(C')}))} \\
\operatorname{Hom}_{\mathsf{D}}(S(C), S(C')) & \xrightarrow[\Psi(C,S(C'))]{} & \operatorname{Hom}_{\mathsf{C}}(C, T(S(C'))).
\end{array}$$

Starting with $\mathrm{id}_{S(C')}$ in the top left-hand corner we obtain

$$\eta(C') \circ f = \Psi(C, S(C'))(S(f)). \tag{40.4}$$

Combining (40.3) and (40.4) tells us that $\eta(C') \circ f = T(S(f)) \circ \eta(C)$, which is what we desired. ∎

Definition 40.20. Let C and D be categories, and let $S\colon \mathsf{C} \to \mathsf{D}$ and $T\colon \mathsf{D} \to \mathsf{C}$ be covariant functors. Suppose (S, T) is an adjoint pair. Given any object D of D, taking $C = T(D)$ we obtain a natural bijection

$$\Psi(T(D), D)\colon \mathrm{Hom}_{\mathsf{D}}(S(T(D)), D) \to \mathrm{Hom}_{\mathsf{C}}(T(D), T(D)),$$

and this allows us to define the **counit** of (S, T) to as the family of maps $\varepsilon\colon S \circ T \to \mathrm{id}_{\mathsf{D}}$ given by

$$\varepsilon(D) := \Psi(T(D), D)^{-1}(\mathrm{id}_{T(D)})\colon S(T(D)) \to D.$$

If C is any object of C and D is any object of D, then the following **counit equation** holds:

$$\Psi(C, D)^{-1}(h) = \varepsilon(D) \circ S(h), \qquad \forall\, h\colon C \to T(D). \tag{40.5}$$

and the same proof as above shows that the counit is a natural transformation $S \circ T \to \mathrm{id}_{\mathsf{D}}$.

Remark 40.21. The unit equation (40.2) and the counit equation (40.5) uniquely determine Ψ. Thus to show two functors (S, T) are an adjoint pair it suffices to construct the unit and the counit.

The next theorem is one of the main properties of adjoint pairs. This is probably the only (semi-)difficult theorem in category theory that we will prove in the entire course.

Theorem 40.22. *Let C and D be categories, and let $S\colon \mathsf{C} \to \mathsf{D}$ and $T\colon \mathsf{D} \to \mathsf{C}$ be covariant functors. Suppose (S, T) is an adjoint pair. Then S preserves colimits and T preserves limits. That is, if $P\colon \mathsf{J} \to \mathsf{C}$ is a diagram in C such that $\operatorname{colim} P$ exists, then $\operatorname{colim}(S \circ P)$ exists as well and there is a natural isomorphism*

$$S(\operatorname{colim} P) \cong \operatorname{colim}(S \circ P).$$

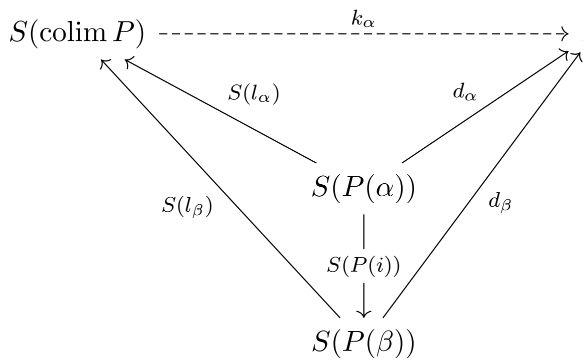
Similarly if $Q\colon \mathsf{J} \to \mathsf{D}$ is a diagram in D such that $\lim Q$ exists, then $\lim(T \circ Q)$ exists as well and there is a natural isomorphism

$$T(\lim Q) \cong \lim(T \circ Q).$$

Proof. We will only prove the colimit statement—the statement about limits is formally dual, and can be obtained by reversing arrows. We will prove the result in four steps.

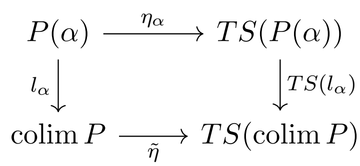
1. Let $l_\alpha\colon P(\alpha) \to \operatorname{colim} P$ denote the morphisms whose existence is guaranteed by the colimit property for P. Assume we are given a solution $(D, \{d_\alpha\})$ for the diagram $S \circ P\colon \mathsf{J} \to \mathsf{D}$. To show that $\operatorname{colim}(S \circ P)$ exists an is equal to $S(\operatorname{colim} P)$

we need to prove the existence of a unique map $k\colon S(\operatorname{colim} P) \to D$ such that the following diagram commutes for each morphism $i\colon \alpha \to \beta$ in J:

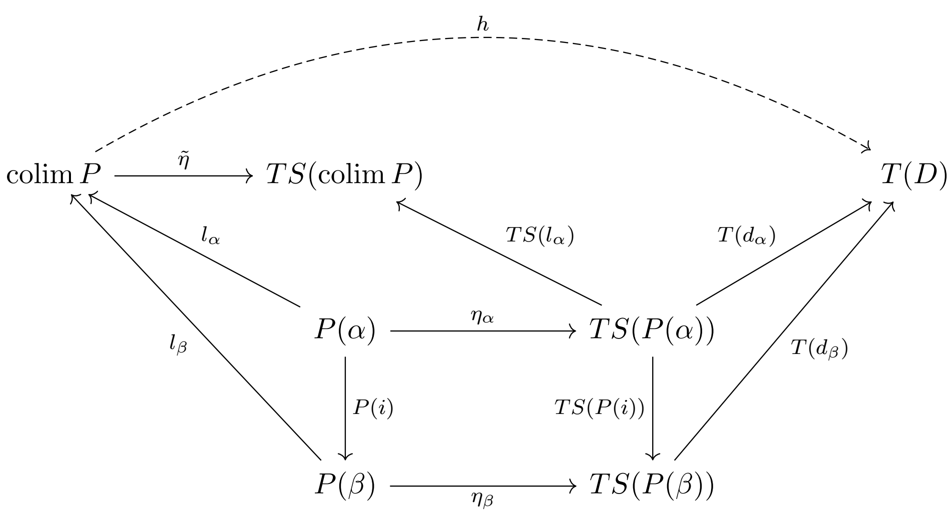


(40.6)

Our plan is to apply T to this diagram and use the unit $\eta\colon \mathrm{id}_{\mathsf{C}} \to T \circ S$ to replace $TS(\operatorname{colim} P)$ and $TS(P(\alpha))$ with $\operatorname{colim} P$ and $P(\alpha)$ respectively. Indeed, naturality of the unit (Lemma 40.19) tells us gives us maps $\tilde{\eta} := \eta(\operatorname{colim} P)\colon \operatorname{colim} P \to TS(\operatorname{colim} P)$ and maps $\eta_\alpha := \eta(P(\alpha))\colon P(\alpha) \to TS(P(\alpha))$ such that the following diagram commutes for every $\alpha \in \operatorname{obj}(\mathsf{J})$:



2. Let us start by applying T to the diagram above and adjoining the original diagram on the left. This leads to the following monstrosity:



(40.7)

The diagram commutes because:

$$\begin{aligned} TS(P(i)) \circ \eta_\alpha &= \eta_\beta \circ P(i), && \text{as } \eta \text{ is natural,} \\ T(d_\alpha) &= T(d_\beta) \circ TS(P(i)), && \text{as } T \text{ is a functor.} \end{aligned}$$

The map $h\colon \operatorname{colim} P \to T(D)$ exists and is unique because $(T(D), \{T(d_\alpha)\})$ is a solution to the diagram P.

3. Let us now define

$$k := \Psi(\operatorname{colim} P, D)^{-1}(h) \in \operatorname{Hom}_{\mathsf{D}}(S(\operatorname{colim} P), D). \tag{40.8}$$

We claim that this choice of k makes the diagram (40.6) commute. Indeed, by (40.1) the following diagram commutes:

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathsf{D}}(S(\operatorname{colim} P), D) & \xrightarrow{\Psi(\operatorname{colim} P, D)} & \operatorname{Hom}_{\mathsf{C}}(\operatorname{colim} P, T(D)) \\
\Big\downarrow {\scriptstyle \operatorname{Hom}(S(l_\alpha), \mathrm{id}_D)} & & \Big\downarrow {\scriptstyle \operatorname{Hom}(l_\alpha, \mathrm{id}_{T(D)})} \\
\operatorname{Hom}_{\mathsf{D}}(S(P(\alpha)), D) & \xrightarrow[\Psi(P(\alpha), D)]{} & \operatorname{Hom}_{\mathsf{C}}(P(\alpha), T(D))
\end{array}$$

Thus starting with k in the top left-hand corner, going clockwise we obtain $h \circ l_\alpha$ and going anticlockwise we obtain $\Psi(P(\alpha), D)(k \circ S(l_\alpha))$. From (40.7) we have

$$h \circ l_\alpha = T(d_\alpha) \circ \eta_\alpha$$

and hence we conclude

$$T(d_\alpha) \circ \eta_\alpha = \Psi(P(\alpha), D)(k \circ S(l_\alpha)). \tag{40.9}$$

Finally we apply the unit equation (40.2) with $g = d_\alpha$ to obtain

$$\Psi(P(\alpha), D)(d_\alpha) = T(d_\alpha) \circ \eta_\alpha.$$

Combining this equation with (40.9) and applying $\Psi(P(\alpha), D)^{-1}$ to both sides we obtain

$$k \circ S(l_\alpha) = d_\alpha.$$

which is exactly what we wanted.

4. To complete the proof we need to show that k is unique. This follows directly from the definition (40.8), since h is unique and $\Psi(\operatorname{colim} P, D)$ is a bijection. ∎

Remark 40.23. There is a partial converse to Theorem 40.22, called the *Freyd Adjoint Functor Theorem*. We won't go into the details however, since we will not need it.

In Lecture 42 we will construct an important adjoint pair of functors on the category hTop_*.

LECTURE 41

Group and cogroup objects

In this lecture we study *group objects* and *cogroup objects*. We begin with some more generalities on products and coproducts. Our first result tells us the notation (f, g) makes sense whenever the relevant (co)product exists.

Proposition 41.1. *Let* C *be a category and let* $A, B \in \operatorname{obj}(\mathsf{C})$.

1. *Suppose the product* $A \sqcap B$ *exists. Then for any* $C \in \operatorname{obj}(\mathsf{C})$, *there is a natural bijection*
$$\operatorname{Hom}(C, A) \times \operatorname{Hom}(C, B) \cong \operatorname{Hom}(C, A \sqcap B).$$
2. *Suppose the coproduct* $A \sqcup B$ *exists. Then for any* $C \in \operatorname{obj}(\mathsf{C})$, *there is a natural bijection*
$$\operatorname{Hom}(A, C) \times \operatorname{Hom}(B, C) \cong \operatorname{Hom}(A \sqcup B, C).$$

Proof. For $\mathsf{C} = \mathsf{Sets}$, the first statement is a formal consequence of Problem Q.3 and Theorem 40.22 (Recall we use $\times$ to denote the product in Sets.) The second statement doesn't quite fit into the this framework, since $\operatorname{Hom}(\square, C)$ is contravariant and Theorem 40.22 only worked with covariant functors.

However in both cases the proof is trivial, and goes as follows (for an arbitrary C): Given a pair $(f, g) \in \operatorname{Hom}(C, A) \times \operatorname{Hom}(C, B)$, the definition of the product as a limit gives us a unique morphism $h \colon C \to A \sqcap B$ such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\;f\;} & A \\ {\scriptstyle g}\downarrow & \overset{h}{\searrow} & \uparrow{\scriptstyle l_A} \\ B & \xleftarrow[\;l_B\;]{} & A \sqcap B \end{array}$$

The desired isomorphism $\operatorname{Hom}(C, A) \times \operatorname{Hom}(C, B) \to \operatorname{Hom}(C, A \sqcap B)$ is then given by
$$(f, g) \mapsto h.$$

To see this is a bijection we define an inverse by sending $h \colon C \to A \sqcap B$ to the ordered pair $(l_A \circ h, l_B \circ h)$. Finally, to check naturality we must show that that if $k \colon C \to D$ is any morphism in C then the following commutes:

$$\begin{array}{ccc} \operatorname{Hom}(C, A) \times \operatorname{Hom}(C, B) & \longrightarrow & \operatorname{Hom}(C, A \sqcap B) \\ {\scriptstyle \operatorname{Hom}(k,A)\times\operatorname{Hom}(k,B)}\uparrow & & \uparrow{\scriptstyle \operatorname{Hom}(k,A\sqcap B)} \\ \operatorname{Hom}(D, A) \times \operatorname{Hom}(D, B) & \longrightarrow & \operatorname{Hom}(D, A \sqcap B) \end{array}$$

This follows readily from the definitions. ∎

From now on we suppress the isomorphism from Proposition 41.1 from our notation whenever possible. Thus for instance if A, B, C are three objects in a category and the product $A \sqcap B$ exists, then for $f \in \mathrm{Hom}(C, A)$ and $g \in \mathrm{Hom}(C, B)$, we denote by (f, g) the morphism in $\mathrm{Hom}(C, A \sqcap B)$ corresponding to the ordered pair (f, g) under the isomorphism form part (1) of Proposition 41.1.

Definition 41.2. Let C be a category. An **initial object** in C is an object A such that for every $C \in \mathrm{obj}(\mathsf{C})$, the set $\mathrm{Hom}(A, C)$ contains a single morphism, which we denote by α_C. A **terminal object** in C is an object Z such that for every $C \in \mathrm{obj}(\mathsf{C})$, the set $\mathrm{Hom}(C, Z)$ contains a single morphism, which we denote by ω_C. An object is a **zero object** if it is both an initial object and a terminal object.

It is immediate that if an initial object exists it is unique up to isomorphism (and similarly for a terminal object). It does not have to be actually unique though: for instance, in Sets, the empty set is the (unique) initial object, and any set with one element is a terminal object. Not all categories have initial and terminal objects; the simplest example is the category of *non-empty* sets, which does not have an initial object.

Remark 41.3. One can think of a terminal project as the limit of an empty diagram (i.e. where the index category has no objects and no morphisms). Similarly one can think of an initial object as the colimit of an empty diagram. This has the following nice consequence: if a category C has a terminal object, and the product $A \sqcap B$ of any two objects of C exists, then by induction, one can show that any finite product $A_1 \sqcap A_2 \cdots \sqcap A_n$ exists ($n \geq 0$.) Similarly if C has an initial object and the coproduct $A \sqcup B$ of any two objects of C exists, then any finite coproduct $A_1 \sqcup A_2 \cdots \sqcup A_n$ exists ($n \geq 0$.)

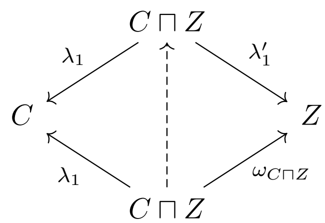
Proposition 41.4. *Let C be a category.*

1. *Suppose an initial object A exists and assume binary*[1] *coproducts exist in C. Then for any $C \in \mathrm{obj}(\mathsf{C})$, the two "injections" $C \to A \sqcup C$ and $C \to C \sqcup A$ (i.e. the maps induced from the colimits) are isomorphisms.*
2. *Suppose a terminal object Z exists and assume that binary products exist in C. Then for any $C \in \mathrm{obj}(\mathsf{C})$, the two "projections" $C \sqcap Z \to C$ and $Z \sqcap C \to C$ (i.e. the maps induced from the limits) are isomorphisms.*

Proof. This time for variety we prove the second statement only. The other is formally dual. Suppose $C \in \mathrm{obj}(\mathsf{C})$, and denote by $\lambda_1 \colon C \sqcap Z \to C$ and $\lambda_1' \colon C \sqcap Z \to Z$ the maps induced from the limit (thus $\lambda_1' = \omega_{C \sqcap Z}$.) We define an inverse to λ_1. For this let $\kappa_1 := (\mathrm{id}_C, \omega_C) \colon C \to C \sqcap Z$, where ω_C is the unique morphism to the terminal object Z. Then clearly $\lambda_1 \circ \kappa_1 = \mathrm{id}_C$. Next, both $\kappa_1 \circ \lambda_1$ and $\mathrm{id}_{C \sqcap Z}$ fit on

[1] That is, the coproduct of two objects exists.

the dashed morphism in the following diagram, and hence by the universal property of a product, they are the same:



Thus κ_1 is an inverse to λ_1. The same argument shows that the other projection $\lambda_2 \colon Z \sqcap C \to C$ has inverse $\kappa_2 = (\omega_C, \mathrm{id}_C)$. ■

Definition 41.5. Let C be a category and $C \in \mathrm{obj}(\mathsf{C})$.

1. Suppose the product $C \sqcap C$ exists. We define the **diagonal** $\Delta_C \in \mathrm{Hom}(C, C \sqcap C)$ to be the map corresponding to $(\mathrm{id}_C, \mathrm{id}_C) \in \mathrm{Hom}(C,C) \times \mathrm{Hom}(C,C)$ in part (1) of Proposition 41.1.
2. Suppose the coproduct $C \sqcup C$ exists. We define the **codiagonal** $\nabla_C \in \mathrm{Hom}(C \sqcup C, C)$ to be the map corresponding to $(\mathrm{id}_C, \mathrm{id}_C) \in \mathrm{Hom}(C,C) \times \mathrm{Hom}(C,C)$ in part (2) of Proposition 41.1.

The next two results are on Problem Sheet Q.

Lemma 41.6. *Let C be a category. Suppose $A_1, A_2, B_1, B_2 \in \mathrm{obj}(\mathsf{C})$ are four objects and $f_i \colon A_i \to B_i$ are morphisms for $i = 1, 2$.*

1. *If the products $A_1 \sqcap A_2$ and $B_1 \sqcap B_2$ exist then there is a unique morphism $f_1 \sqcap f_2 \colon A_1 \sqcap A_2 \to B_1 \sqcap B_2$ such that the following diagram commutes for $i = 1, 2$, where the vertical maps are those induced from the limit:*

$$\begin{array}{ccc} A_1 \sqcap A_2 & \xrightarrow{f_1 \sqcap f_2} & B_1 \sqcap B_2 \\ \downarrow & & \downarrow \\ A_i & \xrightarrow[f_i]{} & B_i \end{array} \tag{41.1}$$

2. *If the coproducts $A_1 \sqcup A_2$ and $B_1 \sqcup B_2$ exist then there is a unique morphism $f_1 \sqcup f_2 \colon A_1 \sqcup A_2 \to B_1 \sqcup B_2$ such that the following diagram commutes for $i = 1, 2$, where the vertical maps are those induced from the colimit:*

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ \downarrow & & \downarrow \\ A_1 \sqcup A_2 & \xrightarrow[f_1 \sqcup f_2]{} & B_1 \sqcup B_2 \end{array} \tag{41.2}$$

Lemma 41.7. *Suppose C is a category and $B, C_1, C_2, D \in \mathrm{obj}(\mathsf{C})$.*

1. *Assume the products $B \sqcap B$ and $C_1 \sqcap C_2$ exist. Suppose $f_i \in \mathrm{Hom}(B, C_i)$ for $i = 1, 2$. Then*
$$(f_1, f_2) = (f_1 \sqcap f_2) \circ \Delta_B.$$

2. *Assume the coproducts* $D \sqcup D$ *and* $C_1 \sqcup C_2$ *exist. Suppose* $g_i \in \mathrm{Hom}(C_i, D)$ *for* $i = 1, 2$. *Then*
$$(g_1, g_2) = \nabla_D \circ (g_1 \sqcup g_2).$$

Warning: It is important to realise that in general the morphisms (f, g), $f \sqcap g$ and $f \sqcup g$ are all different!

Let us now give an alternative way of defining a "group".

Definition 41.8. Let C be a category that possesses a terminal object Z and finite products. A **group object** $G \in \mathrm{obj}(\mathsf{C})$ is an object G together with **multiplication** morphism $m \colon G \sqcap G \to G$ and morphisms $\theta \colon G \to G$ and $e \colon Z \to G$ such that the following diagrams commute:

1. **Associativity:**
$$\begin{array}{ccc} G \sqcap G \sqcap G & \xrightarrow{\mathrm{id}_G \sqcap m} & G \sqcap G \\ {\scriptstyle m \sqcap \mathrm{id}_G}\big\downarrow & & \big\downarrow{\scriptstyle m} \\ G \sqcap G & \xrightarrow[m]{\qquad} & G \end{array}$$
(the notation $G \sqcap G \sqcap G$ is unambiguous by part (3) of Lemma 40.4.)

2. **Identity:**
$$\begin{array}{ccccc} G \sqcap Z & \xrightarrow{\mathrm{id}_G \sqcap e} & G \sqcap G & \xleftarrow{e \sqcap \mathrm{id}_G} & Z \sqcap G \\ & {\scriptstyle \cong}\searrow & \big\downarrow{\scriptstyle m} & \swarrow{\scriptstyle \cong} & \\ & & G & & \end{array}$$
(here the diagonal maps are the isomorphisms from part (1) of Proposition 41.1.)

3. **Inverse:**
$$\begin{array}{ccccc} G & \xrightarrow{(\mathrm{id}_G, \theta)} & G \sqcap G & \xleftarrow{(\theta, \mathrm{id}_G)} & G \\ {\scriptstyle \omega_G}\big\downarrow & & \big\downarrow{\scriptstyle m} & & \big\downarrow{\scriptstyle \omega_G} \\ Z & \xrightarrow[e]{\qquad} & G & \xleftarrow[e]{\qquad} & Z \end{array}$$

The terminology makes sense, as the following easy examples illustrate.

Example 41.9.

1. A group object in Sets is a group.
2. A group object in Groups (!) is an abelian group[2].
3. Every abelian group is a group object in Ab.
4. A group object in Top is a topological group.

[2] This is because if we work in the category Groups, a morphism between two groups is necessarily a homomorphism. Thus if G is a group object in Groups then in particular the group inversion map $\theta \colon G \to G$ is a homomorphism. Since $\theta(gh) = (gh)^{-1} = h^{-1}g^{-1}$, if $\theta(gh) = \theta(g)\theta(h)$ for all g, h then G is abelian.

We will be primarily interested in group objects in the category hTop_*. We will study these next lecture. For now, let us define the dual notion of a *cogroup object*.

Definition 41.10. Let C be a category that possesses an initial object A and finite coproducts. A **cogroup object** $K \in \mathrm{obj}(\mathsf{C})$ is an object together with **comultiplication** morphism $\mu\colon K \to K \sqcup K$ and morphisms $\vartheta\colon K \to K$ and $o\colon K \to A$ such that the following diagrams commute:

1. **Co-associativity:**

$$\begin{array}{ccc} K & \xrightarrow{\quad\mu\quad} & K \sqcup K \\ {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_K \sqcup \mu} \\ K \sqcup K & \xrightarrow[\mu \sqcup \mathrm{id}_K]{} & K \sqcup K \sqcup K \end{array}$$

2. **Co-identity:**

$$\begin{array}{ccccc} K \sqcup A & \xleftarrow{\mathrm{id}_K \sqcup o} & K \sqcup K & \xrightarrow{o \sqcup \mathrm{id}_K} & A \sqcup K \\ & {\scriptstyle \cong}\nwarrow & \big\uparrow{\scriptstyle \mu} & \nearrow{\scriptstyle \cong} & \\ & & K & & \end{array}$$

(here the diagonal maps are the equivalences from part (2) of Proposition 41.1.)

3. **Co-inverse:**

$$\begin{array}{ccccc} K & \xleftarrow{(\mathrm{id}_K, \vartheta)} & K \sqcup K & \xrightarrow{(\mathrm{id}_K, \vartheta)} & K \\ {\scriptstyle \alpha_K}\big\uparrow & & \big\uparrow{\scriptstyle \mu} & & \big\uparrow{\scriptstyle \alpha_K} \\ A & \xleftarrow[o]{} & K & \xrightarrow[o]{} & A \end{array}$$

Cogroup objects are slightly less common "in nature".

Example 41.11.

1. The only cogroup object in Sets and Top is the empty set.
2. A cogroup object in Groups is a free group (this is fairly easy to prove if G is finitely generated, and rather harder in general.)
3. Every abelian group is a cogroup object in Ab.
4. The only cogroup objects in Sets_* and Top_* are the singletons $\{\star\}$,

As with group objects, we will primarily be interested in cogroup objects in hTop_*.

Here is the main result of today's lecture. We remind the reader that we use $\times$ for the product in Sets.

Theorem 41.12. *Let C be a category with a terminal object Z and finite products. An object $G \in \mathrm{obj}(\mathsf{C})$ is a group object if and only if $\mathrm{Hom}(\square, G)\colon \mathsf{C} \to \mathsf{Sets}$ is actually a functor*[3] $\mathrm{Hom}(\square, G)\colon \mathsf{C} \to \mathsf{Groups}$. *If this is the case the the group multiplication $\tilde{m}(C)$ on $\mathrm{Hom}(C, G)$ is given by*

$$\tilde{m}(C)\colon \mathrm{Hom}(C,G) \times \mathrm{Hom}(C,G) \to \mathrm{Hom}(C,G), \qquad \tilde{m}(C)(f,g) := m \circ (f,g), \quad (41.3)$$

[3]This means that $\mathrm{Hom}(C, G)$ is a group for each $C \in \mathrm{obj}(\mathsf{G})$, and if $f\colon A \to B$ is a morphism in C then $\mathrm{Hom}(f, G)\colon \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ is a group homomorphism.

where m is the multiplication on G and we are using the identification of $\mathrm{Hom}(C,G)\times \mathrm{Hom}(C,G)$ *with* $\mathrm{Hom}(C,G\sqcap G)$ *from part (1) of Proposition 41.1.*

The proof will use the following corollary of the Yoneda Lemma (Problem K.5).

LEMMA 41.13. *Let C be a category, and suppose $A,B\in\mathrm{obj}(\mathsf{C})$. Suppose $\Phi\colon \mathrm{Hom}(\square,A)\to \mathrm{Hom}(\square,B)$ is a natural transformation. Then if $C\in\mathrm{obj}(\mathsf{C})$ and $f\in\mathrm{Hom}(C,A)$, one has*

$$\Phi(C)(f)=\Phi(A)(\mathrm{id}_A)\circ f$$

(this makes sense as $\Phi(A)(\mathrm{id}_A)\in\mathrm{Hom}(A,B)$.)

Proof of Theorem 41.12. We will prove the result in four steps.

1. First assume that G is a group object in C. By part (1) of Proposition 41.1, we can identify $\mathrm{Hom}(C,G)\times\mathrm{Hom}(C,G)$ with $\mathrm{Hom}(C,G\sqcap G)$. If we apply $\mathrm{Hom}(C,\square)$ to each of the three diagrams in the definition of a group object, we see that $\mathrm{Hom}(C,G)$ is a group object in Sets, and hence is a group. Now suppose $h\colon A\to B$ is a morphism in C. We claim that $\mathrm{Hom}(h,G)\colon \mathrm{Hom}(B,G)\to\mathrm{Hom}(A,G)$ is a homomorphism, where the multiplication is given by $\tilde m(B)$ and $\tilde m(A)$ respectively, defined as in (41.3). For this we compute:

$$\begin{aligned}\mathrm{Hom}(h,G)(\tilde m(B)(f,g)) &= \mathrm{Hom}(h,G)(m\circ(f,g))\\ &= m\circ(f,g)\circ h\\ &= m\circ(f\circ h,g\circ h)\\ &= m\circ(\mathrm{Hom}(h,G)(f),\mathrm{Hom}(f,G)(g))\\ &= \tilde m(A)(\mathrm{Hom}(h,G)(f),\mathrm{Hom}(h,G)(g)).\end{aligned}$$

2. The converse is harder. Assume that $G\in\mathrm{obj}(\mathsf{C})$ has the property that for every $C\in\mathrm{obj}(\mathsf{C})$, there is a group operation $\tilde m(C)\colon\mathrm{Hom}(C,G)\times\mathrm{Hom}(C,G)\to\mathrm{Hom}(C,G)$. As before we identify $\mathrm{Hom}(C,G)\times\mathrm{Hom}(C,G)$ with $\mathrm{Hom}(C,G\sqcap G)$. With this identifcation, we claim that $\tilde m\colon\mathrm{Hom}(\square,G\sqcap G)\to\mathrm{Hom}(\square,G)$ is a natural transformation. For this we must check that if $f\colon A\to B$ is a morphism in C then the following commutes:

$$\begin{array}{ccc}\mathrm{Hom}(B,G\sqcap G) & \xrightarrow{\tilde m(B)} & \mathrm{Hom}(B,G)\\ {\scriptstyle \mathrm{Hom}(f,G\sqcap G)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(f,G)}\\ \mathrm{Hom}(A,G\sqcap G) & \xrightarrow[\tilde m(A)]{} & \mathrm{Hom}(A,G)\end{array}$$

This however is immediate from the definition.

Thus by the Yoneda Lemma 41.13, for every morphism $h\colon C\to G\sqcap G$ in C, we have

$$\tilde m(C)(h)=m\circ h,$$

where $m\in\mathrm{Hom}(G\sqcap G,G)$ is *defined* by

$$m:=\tilde m(G\sqcap G)(\mathrm{id}_{G\sqcap G}). \tag{41.4}$$

3. We now prove that with m defined as in (41.4), the diagram for the associativity axiom holds. Since $\mathrm{Hom}(C,G)$ is a group for every $C \in \mathrm{obj}(\mathsf{C})$—and thus in particular the multiplication $\tilde{m}(C)$ satisfies the associativity axiom—we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}(C,G) \times \mathrm{Hom}(C,G) \times \mathrm{Hom}(C,G) & \xrightarrow{\tilde{m}(C)\times \mathrm{id}_{\mathrm{Hom}(C,G)}} & \mathrm{Hom}(C,G) \times \mathrm{Hom}(C,G) \\
{\scriptstyle \mathrm{id}_{\mathrm{Hom}(C,G)}\times \tilde{m}(C)}\downarrow & & \downarrow{\scriptstyle \tilde{m}(C)} \\
\mathrm{Hom}(C,G) \times \mathrm{Hom}(C,G) & \xrightarrow[\tilde{m}(C)]{} & \mathrm{Hom}(C,G)
\end{array}$$

Invoking part (1) of Proposition 41.1 again, we can rewrite this diagram as

$$\begin{array}{ccc}
\mathrm{Hom}(C, G \sqcap G \sqcap G) & \longrightarrow & \mathrm{Hom}(C, G \sqcap G) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(C, G \sqcap G) & \longrightarrow & \mathrm{Hom}(C,G)
\end{array}$$

Then arguing as before, we see that the going clockwise and going anticlockwise round the square gives rise to two more equal natural transformations:

$$\mathrm{Hom}(\square, G \sqcap G \sqcap G) \to \mathrm{Hom}(\square, G)$$

Thus by the Yoneda Lemma again, if $k\colon C \to G \sqcap G \sqcap G$ is an element belonging to the top left-hand corner then the image of k in the bottom right-hand corner is of the form $p \circ k$, where:

$$p = \begin{cases} m \circ (m \sqcap \mathrm{id}_G), & \text{going clockwise,} \\ m \circ (\mathrm{id}_G \sqcap m), & \text{going anticlockwise.} \end{cases}$$

Taking $C = G \sqcap G \sqcap G$ and $k = \mathrm{id}_{G\sqcap G\sqcap G}$ it follows that

$$m \circ (m \sqcap \mathrm{id}_G) = m \circ (\mathrm{id}_G \sqcap m),$$

which is what we need for the associativity diagram.

4. Let us now define the two other morphisms $\theta \in \mathrm{Hom}(G,G)$ and $e \in \mathrm{Hom}(Z,G)$ needed for G to be a group object. For this, let

$$\tilde{\theta}(C)\colon \mathrm{Hom}(C,G) \to \mathrm{Hom}(C,G)$$

denote the inversion in the group $\mathrm{Hom}(C,G)$. Then set $\theta := \tilde{\theta}(G)(\mathrm{id}_G)$. Next, let e denote the identity element in the group $\mathrm{Hom}(Z,G)$. The proof that θ satisfies the diagram for the identity axiom and that e satisfies the diagram for the inverse axiom are similar to the proof that m satisfied the diagram for the associativity axiom (actually they are simpler), and we will leave them as exercises. ■

The dual result to Theorem 41.12 is the following statement, whose proof is dual to the one just given.

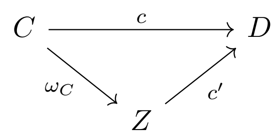
Theorem 41.14. *Let C be a category with an initial object A and finite coproducts. An object $K \in \operatorname{obj}(\mathsf{C})$ is a cogroup object if and only if $\operatorname{Hom}(K, \square)\colon \mathsf{C} \to \mathsf{Sets}$ is actually a functor $\operatorname{Hom}(K, \square)\colon \mathsf{C} \to \mathsf{Groups}$. If this is the case the the group multiplication $\tilde{m}(C)$ on $\operatorname{Hom}(K, C)$ is given by*

$$\tilde{\mu}(C)\colon \operatorname{Hom}(K, C) \times \operatorname{Hom}(K, C) \to \operatorname{Hom}(K, C), \qquad \tilde{\mu}(C)(f, g) := (f, g) \circ \mu,$$

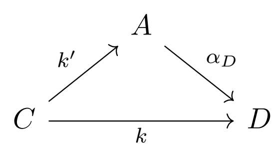
where μ is the comultiplication on G and we are using the identification of $\operatorname{Hom}(K, C) \times \operatorname{Hom}(K, C)$ with $\operatorname{Hom}(K \sqcup K, C)$ from part (2) of Proposition 41.1.

We conclude with a remark that will be useful next lecture.

Definition 41.15. Let C be a category with a terminal object Z. A morphism $c\colon C \to D$ in C is called a **constant** morphism if it factors through Z, that is, there exists a morphism $c'\colon Z \to D$ such that the following commutes:



Similarly if C has an initial object A a morphism $k\colon C \to D$ is a **coconstant** morphism if there exists a morphism $k'\colon C \to A$ such that the following commutes:

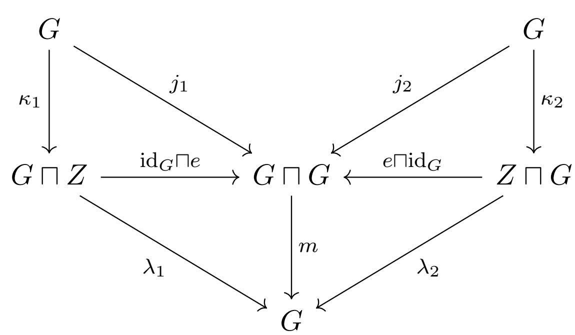


Remark 41.16. One can recast the definition of a group object using constant morphisms. This point of view is less convenient for Theorem 41.12, but will prove more natural in the context of the group objects in hTop_* that we will study next lecture.

Here are the details. Suppose G is a group object in a category C. We use the notation from the proof of Proposition 41.4 for the projections $\lambda_1\colon G \sqcap Z \to G$ and $\lambda_2\colon Z \sqcap G \to G$, together with their inverses κ_1 and κ_2. Define morphisms $j_1, j_2\colon G \to G \sqcap G$ by

$$j_1 := (\operatorname{id}_G \sqcap e) \circ \kappa_1, \qquad j_2 := (e \sqcap \operatorname{id}_G) \circ \kappa_2.$$

Then we can enlarge the identity axiom diagram to see that the following commutes:



Since $\lambda_i \circ \kappa_i = \mathrm{id}_G$, we see that

$$m \circ j_1 = \mathrm{id}_G, \qquad m \circ j_2 = \mathrm{id}_G. \tag{41.5}$$

Moreover there exists a constant map $c\colon G \to G$ such that

$$j_1 = (\mathrm{id}_G, c), \qquad j_2 = (c, \mathrm{id}_G). \tag{41.6}$$

Conversely, if we are given a category C and an object G, together with a morphism $m\colon G \sqcap G \to G$ that satisfies the diagram for the associativity axiom and a morphism $\theta\colon G \to G$ such that:

1. $m \circ (\mathrm{id}_G, \theta) = m \circ (\theta, \mathrm{id}_G)$ are equal constant morphisms,
2. if j_1 and j_2 are defined as in (41.6) with $c = m \circ (\mathrm{id}_G, \theta)$ then (41.5) holds,

then G is a group object. A similar discussion applies to cogroup objects.

LECTURE 42

Loop spaces and reduced suspensions

In this lecture we investigate group and cogroup objects in the category hTop_*. We begin however with an interlude on some point-set topology about function spaces. As with all other point-set topology results, we won't prove them, since the ideas involved have nothing to do with algebraic topology.

DEFINITION 42.1. Let X and Y denote the topological spaces, and write X^Y for the set of all continuous functions[1] from Y into X. Define a topology on X^Y, called the **compact-open topology** by declaring that a subbasis[2] is all the sets of the form

$$(K;U) := \left\{ f \in X^Y \mid f(K) \subset U \right\},$$

where $K \subset Y$ is compact and $U \subset X$ is open.

If X is a metric space then the compact-open topology on X^Y (for any Y) coincides with the topology of uniform convergence on compact subsets.

DEFINITION 42.2. Let X and Y be sets. Define the **evaluation map**

$$\mathrm{ev}\colon \mathrm{Hom}_{\mathsf{Sets}}(Y,X) \times Y \to X$$

by

$$\mathrm{ev}(f,y) := f(y)$$

(here $\mathrm{Hom}_{\mathsf{Sets}}(Y,X)$ denotes the set of *all* functions from Y to X).

The first nice property of the compact-open topology is the following:

PROPOSITION 42.3. *Let X and Y be topological spaces, and assume that Y is locally compact and Hausdorff. Then the restriction of* ev *to $X^Y \times Y$ defines a continuous map* $\mathrm{ev}\colon X^Y \times Y \to X$.

DEFINITION 42.4. Suppose X, Y and Z are sets. If $\zeta\colon Z \times Y \to X$ is an element of $\mathrm{Hom}_{\mathsf{Sets}}(Z \times Y, X)$, define

$$\zeta^\sharp\colon Z \to \mathrm{Hom}_{\mathsf{Sets}}(Y,X), \qquad \zeta^\sharp(z)(y) := \zeta(z,y).$$

Similarly if $\varphi\colon Z \to \mathrm{Hom}_{\mathsf{Sets}}(Y,X)$, define

$$\varphi^\flat\colon Z \times Y \to X, \qquad \varphi^\flat(z,y) := \varphi(z)(y).$$

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[1] We use this notation instead of the more common $C(Y,X)$ or $C^0(Y,X)$ since these are too easy to confuse with the singular (co)chain complex!

[2] This means that the compact-open topology is the smallest topology containing all sets of this form.

It is immediate that in the category Sets, the correspondence $\zeta \mapsto \zeta^\sharp$ defines a bijection

$$\mathrm{Hom}_{\mathsf{Sets}}(Z \times Y, X) \to \mathrm{Hom}_{\mathsf{Sets}}\left(Z, \mathrm{Hom}_{\mathsf{Sets}}(Y, X)\right)$$

(the inverse is given by $\varphi \mapsto \varphi^\flat$). This result is called the **Exponential Law** for sets. It is much less trivial that the same result holds for topological spaces, provided Y and Z are Hausdorff, Y is locally compact, and we work with the compact-open topology.

THEOREM 42.5 (Exponential Law). *Let X, Y and Z be topological spaces.*

1. *Assume that Y is locally compact and Hausdorff. Then the operation $\zeta \mapsto \zeta^\sharp$ defines a bijection*
$$X^{Z\times Y} \to (X^Y)^Z,$$
with inverse given by $\varphi \mapsto \varphi^\flat$.
2. *If instead we assume that Z is Hausdorff (and no assumptions on Y) then the map $\zeta \mapsto \zeta^\sharp$ is continuous in the compact-open topology.*
3. *If both conditions are fulfilled, i.e. if Y is locally compact Hausdorff and Z is Hausdorff, then the map $\zeta \mapsto \zeta^\sharp$ is also an open map, and thus a homeomorphism.*

For us, the most useful corollary of (part (1) of) Theorem 42.5 is the following.

COROLLARY 42.6. *Let X, Y and Z be spaces, and assume that Y is locally compact and Hausdorff. A function $\varphi\colon Z \to X^Y$ is continuous if and only if the composition* $\mathrm{ev} \circ (\varphi \times \mathrm{id}_Y)$ *is continuous:*

$$Z \times Y \xrightarrow{\varphi\times\mathrm{id}_Y} X^Y \times Y \xrightarrow{\mathrm{ev}} X$$

Indeed, denoting the composition by ζ, one has $\varphi = \zeta^\sharp$, so the corollary follows from Theorem 42.5.

REMARK 42.7. Suppose $f, g\colon X \to Y$ are homotopic, where X is locally compact and Hausdorff. If we think of a homotopy $F\colon f \simeq g$ as a map $F\colon I \times X \to Y$ (instead of the more usual $X \times I \to Y$) then $F^\sharp\colon I \to Y^X$ is a path in Y^X from f to g. Conversely every such path determines a homotopy. Thus $[X, Y] = \pi_0(Y^X)$.

REMARK 42.8. The first statement of Theorem 42.5 can be alternatively phrased as follows: if Y is a locally compact Hausdorff space then $(\square \times Y, \square^Y)$ form an adjoint pair on Top.

Recall from Lecture 4 that an object of hTop_* is a pair (X, x) where X is a topological space and $x \in X$ is the *basepoint*. A morphism $[f]$ from (X, x) to (Y, y) is a homotopy class of pointed maps $f\colon (X, x) \to (Y, y)$, where the homotopies are taken relative to x.

Let us begin in Top_*.

Lemma 42.9. *The category* Top_* *has a zero object. It has finite products and coproducts.*

Proof. Let $*$ be a space with one point. Then $(*, *)$ (as a pointed space) is a zero object in Top_*. If (X_i, x_i) are two pointed spaces then $(X_1 \times X_2, (x_1, x_2))$ is the product and the wedge $(X_1 \vee X_2, (x_1, x_2))$ (cf. Definition 18.10) is the coproduct. ∎

Remark 42.10. The wedge $(X_1 \vee X_2, (x_1, x_2))$ is homeomorphic to the subset $(X_1 \times \{x_2\}) \cup (\{x_1\} \times X_2)$ of $(X_1 \times X_2, (x_1, x_2))$. Thus in Top_*, the coproduct can always be embedded in the product. This is not true in general. For instance, in Groups, if G_1 and G_2 are finite groups then their product is the direct product (which is also finite), but their coproduct is the free product (which is infinite if G_1 and G_2 both have more than one element.)

In general there is no easy way to pass from (co)limits in a category C to (co)limits in a quotient category of C. However in the case of hTop_* it is easy to see that the same constructions work. Thus $(*, *)$ is a zero object, the product is the normal cartesian product and the coproduct is the wedge. Indeed, if $F_i\colon f_i \simeq g_i$ is a pointed homotopy $Y \times I \to X_i$ then $(F_1, F_2)\colon Y \times I \to X_1 \times X_2$ is a pointed homotopy $(f_1, f_2) \simeq (g_1, g_2)$. This shows that $(X_1 \times X_2, (x_1, x_2))$ also solves the universal property for a product in hTop_*, and the other statements follow similarly. Thus:

Corollary 42.11. *The category* hTop_* *has a zero object. It has finite products and coproducts.*

Notation: For the rest of this lecture (and indeed, the rest of the course) we will almost exclusively be concerned with hTop_*. However, constantly displaying the basepoint is cumbersome (for instance, it stops commutative diagrams from fitting onto the page). Thus we will often omit it from the notation, and so a pointed space (X, x) will often (but not always) be written simply as X. When talking about Top_* and hTop_*, *all maps are implicitly assumed to be pointed*, even if the notation does not reflect this.

This notation applies in particular to morphism sets in hTop_*. If X and Y are two pointed spaces then we will write $[X, Y]_*$ for the set of pointed maps from X to Y.

Definition 42.12. A pointed space (X, x_0) is a H**-group** if there are continuous pointed maps $m\colon X \times X \to X$ and $\theta\colon X \to X$ together with pointed homotopies

$$m \circ (\mathrm{id}_X \times m) \simeq m \circ (m \times \mathrm{id}_X), \qquad \text{rel } x_0,$$

and

$$m \circ j_1 \simeq \mathrm{id}_X \simeq m \circ j_2, \qquad \text{rel } x_0, \tag{42.1}$$

where $j_i\colon X \to X \times X$ is the pointed "injection" given by $j_1(x) := (x, x_0)$ and $j_2(x) := (x_0, x)$, and finally such that

$$m \circ (\mathrm{id}_X, \theta) \simeq c \simeq m \circ (\theta, \mathrm{id}_X), \qquad \text{rel } x_0,$$

where $c\colon X \to X$ is the constant map at x_0.

For the dual definition, let us first fix notation. We identify the pointed wedge $X \vee X$ with the subspace $(X \times \{x_0\}) \cup (\{x_0\} \times X)$ of $X \times X$, and denote by $q_i \colon X \vee X \to X$ the restriction of the projection $p_i \colon X \times X \to X$ to $X \vee X \subset X \times X$. Thus $q_1(x, x_0) = x = q_2(x_0, x)$ for all $x \in X$. Recall also from the end of Lecture 20 that if

$$f \colon (X, x_0) \to (Z, z_0), \qquad g \colon (Y, y_0) \to (Z, z_0)$$

are two pointed continuous maps, there is a well-defined pointed continuous map

$$f \vee g \colon X \vee Y \to Z \vee Z$$

defined by

$$(f \vee g)(x, y) := \begin{cases} (f(x), z_0), & y = y_0, \\ (z_0, g(y)), & x = x_0. \end{cases}$$

(Thus $\vee$ coincides with the $\sqcup$ notation used in the last lecture (41.2) for the coproduct.)

Definition 42.13. A pointed space (X, x_0) is a ***H*-cogroup** if there are continuous pointed maps $\mu \colon X \to X \vee X$ and $\vartheta \colon X \to X$ together with pointed homotopies

$$(\mathrm{id}_X \vee \mu) \circ \mu \simeq (\mu \vee \mathrm{id}_X) \circ \mu,$$

and

$$q_1 \circ \mu \simeq \mathrm{id}_X \simeq q_2 \circ \mu,$$

and finally such that

$$(\mathrm{id}_X, \vartheta) \circ \mu \simeq c \simeq (\vartheta, \mathrm{id}_X) \circ \mu,$$

where $c \colon X \to X$ is the constant map at x_0.

The next result is just rephrasing the definitions.

Proposition 42.14. *The group objects in* hTop_* *are the* H*-groups. The cogroup objects in* hTop_* *are the* H*-cogroups.*

Proof. This is immediate from Remark 41.16, since the constant map $c \colon X \to X$ at x_0 is the desired constant (and coconstant) morphism from (X, x_0) to itself in hTop_*. ∎

Remark 42.15. There is a weaker notion of an H-group, which is (rather unhelpfully called) an ***H*-space**. This is a pointed space (X, x_0) together with a continuous pointed map $m \colon X \times X \to X$ such that both $x \mapsto m(x, x_0)$ and $x \mapsto m(x_0, x)$ are homotopic to the identity relative to x_0. In Problem C.5 you proved that an H-space has abelian fundamental group. Similarly there is a weaker notion of an H-cogroup, where one only insists that the comultiplication is homotopic to the identity relative to the basepoint in both variables, and this is called an H-cospace.

Definition 42.16. Let (X, x_0) be a pointed space. The **loop space** of (X, x_0) is the pointed space

$$\Omega(X, x_0) := \left\{ u \in X^I \mid u(0) = u(1) = x_0 \right\},$$

where the basepoint is the constant path at x_0.

We often write ΩX instead of $\Omega(X, x_0)$ (this is somewhat naughty, as the loop space does depend on the choice of x_0).

THEOREM 42.17. *Loop space defines a functor* $\Omega\colon \mathsf{hTop}_* \to \mathsf{hTop}_*$.

Proof. It suffices by Problem A.3 to show that there is a functor $\Omega\colon \mathsf{Top}_* \to \mathsf{Top}_*$ that has the property that if $f \simeq g$ (via a pointed homotopy) then $\Omega(f) \simeq \Omega(g)$. If $f\colon (X, x_0) \to (Y, y_0)$, define $\Omega(f)\colon \Omega(X, x_0) \to \Omega(Y, y_0)$ by $u \mapsto f \circ u$. We need to check is that $\Omega(f)$ is continuous. Since $\Omega(X, x_0)$ is a subspace of X^I, it suffices to show that $u \mapsto f \circ u$ is continuous as a map $X^I \to Y^I$ (this is not completely trivial!) For this consider the commutative diagram

$$\begin{array}{ccc} X^I \times I & \xrightarrow{(u,s)\mapsto(f\circ u,s)} & Y^I \times I \\ {\scriptstyle \mathrm{ev}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Since I is locally compact and Hausdorff, both maps ev are continuous. Thus going anti-clockwise is continuous around the square, and hence going clockwise is continuous. Thus $u \mapsto f \circ u$ is continuous by Corollary 42.6.

Suppose now $f, g\colon (X, x_0) \to (Y, y_0)$ are pointed continuous maps and $F\colon f \simeq g$ is a homotopy rel x_0. Define $G\colon \Omega X \times I \to \Omega Y$ by $G(u, t) := F(u(\cdot), t)$, and define $r\colon X^I \times I \times I \to X^I \times I \times I$ by $r(u, t, s) = (u, s, t)$. Then r is (obviously) continuous, and we have a commutative diagram

$$\begin{array}{ccc} X^I \times I \times I & \xrightarrow{G\times\mathrm{id}_I} & Y^I \times I \\ {\scriptstyle (\mathrm{ev}\times\mathrm{id}_I)\circ r}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ X \times I & \xrightarrow[F]{} & Y \end{array}$$

Starting in the top left-hand corner and going anticlockwise is continuous (since ev is continuous)—this is just the map $(u, t, s) \mapsto F(u(s), t))$, and hence $\mathrm{ev}\circ(G\times\mathrm{id}_I)$ is also continuous. Thus by Corollary 42.6 again, G is continuous. Since $G\colon \Omega(f) \simeq \Omega(g)$, the proof is complete. ∎

THEOREM 42.18. *If* (X, x_0) *is a pointed space then* $\Omega(X, x_0)$ *is an* H*-group.*

Proof. As in Definition 3.9, we define $m\colon \Omega X \times \Omega X \to \Omega X$ by

$$m(u, v) := u * v, \qquad (u * v)(s) := \begin{cases} u(2s), & 0 \le s \le \frac{1}{2}, \\ v(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

To prove that m is continuous, consider the composition

$$\Omega X \times \Omega X \times I \xrightarrow{m\times\mathrm{id}_I} \Omega X \times I \xrightarrow{\mathrm{ev}} X.$$

On $\Omega X \times \Omega X \times [0, 1/2]$, this agrees with the composition

$$\Omega X \times \Omega X \times [0, 1/2] \xrightarrow{\pi'\times 2} \Omega X \times I \xrightarrow{\mathrm{ev}} X,$$

where $\pi'\colon \Omega X\times\Omega X\to\Omega X$ is projection on the first factor and $2\colon [0,1/2]\to I$ is the map $s\mapsto 2s$. This composition is continuous, and hence on $\Omega X\times\Omega X\times[0,1/2]$ the composition $\mathrm{ev}\circ(m\times\mathrm{id}_I)$ is continuous. A similar argument shows that $\mathrm{ev}\circ(m\times\mathrm{id}_I)$ is continuous on $\Omega X\times\Omega X\times[1/2,1]$. Thus by Corollary 42.6, we see that m is continuous.

Now let us prove that m satisfies the homotopy associativity requirement. This argument is basically the same as Proposition 3.17, we just need to be a little more careful about checking everything really is continuous. Consider first the function

$$H\colon \Omega X\times\Omega X\times\Omega X\times I\times I\to X$$

given by

$$H(u,v,w,t,s):=\begin{cases} u\left(\frac{4s}{t+1}\right), & 0\le s\le \frac{t+1}{4},\\ v(4s-t-1), & \frac{t+1}{4}\le s\le\frac{t+2}{4},\\ w\left(\frac{4s-2-t}{2-t}\right), & \frac{t+2}{4}\le s\le 1.\end{cases}$$

By Corollary 42.6, $F:=H^\sharp$ is a continuous function $\Omega X\times\Omega X\times\Omega X\times I\to\Omega X$. By construction F is a pointed homotopy $m\circ(m\times\mathrm{id}_{\Omega X})\simeq m\circ(\mathrm{id}_{\Omega X}\circ m)$.

As in Definition 3.15, let $e_0\in\Omega X$ denote the constant path at x_0. Define $E\colon \Omega X\times I\to\Omega X$ by $E(u,t):=u_t$, where

$$u_t(s):=\begin{cases} u\left(\frac{2s}{t+1}\right), & 0\le s\le\frac{t+1}{2},\\ x_0, & \frac{t+1}{2}\le s\le 1.\end{cases}$$

Then E is continuous as $\mathrm{ev}\circ(E\times\mathrm{id}_I)$ is, and hence E is a homotopy from the $m\circ j_1\colon u\mapsto u*e_0$ to the identity. A similar argument shows that $m\circ j_2\colon u\mapsto e_0*u$ is homotopic to the identity.

Finally, define $\theta\colon\Omega X\to\Omega X$, $\theta(u)=\bar u$, where as in Definition 3.14, $\bar u(s):=u(1-s)$. If $K\colon\Omega X\times I\times I\to X$ is defined by

$$K(u,t,s):=\begin{cases} x_0, & 0\le s\le\frac{t}{2},\\ u(2s-t), & \frac{t}{2}\le s\le\frac12,\\ u(2-2s-t), & \frac12\le s\le\frac{2-t}{2},\\ x_0, & \frac{2-t}{2}\le s\le 1,\end{cases}$$

then $K^\sharp$ is continuous (by Corollary 42.6 again), and defines a pointed nullhomotopy for $u\mapsto m(u,\theta(u))$. Similarly $u\mapsto m(\theta(u),u)$ is nullhomotopic. This completes the proof. ■

Corollary 42.19. *For any pointed space X, there is a contravariant functor*

$$[\square,\Omega X]_*\colon \mathsf{hTop}_*\to\mathsf{Groups}.$$

If Y is another pointed space and $[\varphi],[\psi]\in[Y,\Omega X]_*$ then their product is $\tilde m(Y)([\varphi],[\psi]):=[\varphi*\psi]$.

Proof. Proposition 42.14 and Theorem 42.18 tell us that ΩX is a group object in hTop_*. Thus Theorem 41.12 tells us that $[\square, \Omega X]_*$ takes values in Groups, and (41.3) gives the group multiplication. ■

How about H-cogroups? These are the so-called reduced suspensions. The *suspension* SX of a topological space X is the quotient space of $X \times I$ where we identify $X \times \{0\}$ and $X \times \{1\}$ to a point. The *reduced suspension* is the pointed version of this:

Definition 42.20. Let (X, x_0) be a pointed space. Define the **reduced suspension**

$$\Sigma X := \Big(X \times I\Big) \Big/ \Big((X \times \partial I) \cup (\{x_0\} \times I)\Big).$$

We think of ΣX as a pointed space, where the basepoint is the identified subset. See Figure 42.1

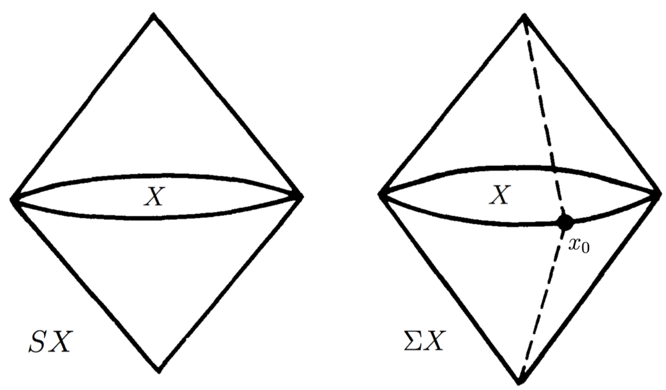


Figure 42.1: The suspension SX and the reduced suspension ΣX.

We will write $[x, t]$ for the equivalence class of $(x, t) \in X \times I$ in ΣX. By a slighty abuse of notation we write simply x_0 instead of $[x_0, t]$ for the basepoint.

Theorem 42.21. *Reduced suspension defines a functor* $\Sigma \colon \mathsf{hTop}_* \to \mathsf{hTop}_*$.

Proof. Once again, by Problem A.3, it suffices to show that Σ defines a functor on Top_* that respects pointed homotopies. If $f \colon (X, x_0) \to (Y, y_0)$, define $\Sigma(f) \colon \Sigma X \to \Sigma Y$ by $\Sigma(f)[x, t] := [f(x), t]$. The fact that Σ respects pointed homotopies is a consequence of general fact that if $f, g \colon (W, W') \to (Z, Z')$ are maps of pairs that are homotopic through maps of pairs, then the induced maps $(W/W', *) \to (Z/Z', *)$ are also homotopic. ■

We could prove directly that ΣX is always a H-cogroup, but instead we will give a pleasing argument using Theorem 40.22 and Theorem 41.16.

Theorem 42.22. (Σ, Ω) *form an adjoint pair on* hTop_*.

Proof. We want to define a natural bijection

$$\Psi(X,Y)\colon [\Sigma X, Y]_* \mapsto [X, \Omega Y]_*.$$

Let the basepoints of X and Y be x_0 and y_0 respectively. We start from the observation that if we restrict the (inverse) of the bijection given by the Exponential Law (Theorem 42.5):

$$\varphi\colon X \to Y^I \qquad \mapsto \qquad \varphi^\flat\colon X \times I \to Y$$

to those φ such that $\varphi(x_0)$ is the constant loop at y_0 and $\varphi(x)(0) = y_0 = \varphi(x)(1)$ for all $x \in X$ then $\varphi^\flat(x,0) = \varphi^\flat(x,1) = y_0 = \varphi^\flat(x_0,t)$. In other words, if $\varphi\colon (X,x_0) \to \Omega(Y,y_0)$ is a pointed map, then $\varphi^\flat$ factors to define a map $(\Sigma X, x_0) \to (Y, y_0)$. Thus we have bijections

$$\tilde{\Psi}(X,Y)\colon \mathrm{Hom}_{\mathsf{Top}_*}((\Sigma X, x_0),(Y,y_0)) \to \mathrm{Hom}_{\mathsf{Top}_*}((X,x_0),\Omega(Y,y_0))$$

given by $\zeta \mapsto \zeta^\sharp$ (with inverse $\varphi \mapsto \varphi^\flat$.)

Similarly a pointed homotopy $\Sigma X \times I \to Y$ determines (and is determined by) a pointed homotopy $X \times I \to \Omega Y$. Thus $\tilde{\Psi}(X,Y)$ induces a bijection on the homotopy classes. The fact that the required diagrams commute to make Ψ natural is immediate from the definitions. ■

Our final result needs a preliminary lemma.

Lemma 42.23. *Let C be a category with a terminal object and finite products. Suppose G, G' are two group objects in C with multiplications m and m' respectively, and suppose $h \in \mathrm{Hom}(G,G')$ is a morphism with the property that*

$$h \circ m = m' \circ (h \sqcap h). \tag{42.2}$$

Then for any object C of C, the map $\mathrm{Hom}(C,h)\colon \mathrm{Hom}(C,G) \to \mathrm{Hom}(C,G')$ is a group homomorphism (note both $\mathrm{Hom}(C,G)$ and $\mathrm{Hom}(C,G')$ are groups by Theorem 41.12.)

Proof. The proof is almost the same as the proof of the easy half of Theorem 41.12. Denote by $\tilde{m}(C)$ and $\tilde{m}'(C)$ the group multiplications on $\mathrm{Hom}(C,G)$ and $\mathrm{Hom}(C,G')$ respectively (as defined in (41.3).) Then if $f, g \in \mathrm{Hom}(C,G)$ we have

$$\begin{aligned}
\mathrm{Hom}(C,h)(\tilde{m}(C)(f,g)) &= \mathrm{Hom}(C,h)(m(f,g)) \\
&= h \circ m \circ (f,g) \\
&\overset{(\star)}{=} m' \circ (h \circ f, h \circ g) \\
&= \tilde{m}'(C)(h \circ f, h \circ g) \\
&= \tilde{m}'(C)(\mathrm{Hom}(C,h)(f), \mathrm{Hom}(C,h)(g)),
\end{aligned}$$

where $(\star)$ used the hypothesis (42.2). ■

Corollary 42.24. *If X is a pointed space then ΣX is a cogroup object in hTop_*.*

Proof. By Theorem 41.14, we need only prove that the functor $[\Sigma X, \Box]_*$ takes values in **Groups**. Let Y be a pointed space. By Theorem 42.22, there is a bijection $\Psi(X,Y)\colon [\Sigma X, Y]_* \mapsto [X, \Omega Y]_*$ given by $\zeta \mapsto \zeta^\sharp$. By Corollary 42.19 the set $[X, \Omega Y]_*$ is a group, and we can use this to *define* a group structure on $[\Sigma X, Y]_*$ by

$$\tilde{\mu}(Y)([\zeta],[\xi]) := \Psi(X,Y)^{-1}[\zeta^\sharp * \xi^\sharp].$$

It remains to show that if $f\colon Y \to Z$ is continuous then the induced map $[\Sigma X, Y]_* \to [\Sigma X, Z]_*$ is a homomorphism. For this we consider the adjoint diagram:

$$\begin{array}{ccc}
[\Sigma X, Y]_* & \xrightarrow{\Psi(X,Y)} & [X, \Omega Y]_* \\
\Big\downarrow {\scriptstyle \mathrm{Hom}(\mathrm{id}_{\Sigma X}, f)} & & \Big\downarrow {\scriptstyle \mathrm{Hom}(\mathrm{id}_X, \Omega(f))} \\
[\Sigma X, Z]_* & \xrightarrow[\Psi(X,Z)]{} & [X, \Omega Z]_*
\end{array}$$

The map $\Omega(f)\colon \Omega Y \to \Omega Z$ satisfies the hypothesis (42.2) in Lemma 42.23, since $\Omega(f)(u*v) = (f \circ u) * (f \circ v)$ by definition. Thus Lemma 42.23 tells us that the right-hand vertical map is a homomorphism. Since the horizontal maps are isomorphisms, the left-hand vertical map is also a homomorphism. The proof is complete. ∎

Remark 42.25. It is sometimes useful to know an explicit formula for the comultiplication $\mu\colon \Sigma X \to \Sigma X \vee \Sigma X$ (we would have needed this in order to prove Corollary 42.24 directly). The formula is:

$$\mu([x,t]) := \begin{cases} ([x,2t],*), & 0 \le t \le 1/2, \\ (*,[x,2t-1]), & 1/2 \le t \le 1. \end{cases}$$

Thus μ is "pinching" the reduced suspension. See Figure 42.2. For $X = S^n$ this is the same as the map we called "Pinch" at the end of Lecture 20.

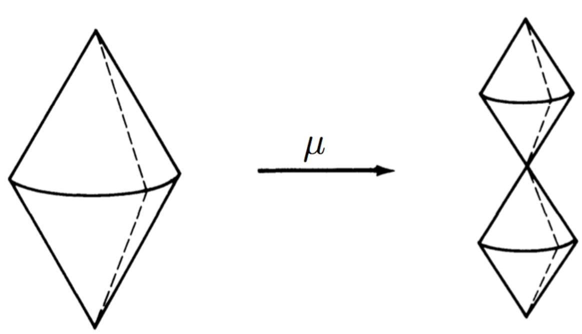


Figure 42.2: The comulitplication μ.

LECTURE 43

Higher homotopy groups

In this lecture we finally get round to defining the *higher homotopy groups*. Let us regard S^n as subspace of $\mathbb{R}^{n+1}$ in the usual way. We choose the point $* := (1, 0, \dots, 0) \in S^n$ for our basepoint (although we will almost always omit this form the notation.)

DEFINITION 43.1. Given a pointed space (X, x_0) and an integer $n \geq 0$, we define the **nth homotopy group**

$$\pi_n(X, x_0) := [(S^n, *), (X, x_0)] = [S^n, X]_*.$$

For $n = 1$, this is the none other than the **fundamental group**—the equivalence between this definition and our original one (Definition 4.1) was shown in Problem B.5. For $n = 0$, it is clear that $\pi_0(X, x_0)$ is the set $\pi_0(X)$ from Definition 3.6, where the basepoint is chosen to be the path component of X containing x_0. For $n \geq 2$ one usually calls $\pi_n(X, x_0)$ a **higher homotopy group**. Of course, the word "group" needs justifying:

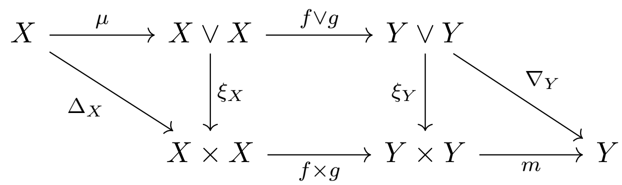
PROPOSITION 43.2. *For every pointed space (X, x_0), $\pi_0(X, X_0)$ is a pointed set, and for every $n \geq 1$, $\pi_n(X, x_0)$ is a group.*

Proof. Problem Q.7 tells us that S^n is a cogroup object in hTop_* for all $n \geq 1$. Thus Theorem 41.14 tells us that $[S^n, \square]_*$ takes values in Groups for all $n \geq 1$. ■

In fact, for $n \geq 2$ the group $\pi_n(X, x_0)$ is always abelian, as we will shortly prove. In the following, whenever possible we will start to omit the basepoint from our notation and just write $\pi_n(X)$ instead of $\pi_n(X, x_0)$. As with all our other "basepoint-omitting" conventions, this is somewhat imprecise. However as we will see below, if x_0 and x_1 lie in the same path component of X then there is always an isomorphism $\pi_n(X, x_0) \cong \pi_n(X, x_1)$.

PROPOSITION 43.3. *Suppose X is an H-cogroup and Y is H-group. Then the two group operations on $[X, Y]_*$ (one coming from the comultiplication μ on X and one coming from the multiplication m on Y) coincide.*

Proof. Let $\xi_X \colon X \vee X \hookrightarrow X \times X$ denote the inclusion, where as usual we think of $X \vee X$ as the subspace $(X \times \{x_0\}) \cup (\{x_0\} \times X)$ of $X \times X$, and let ξ_Y be defined similarly. Let $f, g \colon (X, x_0) \to (Y, y_0)$ be pointed maps. By Problem Q.6 the following diagram commutes up to homotopy:



The group multiplication determined by m is given by

$$\begin{aligned}[f] *_m [g] &:= [m \circ (f, g)] \\ &= [m \circ (f \times g) \circ \Delta_X]\end{aligned}$$

(using part (1) of Problem Q.5). The group multiplication determined by μ is given by

$$\begin{aligned}[f] *_\mu [g] &:= [(f, g) \circ \mu] \\ &= [\nabla_Y \circ (f \vee g) \circ \mu]\end{aligned}$$

(using part (2) of Problem Q.5). Thus commutativity of the diagram above (up to homotopy) tells us that

$$[f] *_m [g] = [f] *_\mu [g]$$

as required. ■

We can now prove:

THEOREM 43.4. *If X is a pointed space and $1 \leq k < n$, one has*

$$\pi_n(X) \cong \pi_{n-k}(\Omega^k X),$$

where Ω^k is the composition of Ω with itself k times.

Proof. By Problem Q.7 and Theorem 42.22 and Proposition 43.3, applied repeatedly, we have

$$\begin{aligned}\pi_n(X) &= [S^n, X]_* \\ &= [\Sigma^n S^0, X]_* \\ &= [\Sigma^{n-k} S^0, \Omega^k X]_* \\ &= [S^{n-k}, \Omega^k X]_* \\ &= \pi_{n-k}(\Omega^k X).\end{aligned}$$

■

COROLLARY 43.5. *If $n \geq 2$ then $\pi_n(X)$ is always abelian.*

Proof. By Theorem 43.4 for $n \geq 2$ we have $\pi_n(X) \cong \pi_1(\Omega^{n-1} X)$. Since $\Omega^{n-1} X$ is an H-group by Theorem 42.18, it is also in particular an H-space (Remark 42.15). Thus by Problem C.5, $\pi_1(\Omega^{n-1} X)$ is abelian. ■

Let us now give another proof of Corollary 43.5 that does not use Problem C.5. This starts from the following statement.

The following easy piece of algebra is on Problem Sheet R.

LEMMA 43.6. *Let A be a set, and assume A is equipped with two binary operations $*$ and $\circ$ such that:*

1. *$*$ and $\circ$ have a common two-sided unit,*
2. *$*$ and $\circ$ are mutually distributive, that is,*

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d), \qquad \forall\, a, b, c, d \in A.$$

Then $*$ and $\circ$ coincide and each is commutative and associative.

This gives us the desired second proof of Corollary 43.5:

Second proof of Corollary 43.5. As in the proof of Theorem 43.4, we have

$$\pi_n(X) \cong [\Sigma S^{n-2}, \Omega X]_*$$

for $n \geq 2$. The proof of Proposition 43.3 tells us the that the hypotheses of Lemma 43.6 are satisfied. Indeed, the constant map is the desired two-sided identity, and to check mutual distributivity we observe:[1]

$$\begin{aligned}\big([f] *_m [g]\big) *_\mu \big([h] *_m [k]\big) &= \big[m \circ (f \times g) \circ \Delta_X\big] *_\mu \big[m \circ (h \times k) \circ \Delta_X\big] \\ &= \big[\nabla_Y \circ \big(m \circ (f \times g) \circ \Delta_X\big) \vee \big(m \circ (h \times k) \circ \Delta_X\big) \circ \mu\big] \\ &= \big[m \circ \big((m \circ (f \times g) \circ \Delta_X) \times (m \circ (h \times k) \circ \Delta_X)\big) \circ \Delta_X\big] \\ &= \big[m \circ \big((\nabla_Y \circ (f \vee g) \circ \mu) \times (\nabla_Y \circ (h \vee k) \circ \mu)\big) \circ \Delta_X\big] \\ &= \big[\nabla_Y \circ (f \vee g) \circ \mu\big] *_m \big[\nabla_Y \circ (h \vee k) \circ \mu\big] \\ &= \big([f] *_\mu [g]\big) *_m \big([h] *_\mu [k]\big).\end{aligned}$$

■

Thus π_n is a functor $\mathsf{hTop}_* \to \mathsf{Sets}$ for $n = 0$, $\mathsf{hTop}_* \to \mathsf{Groups}$ for $n = 1$, and $\mathsf{hTop}_* \to \mathsf{Ab}$ for $n \geq 2$. Explicitly, if $f\colon (X, x_0) \to (Y, y_0)$ is a pointed map, then

$$\pi_n(f)\colon \pi_n(X, x_0) \to \pi_n(Y, y_0), \qquad [u] \mapsto [f \circ u].$$

We will see that these functors satisfy pointed versions of most of the Eilenberg-Steenrod axioms (the homotopy axiom is immediate from the fact that we have *defined* these functors on hTop_*). We will prove the dimension axiom below. Next lecture we will prove the long exact sequence axiom. The excision axiom does *not* hold in general (in Lecture 46 we will discuss the *Blakers-Massey Theorem*, which gives conditions for excision to hold.)

PROPOSITION 43.7 (The dimension axiom). *If X is a one point space then $\pi_n(X) = 0$ for all $n \geq 0$.*

[1] Thanks to Raccoon for this computation!

Proof. There is only one function $S^n \to *$, and thus $[S^n, *]_*$ only has one element. ■

Let us now discuss a more straightforward approach to defining the higher homotopy groups. This approach is less conceptually satisfying than the one we gave above, but it has the advantage of giving us explicit formulae, which will be helpful in further computations. Let us denote by I^n the unit cube $[0,1]^n$ and by ∂I^n its boundary:

$$\partial I^n = \{(s_1, \dots, s_n) \in I^n \mid \text{at least one } s_i \text{ is equal to 0 or 1}\}.$$

The following result should now be straightforward for all of you.

LEMMA 43.8. *If $n \geq 1$ then there exists a homeomorphism $\tau \colon I^n/\partial I^n \to S^n$. If $n \geq 2$ then one can build a homeomorphism $\sigma \colon I^n/\partial I^n \to \Sigma S^{n-1}$ from τ via*

$$\sigma \colon [s_1, \dots, s_n] \mapsto [\tau[s_1, \dots, s_{n-1}], s_n].$$

A pointed function $u \colon (I^n, \partial I^n) \to (X, x_0)$ induces a pointed map $\bar{u} \colon I^n/\partial I^n \to X$. If two such maps u, v are homotopic rel ∂I^n then there is a pointed homotopy $\bar{u} \simeq \bar{v}$. It follows that there is a bijection

$$J \colon [(I^n, \partial I^n), (X, x_0)] \to [(\Sigma S^{n-1}, *), (X, x_0)] = \pi_n(X, x_0), \qquad J[u] := [\bar{u} \circ \sigma^{-1}].$$

This allows us to endow $[(I^n, \partial I^n), (X, x_0)]$ with a group structure, and—more importantly—it gives us an explicit formula for the group addition:

$$(u+v)(s_1, \dots, s_n) := \begin{cases} u(s_1, \dots, s_{n-1}, 2s_n), & 0 \leq s_n \leq 1/2, \\ v(s_1, \dots, s_{n-1}, 2s_n - 1), & 1/2 \leq s_n \leq 1. \end{cases}$$

Of course, for $n = 1$ this is just the usual concatenation of paths. We use the notation $u + v$ instead of $u * v$ for $n \geq 2$ to emphasise that this is an abelian group operation. The formula also gives the a third "picture proof" (Figure 43.1) that $\pi_n(X)$ is abelian for $n \geq 2$, where the shaded regions are places where the homotopy is constant.

We now aim to prove the aforementioned statement that if X is path connected then $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for any two points $x_0, x_1 \in X$. The main result of today's lecture is the following theorem. Recall $\Pi(X)$ denotes the *fundamental groupoid* of X, whose objects are the points of x and whose morphisms are the path classes (cf. Proposition 3.18).

THEOREM 43.9. *Let X be path connected. Then for all $n \geq 1$, there is a covariant functor $T \colon \Pi(X) \to \mathsf{Groups}$.*

REMARK 43.10. In general a functor $T \colon \Pi(X) \to \mathsf{Groups}$ is said to be a **local system** of groups on X. (Similarly a functor $T \colon \Pi(X) \to \mathsf{Rings}$ is a local system of rings.) Thus Theorem 43.9 says that the homotopy groups form a local system. Note also that in the case $n = 1$ the theorem is immediate from the definition, since $\pi_1(X, x)$ is to the morphism set $\operatorname{Hom}_{\Pi(X)}(x, x)$.

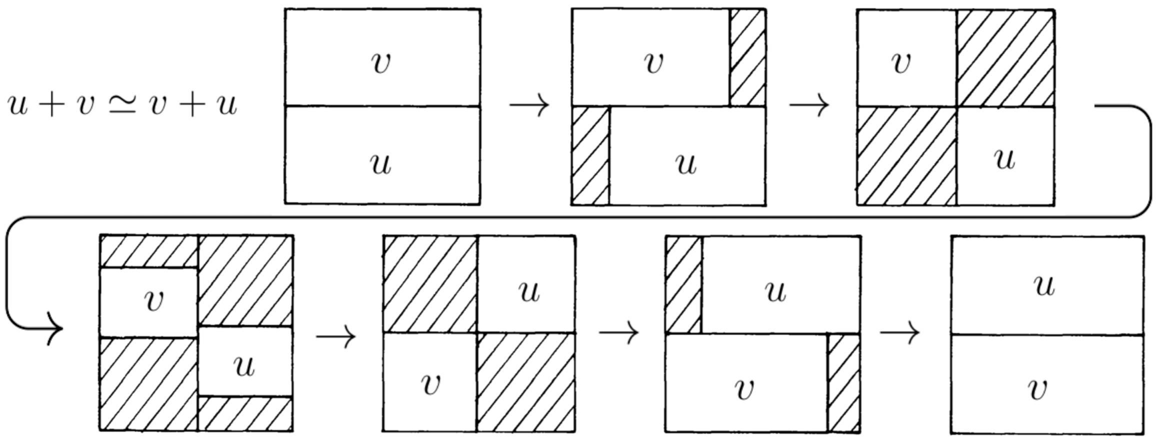


Figure 43.1: Proof that π_2 is abelian.

The proof of Theorem 43.9 will take some time. We begin with a collection of preliminary results. In order to keep the notation transparent, for the rest of this lecture the letters u, v represent elements of π_n and the letters w, z represent elements of π_1 (the reason for this distinction will shortly become clear).

Definition 43.11. Suppose $U\colon I^n \times I \to X$ is a free homotopy. Let $0 = (0, 0, \ldots, 0)$ denote the origin in I^n. Then $w(t) := U(0, t)$ is a path $I \to X$. We say that U is a homotopy **along** w. If in addition $U(x, t) = w(t)$ for all $x \in \partial I^n$ then we say that U is a **level homotopy along** w. Thus U is a homotopy relative to ∂I^n if and only if U is a level homotopy along the constant path at $U(0, 0)$.

Definition 43.12. Define a **stereographic retraction**

$$r\colon I^n \times I \to \big(I^n \times \{0\}\big) \cup \big(\partial I^n \times I\big) \tag{43.1}$$

as follows. Think of $I^n \times I$ as being embedded in $\mathbb{R}^{n+1}$, and let N denote the "north star", which for us will be the point $N = (1/2, \ldots, 1/2, , 2) \in \mathbb{R}^{n+1}$. Now if $y \in I^n \times I$, consider the line from N to y. This intersects $\big(I^n \times \{0\}\big) \cup \big(\partial I^n \times I\big)$ in a unique point, which we call $r(y)$.

For $n = 2$ this is easy to picture—the right-hand side is a "box" without a lid. Think of the north star as floating above the box and illuminating the contents. Given a point y in the cube, the light beam from the north star intersects the box in a unique point, and this is $r(y)$.

Let us also denote by R the analogously defined stereographic retraction

$$R\colon I^n \times I \times I \to \big(I^n \times I \times \{0\}\big) \cup \big(\partial I^n \times I \times I\big), \tag{43.2}$$

where we multiply both sides by another factor I.

To keep the notation concise, let us temporarily denote by $P(X, x_0)$ the space of all continuous maps $u\colon (I^n, \partial I^n) \to (X, x_0)$. If w is a path in X from x_0 to x_1 and u is an element of $P(X, x_0)$, then we can "glue" them together to form an element of $P(X, x_1)$ as follows:

1. Firstly define $U' \colon \big(I^n \times \{0\}\big) \cup \big(\partial I^n \times I\big) \to X$ by setting

$$U'(x,0) = u(x), \qquad x \in I^n$$

and

$$U'(x,s) = w(s), \qquad (x,s) \in \partial I^n \times I.$$

This does indeed define a continuous function on $I^n \times I \to \big(I^n \times \{0\}\big) \cup \big(\partial I^n \times I\big)$ since $u \in P(X, x_0)$. (Here we are using the Gluing Lemma 2.2.)

2. Now set $U := U' \circ r \colon I^n \times I \to X$. Then U is a level homotopy along w with $U(\cdot, 0) = u$.

3. Finally define $t_w(u) := U(\cdot, 1)$. Then $t_w(u)$ belongs to $P(X, x_1)$, and we obtain a map

$$t_w \colon P(X, x_0) \to P(X, x_1), \qquad u \mapsto t_w(u).$$

Note that (by construction) U is a level homotopy $u \simeq t_w(u)$ along w.

The following lemma explains why this construction is useful. It tells us that a level homotopy along a contractible loop can be changed into a relative homotopy (i.e. a level homotopy along a constant loop.)

LEMMA 43.13. *Suppose w is a loop in X based at x_0 which is nullhomotopic rel ∂I (thus $[w] = 0 \in \pi_1(X, x_0)$.) Suppose $U \colon u \simeq v$ is a level homotopy along w. Then $u \simeq v$ rel ∂I^n.*

Proof. Let $W \colon I \times I \to X$ be a homotopy rel ∂I from w to the constant loop at x_0. We glue W and U together to define a function

$$V' \colon \big(I^n \times I \times \{0\}\big) \times \big(\partial I^n \times I \times I\big) \to X$$

by

$$V'(x,t,0) = U(x,t), \qquad (x,t) \in I^n \times I,$$

and

$$V'(x,t,s) = W(t,s), \qquad (x,t,s) \in \partial I^n \times I \times I.$$

The function V' is continuous because the two definitions agree on the overlap $\partial I^n \times I \times \{0\}$, since U is a level homotopy. Now set $V := V' \circ R$, where R is the stereographic retraction from (43.2). We now build from V a new homotopy Q by setting:

$$Q(x,t) := \begin{cases} V(x, 0, 4t), & 0 \le t \le \frac{1}{4}, \\ V(x, 4t-1, 1), & \frac{1}{4} \le t \le \frac{1}{2}, \\ V(x, 1, 2-2t), & \frac{1}{2} \le t \le 1. \end{cases}$$

Again, Q is continuous by the Gluing Lemma 2.2. Since

$$Q(x,0) = V(x,0,0) = U(x,0) = u(x), \qquad Q(x,1) = V(x,1,0) = U(x,1) = v(x),$$

and $Q(x,t) = x_0$ for all $(x,t) \in \partial I^n \times I$, the homotopy Q shows that $u \simeq v$ rel ∂I^n, which is what we wanted to prove. ∎

COROLLARY 43.14. *Suppose w and z are two paths in X from x_0 to x_1 and $V\colon u \simeq v$ is a level homotopy along z. If $w \simeq z$ rel ∂I then $t_w(u) \simeq v$ rel ∂I^n.*

Proof. Let $U\colon I^n \times I \to X$ be the homotopy that defines $t_w(u)$ (i.e. $U = U' \circ r$ as specified in (2) above.) Concatenate U and V in the normal way:

$$(x,t) \mapsto \begin{cases} U(x, 1-2t), & 0 \le t \le \frac{1}{2}, \\ V(x, 2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

This defines a homotopy $t_w(u) \simeq v$ which moreover is a level homotopy along $\bar{w} * z$. Since by assumption $\bar{w} * z$ is nullhomotopic, Lemma 43.13 tells us that $t_w(u) \simeq v$ rel ∂I^n. ■

We now prove Theorem 43.9.

Proof of Theorem 43.9. Define T on objects by $T(x) := \pi_n(X, x)$. Define T on morphisms by setting

$$T[w] := [t_w(u)],$$

There are several things we need to check in order to complete the proof, and we will carry out the argument in three steps.

1. We first show that if $[w]$ is a path class in X from x_0 to x_1 and $[u] \in \pi_n(X, x_0)$ then $[t_w(u)]$ is a well-defined element of $\pi_n(X, x_1)$. Corollary 43.14 tells us that $[t_w(u)]$ depends only on the path class $[w]$ of w. To see that $[t_w(u)]$ only depends on the homotopy class $[u]$ of u, we argue as follows. Suppose $U\colon u \simeq v$ rel ∂I^n, that is, U is a level homotopy along the constant path e_0 based at x_0. Let V be the homotopy defining $t_w(v)$ (thus V is a level homotopy $v \simeq t_w(v)$). Concatenating U and V together gives a level homotopy $u \simeq t_w(v)$ along $e_0 * w$. Since $e_0 * w \simeq w$ rel ∂I, Corollary 43.14 tells us that $t_w(u) \simeq t_w(v)$ rel ∂I^n.

2. We now prove that T does indeed take values in Groups. For this it suffices to show that if w is a path from x_0 to x_1 and $u, v \in P(X, x_0)$, then (as elements of $P(X, x_1)$, one has:

$$t_w(u * v) \simeq t_w(u) * t_w(v) \qquad \text{rel } \partial I^n.$$

For this let $U\colon u \simeq t_w(u)$ and $V\colon v \simeq t_w(v)$ denote the level homotopies that define $t_w(u)$ and $t_w(v)$ respectively. Then define $Q\colon I^n \times I \to X$ by

$$Q(s_1, \dots, s_n, s) := \begin{cases} U(s_1, \dots, s_{n-1}, 2s_n, s), & 0 \le s_n \le \frac{1}{2}, \\ V(s_1, \dots, s_{n-1}, 2s_n - 1, s), & \frac{1}{2} \le s_n \le 1. \end{cases}$$

If $s_n = \frac{1}{2}$ then

$$(s_1, \dots, s_{n-1}, 2s_n) = (s_1, \dots, s_{n-1}, 1) \in \partial I^n$$

and

$$(s_1, \dots, s_{n-1}, 2s_n - 1) = (s_1, \dots, s_{n-1}, 0) \in \partial I^n,$$

and hence both U and V give the same value for all $s \in I$, namely $w(s)$. Thus Q is continuous, and so Q is a level homotopy $u * v \simeq t_w(u) * t_w(v)$ along w. By Corollary 43.14 (applied with $z := w$) we obtain $t_w(u * v) \simeq t_w(u) * t_w(v)$ rel ∂I^n as required.

3. Finally let us check that T really is a functor. If e_0 is the constant path at x_0 then it is clear that $T(e_0)$ acts as the identity on $\pi_n(X, x_0)$, so we need only check that if w and z are paths in X with $w(1) = z(0)$ then[2]

$$T([w * z]) = T[z] \circ T[w] \tag{43.3}$$

To see this, take $u \in P(X, w(0))$. Then there are level homotopies $u \simeq t_w(u)$ along w and $t_w(u) \simeq t_z(t_w(u))$ along z, and concatenating gives us a level homotopy $u \simeq t_z(t_w(u))$ along u. Applying Corollary 43.14 again, we obtain $t_{w*z}(u) \simeq t_z(t_w(u))$ rel ∂I^n, which is what we wanted to prove. ■

Theorem 43.9 has several formal consequences. The first is the following version of Proposition 4.13.

Proposition 43.15. *Suppose $f_0, f_1 \colon X \to Y$ are continuous maps and $F \colon f_0 \simeq f_1$ is a free homotopy from f_0 to f_1. Choose $x_0 \in X$ and let w denote the path in Y given by $w(t) = F(x_0, t)$. Then there is a commutative diagram:*

$$\begin{array}{ccc}
\pi_n(X, x_0) & \xrightarrow{\pi_n(f_0)} & \pi_n(Y, f_0(x_0)) \\
 & \searrow^{\pi_n(f_1)} & \downarrow T[w] \\
 & & \pi_n(Y, f_1(x_0)),
\end{array}$$

Proof. If $u \in P(X, x_0)$ then define $U \colon I^n \times I \to Y$ by $U(x, t) := F(u(x), t)$. Then U is a level homotopy along w from $f_0 \circ u$ to $f_1 \circ u$. By Corollary 43.14 again, we have $t_w(f_0 \circ u) \simeq f_1 \circ u$ rel ∂I^n, which gives commutativity of the diagram. ■

By arguing as in the proof of Proposition 4.15 we then obtain:

Proposition 43.16. *Suppose $f \colon X \to Y$ is a homotopy equivalence. Then for any $x_0 \in X$ the induced map $\pi_n(f) \colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ is an isomorphism.*

And then as in Corollary 4.16 we have:

Corollary 43.17. *Suppose X has the same homotopy type as a path connected space Y. Then for all $x_0 \in X$ and $y_0 \in Y$ one has $\pi_n(X, x_0) \cong \pi_n(Y, y_0)$. If X is a contractible space then $\pi_n(X, x_0) = \{0\}$ for all $x_0 \in X$.*

We conclude this lecture by discussing the homotopy groups of spheres.

Theorem 43.18 (Homotopy groups of spheres).

1. *One has $\pi_n(S^1) = 0$ for all $n \geq 2$.*
2. *If $0 < k < n$ then $\pi_k(S^n) = 0$.*
3. *For all $n \geq 1$ one has $\pi_n(S^n) \neq 0$.*

[2]This equation looks contravariant at first sight, but this is just due to our convention that $w * z$ is the path that first follows w and then follows z.

Proof (Sketch). The first statement follows from the **long exact sequence** of homotopy groups for fibrations that we will prove in Lecture 45 (specifically, Corollary 45.18.) Indeed, there is a fibration $\mathbb{Z} \to \mathbb{R} \to S^1$. Since both $\mathbb{R}$ and every path connected component of $\mathbb{Z}$ is contractible, Corollary 43.17 tells us that $\pi_n(\mathbb{R}) = \pi_n(\mathbb{Z}) = 0$ for all $n \geq 1$. Thus the long exact sequence tells us that $\pi_n(S^1) = 0$ for all $n \geq 2$.

The second statement follows from the fact that if $0 < k < n$ then any continuous map $f\colon S^k \to S^n$ is nullhomotopic relative to the basepoints. This follows from the Cellular Approximation Theorem that we will discuss in Lecture 46, see Corollary 46.14.

The fact that $\pi_n(S^n) \neq 0$ follows immediately from the fact that $H_n(S^n) \neq 0$. Indeed, since H_n is a functor, the latter tells us that the identity map on S^n is not freely nullhomotopic to a constant. Thus the identity map is also not relatively homotopic to a constant, and thus $[\mathrm{id}_{S^n}] \neq 0 \in [S^n, S^n]_*$. ∎

REMARK 43.19. In fact, as we will see in Lecture 46, one actually always has

$$\pi_n(S^n) \cong \mathbb{Z}, \qquad \forall\, n \geq 1.$$

It is however an **open problem** to compute the homotopy groups of spheres in general—S^1 is the only sphere for which all the homotopy groups are known. In Lecture 45 we will prove give a few more "ad hoc" computations: for instance, $\pi_3(S^2) \cong \mathbb{Z}$. But the groups can be more complicated! Indeed, one has (!!)

$$\pi_{14}(S^4) \cong \mathbb{Z}_2 \times \mathbb{Z}_{12} \times \mathbb{Z}_{120}, \qquad \text{and} \qquad \pi_{15}(S^4) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{84}.$$

LECTURE 44

The Puppe Sequence

In this lecture we prove the analogue of the *long exact sequence* axiom for homotopy groups. This requires us to define the *relative homotopy groups* $\pi_n(X, X')$ for $X' \subset X$ (the standard homotopy groups correspond to X' being a single point x_0). The long exact sequence axiom will take the following form: if $X' \subset X$ and $x_0 \in X'$ then the pointed inclusion $(X', x_0) \hookrightarrow (X, x_0)$ gives rise to an long exact sequence

$$\cdots \to \pi_n(X') \to \pi_n(X) \to \pi_n(X, X') \to \pi_{n-1}(X') \to \cdots$$

where we abbreviate (as usual) $\pi_n(X) = \pi_n(X, x_0)$ and $\pi_n(X') = \pi_n(X', x_0)$.

A first obstacle to this making sense is that for small n, $\pi_n(X)$ and $\pi_n(X, X')$ are not groups (just pointed sets) and thus we need to redefine what exactness actually means in this setting.

Definition 44.1. A sequence of pointed sets and pointed functions

$$(X, x) \xrightarrow{f} (Y, y) \xrightarrow{g} (Z, z)$$

is said to be **exact in** Sets_* if

$$\operatorname{im} f = \ker g := g^{-1}(z).$$

If the pointed sets are groups, and the basepoints are chosen to be the identity elements, then this recovers the standard notion of exactness. Note that here the choice of basepoint is crucial, since "$\ker g$" depends on the basepoint.

We now use Definition 44.1 to define what it means for a sequence to be exact in hTop_*.

Definition 44.2. Let

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$$

be a sequence of pointed spaces and pointed maps. We say this sequence is **exact in** hTop_* if for every pointed space Y the induced sequence of pointed homotopy classes:

$$\cdots [Y, X_{n+1}]_* \to [Y, X_n]_* \to [Y, X_{n-1}]_* \to \cdots$$

is exact in Sets_*. Here the basepoint of $[Y, X_n]_*$ is the pointed homotopy class of the constant map that sends all of Y to the basepoint of X_n.

Definition 44.3. Let $f\colon (X, x_0) \to (Y, y_0)$ be a pointed map. The **mapping fibre** of f is the pointed space

$$Mf := \left\{(x, w) \in X \times Y^I \mid w(0) = y_0,\ w(1) = f(x)\right\},$$

where the basepoint is $\varepsilon_0 := (x_0, e_{y_0})$, where e_{y_0} is the constant path at y_0.

From a categorical point of view, Mf is the pullback in Top_* of the diagram

$$\begin{array}{ccc} Mf & \dashrightarrow & Y^I \\ \downarrow & & \downarrow \mathrm{ev}_1 \\ X & \xrightarrow[f]{} & Y \end{array}$$

where $\mathrm{ev}_1(w) := w(1)$. We now define a collection of related maps.

1. There is an injection

$$\ell_1\colon \Omega Y \to Mf, \qquad \ell_1(w) = (x_0, w).$$

2. There is also a projection

$$f_1\colon Mf \to X, \qquad f_1(x, w) := x.$$

   Calling this projection "f_1" may appear strange at first, but it will shortly seem to be convenient.

3. We can iterate the construction of Mf and define

$$Mf_1 := \left\{(x, w, z) \in Mf \times X^I \mid z(0) = x_0,\ z(1) = f_1(x, w) = x\right\}$$

   This is again a pointed space, where the basepoint is (x_0, e_{y_0}, e_{x_0}). There is another injection

$$\ell_2\colon \Omega Y \to Mf_1, \qquad \ell_2(w) = (x_0, w, e_{x_0}).$$

4. There is another projection

$$f_2\colon Mf_1 \to Mf, \qquad f_2(x, w, z) = (x, w).$$

5. We can then iterate a third time to form

$$Mf_2 := \left\{(x, w, z, \zeta) \in Mf_1 \times (Mf)^I \mid \zeta(0) = \varepsilon_0,\ \zeta(1) = f_2(x, w, z) = (x, w)\right\},$$

   which is a pointed space with basepoint $(x_0, e_{y_0}, e_{x_0}, \varepsilon_0)$. There is an third injection

$$\ell_3\colon \Omega X \to Mf_2, \qquad \ell_3(z) = (x_0, e_{y_0}, z, \varepsilon_0),$$

   where ε_0 is the basepoint of Mf. (**Warning**: the domain of ℓ_3 is different to the domain of ℓ_1 and ℓ_2!)

6. There is another projection

$$f_3 \colon Mf_2 \to Mf_1, \qquad f_3(x, w, z, \zeta) = (x, w, z).$$

7. We now iterate a fourth time... Only joking, we could keep going indefinitely, but three iterations will suffice for the time being.

These maps fit together nicely to build the following commutative diagram in hTop_*.

Proposition 44.4. *Let $f \colon X \to Y$ be a pointed map. Then the following diagram is commutative in hTop_*:*

$$\begin{array}{ccccccccc}
\Omega X & \xrightarrow{\Omega(f)} & \Omega Y & \xrightarrow{\ell_1} & Mf & \xrightarrow{f_1} & X & \xrightarrow{f} & Y \\
\big\downarrow{\scriptstyle \ell_3 \circ \theta} & & \big\downarrow{\scriptstyle \ell_2} & & \big\downarrow{\scriptstyle \mathrm{id}_{Mf}} & & \big\downarrow{\scriptstyle \mathrm{id}_X} & & \big\downarrow{\scriptstyle \mathrm{id}_Y} \\
Mf_2 & \xrightarrow[f_3]{} & Mf_1 & \xrightarrow[f_2]{} & Mf & \xrightarrow[f_1]{} & X & \xrightarrow[f]{} & Y
\end{array}$$

where the map $\theta \colon \Omega X \to \Omega X$ appearing on the left-most vertical arrow is the inversion $\theta(z) = \bar{z}$.

Proof. I leave it up to you to check that the two squares on the right commute. The second square actually commutes in Top_* (and thus also in hTop_*):

$$f_2 \circ \ell_2(w) = f_2(x_0, w, e_{x_0}) = (x_0, w) = \ell_1(w).$$

The left-most square only commutes up to homotopy. Indeed, if $z \in \Omega X$ then going clockwise,

$$\ell_2 \circ \Omega(f)(z) = \ell_2(f \circ z) = (x_0, f \circ z, e_{x_0}).$$

Going anti-clockwise,

$$f_3 \circ \ell_3 \circ \theta(z) = f_3 \circ \ell_3(\bar{z}) = f_3(x_0, e_{y_0}, \bar{z}, \varepsilon_0) = (x_0, e_{y_0}, \bar{z}).$$

To define a homotopy between the two, let us introduce the following notation: if $u \colon I \to Z$ is a path and $t \in I$, let $u_t \colon I \to Z$ denote the path $u_t(s) = u(st)$, and define $\bar{u}_t$ to be the path[1] $\bar{u}_t(s) := \bar{u}(st) = u(1 - st)$.

Now define $F \colon \Omega(X, x_0) \times I \to Mf_1$ by

$$F(z, t) := (z(1 - t), f \circ z_{1-t}, \bar{z}_t).$$

Then F is continuous and $F(z, 0) = (x_0, f \circ z, e_{x_0})$ and $F(z, 1) = (x_0, e_{y_0}, z)$, so that $\ell_2 \circ \Omega(f) \simeq f_3 \circ \ell_3 \circ \theta$. Finally, $F(e_{x_0}, t) = (x_0, e_{y_0}, e_{x_0})$ for all $t \in I$, and hence F is a homotopy relative to the basepoint $e_{x_0} \in \Omega(X, x_0)$. This completes the proof. ■

Proposition 44.5. *The two injections ℓ_2 and ℓ_3 induce isomorphisms $[\ell_2]$ and $[\ell_3]$ in hTop_*.*

[1]Warning: in general $\bar{u}_t \neq \overline{u_t}$, as the latter is the path $s \mapsto u(t(1 - s))$.

Proof. We define a continuous map $F\colon Mf_1 \times I \to Mf_1$ such that

$$F(x,w,z,0) = (x,w,z), \qquad F(x,w,z,1) \in \ell_2(\Omega(Y,y_0)), \qquad F(x_0, e_{y_0}, e_{x_0}, t) = (x_0, e_{y_0}, e_{x_0}).$$

In fact, we will construct F as a concatenation $F_1 * F_2$ of two other homotopies. For this, first let F_1 denote a homotopy that begins with (x,w,z) and ends at $(x, w * e_{f(x)}, z)$. Next, let F_2 denote the homotopy given by

$$F_2(x,w,z,t) = (z(1-t), w * (f \circ \bar{z}_t), z_{1-t}),$$

where we are using the same notation as in the proof of Proposition 44.4, i.e. $z_{1-t}(s) := z(s(1-t))$. Then $F_2(x,w,z,0) = (x_0, w * e_{f(x)}, z)$, so that the composition $F_1 * F_2$ is well-defined. Moreover

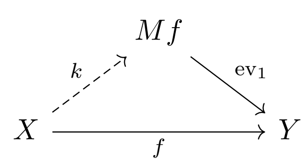
$$F_2(x,w,z,1) = (x_0, w * (f \circ \bar{z}), e_{y_0}) \in \ell_2(\Omega(Y,y_0)).$$

Finally, (x, e_{y_0}, e_{x_0}) really is fixed throughout the entire homotopy, since $e_{y_0} * e_{y_0} = e_{y_0}$.

The homotopy F exhibits $\ell_2(\Omega(Y,y_0))$ as a pointed deformation retract of Mf_1. Thus $[\ell_2]$ is an isomorphism in hTop_*. The proof that $[\ell_3]$ is an isomorphism is similar, and I will leave this as an exercise. ■

The next result gives us a way of proving exactness in hTop_*.

LEMMA 44.6. *Let $f\colon (X,x_0) \to (Y,y_0)$ be a pointed map, and let $\mathrm{ev}_1\colon Mf \to Y$ denote the map $\mathrm{ev}_1(x,w) = w(1)$. Then f is nullhomotopic rel x_0 if and only if there exists a pointed map $k\colon X \to Mf$ such that $f = \mathrm{ev}_1 \circ k$:*



Proof. Assume there exists a continuous map $F\colon X \times I \to Y$ such that $F(x,0) = y_0$, $F(x,1) = f(x)$ and $F(x_0,t) = y_0$ for all t. Let $F_x\colon I \to Y$ be the map $F_x(t) = F(x,t)$. Then define $k\colon X \to Mf$ by

$$k(x) = (x, F_x).$$

This k has the desired properties.

Conversely, if such a pointed map k exists, then writing $k(x) = (k_1(x), w_x)$, one necessarily has $w_{x_0} = e_{y_0}$, and commutativity forces $w_x(1) = f(x)$ for each x. Our desired pointed homotopy is then given by

$$F\colon X \times I \to Y, \qquad F(x,t) = w_x(t).$$

■

LEMMA 44.7. *Let $f\colon (X,x_0) \to (Y,y_0)$ be a pointed map. Then the sequence*

$$Mf \xrightarrow{f_1} X \xrightarrow{f} Y$$

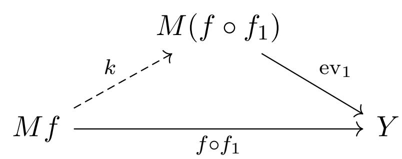
is exact in hTop_*.

Proof. Let Z be any pointed space, and consider the induced maps in Sets_*:

$$[Z, Mf]_* \xrightarrow{\alpha} [Z, X]_* \xrightarrow{\beta} [Z, Y]_*$$

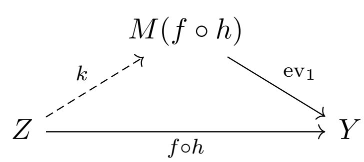
Here α sends a class $[g] \in [Z, Mf]_*$ to the class $[f_1 \circ g] \in [Z, X]_*$, and β is defined similarly. We take the basepoint in $[Z, Y]_*$ to be the class of the constant map $Z \to y_0$. Thus the kernel of β consists of all (homotopy classes) of maps $h\colon Z \to X$ such that $f \circ h$ is pointedly nullhomotopic. To prove exactness, we need to show:

- $\operatorname{im}\alpha \subseteq \ker\beta$: Define a map $k\colon Mf \to M(f \circ f_1)$ by setting $k(x, w) = (x, w, w)$. (Here $M(f \circ f_1) \subset Mf \times Y^I$ as $f \circ f_1\colon Mf \to Y$). With ev_1 defined as in Lemma 44.6 (but as a map $M(f \circ f_1) \to Y$ instead), it is clear this diagram commutes:

  Thus $f \circ f_1$ is pointedly nullhomotopic. It follows that $f \circ f_1 \circ g$ is also pointedly nullhomotopic for every $[g] \in [Z, Mf]_*$.

- $\ker\beta \subseteq \operatorname{im}\alpha$: Suppose $[h] \in [Z, X]_*$ and suppose $[h] \in \ker\beta$. Thus there exists $F\colon f \circ h \simeq c \text{ rel } x_0$, where c is the constant map at y_0. Define $F_z\colon I \to Y$ by $F_z(t) = F(z, t)$, and then define, as in the proof of Lemma 44.6, $k\colon Z \to M(f \circ h)$ by $k(z) = (z, F_z)$, so that the following commutes:

  Now by definition, $M(f \circ h) \subset Z \times Y^I$, and the map $h \times \mathrm{id}_{Y^I}\colon Z \times Y^I \to Z \times Y^I$ restricts to define a map $\tilde{h}\colon M(f \circ h) \to Mf$. Thus $\tilde{h} \circ k\colon Z \to Mf$, and since $f_1 \circ \tilde{h} \circ k = h$, we have that $[h] = \alpha[\tilde{h} \circ k]$.

■

Corollary 44.8. *If $f\colon X \to Y$ is a pointed map then the sequence*

$$\cdots \to Mf_2 \xrightarrow{f_3} Mf_1 \xrightarrow{f_2} Mf \xrightarrow{f_1} X \xrightarrow{f} Y$$

is exact in hTop_*.

Proof. Iterate Lemma 44.7. ■

Corollary 44.9. *If $f\colon X \to Y$ is a pointed map then the sequence*

$$\Omega X \xrightarrow{\Omega(f)} \Omega Y \xrightarrow{\ell_1} Mf \xrightarrow{f_1} X \xrightarrow{f} Y$$

is exact in hTop_*.

Proof. We apply $[Z, \square]_*$ to the commutative diagram from Proposition 44.4 to form a new commutative square:

$$\begin{array}{ccccccccc} [Z,\Omega X]_* & \longrightarrow & [Z,\Omega Y]_* & \longrightarrow & [Z,Mf]_* & \longrightarrow & [Z,X]_* & \longrightarrow & [Z,Y]_* \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ [Z,Mf_2]_* & \longrightarrow & [Z,Mf_1]_* & \longrightarrow & [Z,Mf]_* & \longrightarrow & [Z,X]_* & \longrightarrow & [Z,Y]_* \end{array}$$

The bottom row is exact in Sets_* by Corollary 44.8. Since the vertical maps are isomorphisms by Proposition 44.5, it follows that the top row is also exact in Sets_*. ∎

The next result tells us that Ω is an exact functor on hTop_*.

Proposition 44.10. *If $X \to Y \to Z$ is an exact sequence in hTop_* then so is the "looped" sequence $\Omega X \to \Omega Y \to \Omega Z$.*

Proof. We use the fact that (Σ, Ω) form an adjoint pair. For every pointed space W, there is a commutative diagram in which the vertical maps are pointed bijections:

$$\begin{array}{ccccc} [\Sigma W,X]_* & \longrightarrow & [\Sigma W,Y]_* & \longrightarrow & [\Sigma W,Z]_* \\ \downarrow & & \downarrow & & \downarrow \\ [W,\Omega X]_* & \longrightarrow & [W,\Omega Y]_* & \longrightarrow & [W,\Omega Z]_* \end{array}$$

The top row is exact in Sets_* by hypothesis. Thus the bottom row is exact in Sets_* too. Since W was arbitrary, it follows that $\Omega X \to \Omega Y \to \Omega Z$ is exact in hTop_*. ∎

Putting what we have done so far together gives the following theorem, which is the main result of today's lecture.

Theorem 44.11 (The Puppe Sequence). *If $f\colon X \to Y$ is a pointed map then there is a long exact sequence in hTop_* of the form*

$$\cdots \to \Omega^n(Mf) \xrightarrow{\Omega^n(f_1)} \Omega^n X \xrightarrow{\Omega^n(f)} \Omega^n Y \xrightarrow{\Omega^{n-1}(\ell_1)} \Omega^{n-1}(Mf) \to \cdots$$

which ends with $Mf \xrightarrow{f_1} X \xrightarrow{f} Y$.

Proof. By Corollary 44.9 the sequence

$$\Omega X \to \Omega Y \to Mf \to X \to Y$$

is exact in hTop_*, and by Proposition 44.10 the looped sequence

$$\Omega^2 X \to \Omega^2 Y \to \Omega(Mf) \to \Omega X \to \Omega Y$$

is also exact in hTop_*. Since these sequences overlap, we may splice them together. An inductive argument completes the proof. ∎

Remark 44.12. It will not surprise you to learn that there is a dual formulation of this result. We won't need it, so I won't go over the proofs, but let me summarise the ideas. Firstly, a sequence of pointed spaces and pointed maps

$$\cdots \to X_{n+1} \to X_n \to X_{n-1} \to \cdots$$

is said to be **coexact** in hTop_* if for any pointed space Z, the reversed sequence

$$\cdots \leftarrow [X_{n+1}, Z]_* \leftarrow [X_n, Z]_* \leftarrow [X_{n-1}, Z]_* \leftarrow \cdots$$

is exact in Sets_*. Given $f\colon X \to Y$, define the[2] **mapping cone** Cf, which is defined as the adjunction space $(X \wedge I) \cup_f Y$, where $X \wedge I$ is the *smash product* (cf. Problem Q.7). There is a natural map $Cf \to \Sigma X$ (this plays the role that $\ell_1\colon \Omega Y \to Mf$ did above), and one obtains the following **coexact Puppe Sequence**:

$$X \xrightarrow{f} Y \to Cf \to \Sigma X \to \Sigma Y \to \Sigma(Cf) \to \Sigma^2 X \to \Sigma^2 Y \to \Sigma^2(Cf) \to \cdots$$

The long exact sequence for homotopy groups and the homotopy long exact sequence for fibrations are basically special cases of the Puppe Sequence. In this lecture we will discuss the former, and leave the homotopy sequence for fibrations till the next lecture.

Corollary 44.13. *Let (X, x_0) be a pointed space, and let $X' \subset X$ be a subspace containing x_0. Let $\imath\colon (X', x_0) \hookrightarrow (X, x_0)$ be the pointed inclusion. Then there is an exact sequence in Sets_*:*

$$\cdots \pi_{n+1}(X') \to \pi_{n+1}(X) \to [S^0, \Omega^n(M\imath)]_* \to \pi_n(X') \to \cdots$$

which ends with $[S^0, M\imath]_* \to \pi_0(X') \to \pi_0(X)$.

Proof. Here we have just applied the functor $[S^0, \square]$ to the Puppe Sequence for the inclusion $\imath\colon X' \hookrightarrow X$, and used the definition of π_n, recalling that $\pi_{n+1}(X') = \pi_1(\Omega^n X')$. ■

To transform this sequence into our desired long exact sequence, we want write the terms $[S^0, \Omega^n(M\imath)]_*$ in a more convenient form, and at the same time get a nice description of the "connecting homomorphism" $[S^0, \Omega^n(M\imath)]_* \to \pi_n(X')$. This requires us to define the *relative homotopy groups*.

Definition 44.14. Let $X' \subset X$ and let $x_0 \in X'$. Then (X, x_0) and (X', x_0) are pointed spaces. We refer to (X, X', x_0) (or sometimes just (X, X')) as a **pointed pair**. A **pointed pair map** $f\colon (X, X') \to (Y, Y')$ is a pointed map that also has $f(X') \subset Y'$. There is also a notion of a **pointed pair homotopy**, which is defined as you would guess. We write $[(X, X'), (Y, Y')]_*$ for the space of homotopy classes of pointed pair maps.

Now think of $S^{n-1} \subset B^n$, and choose $* := (1, 0, \ldots, 0) \in S^{n-1}$.

[2]**Warning:** this is not quite the same thing as the "mapping cone" from Definition 27.2, which was concerned with chain maps between chain complexes. The two concepts are related though.

Definition 44.15. Given a pointed pair (X, X', x_0) and an integer $n \geq 0$, we define the n**th relative homotopy group**

$$\pi_n(X, X', x_0) := [(B^n, S^{n-1}, *), (X, X', x_0)]_* = [(B^n, S^{n-1}), (X, X')]_*.$$

We usually abbreviate $\pi_n(X, X', x_0)$ just as $\pi_n(X, X')$.

The name "group" is a bit of a misnomer, since (at least at the moment) there is no obvious group structure on $\pi_n(X, X')$ (it is just a pointed set). This will be rectified in Corollary 44.18 below. There is a well-defined map (of pointed sets)

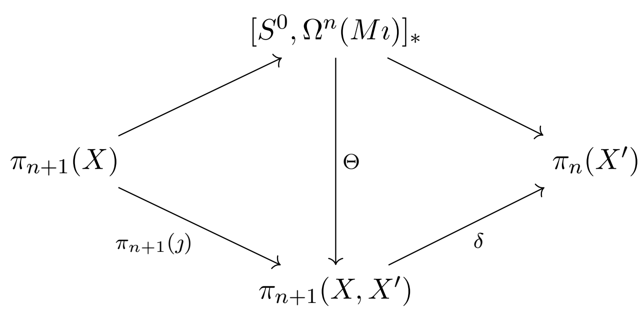
$$\delta \colon \pi_{n+1}(X, X') \to \pi_n(X'), \qquad [u] \mapsto [u|_{S^n}]. \tag{44.1}$$

Remark 44.16. One can also describe $\pi_n(X, X')$ as $[(I^n, \partial I^n), (X, X')]_*$, and we will switch back and forth between the two definitions next lecture. It is clear that if X' is itself a point x_0 then $\pi_n(X, x_0, x_0) = \pi_n(X, x_0) = \pi_n(X)$ is just the usual higher homotopy group, since:

$$[(B^n, S^{n-1}), (X, x_0)] = [(B^n/S^{n-1}, *), (X, x_0)] = [(S^n, *), (X, x_0)] = \pi_n(X, x_0).$$

Here is the promised identification.

Theorem 44.17. *Let (X, X') be a pointed pair with inclusion $\imath \colon X' \hookrightarrow X$. There is a bijection $\Theta \colon [S^0, \Omega^n(M\imath)]_* \to \pi_{n+1}(X, X')$. Moreover the following diagram commutes, where the top two maps are from Corollary 44.13, the map $\jmath$ is the inclusion $(X, x_0) \hookrightarrow (X, X')$ (and we identify $\pi_{n+1}(X, x_0)$ with $\pi_{n+1}(X, x_0, x_0)$ as above) and δ was defined in (44.1).*



Proof. We will prove the theorem in three steps.

1. In this step we define Θ. Using adjointness of (Σ, Ω), we can rewrite

$$[S^0, \Omega^n(M\imath)]_* \cong [S^n, M\imath]_* = \pi_n(M\imath).$$

Thus given a pointed map $a \colon S^n \to M\imath$, we want to build a pointed map of pairs $A \colon (B^{n+1}, S^n) \to (X, X')$. Such a map a is of the form

$$a(y) = (x_y, w_y) \in X' \times X^I, \qquad w_y(0) = x_0,\ w_y(1) = x_y, \qquad y \in S^n.$$

An arbitrary non-zero element of B^{n+1} can be written uniquely as $t \cdot y$ for some $y \in S^n$. We then define $A \colon B^{n+1} \to X$ by $A(0) = x_0$ and $A(t \cdot y) = w_y(t)$. This is continuous as $w_y(0) = x_0$ for all $y \in S^n$. Moreover $A(1 \cdot y) = w_y(1) = x_y \in X'$, and thus A is a map of pairs $(B^{n+1}, S^n) \to (X, X')$. Thus our desired map Θ is $[a] \mapsto [A]$.

2. In this step we show that Θ is well-defined. Suppose $a \simeq b$, and $F\colon S^n \times M\imath \to X$ is a pointed homotopy from a to b. Thus

$$F(y, 0) = a(y), \qquad F(y, 1) = b(y), \qquad \forall\, y \in S^n,$$

and if $*$ is our basepoint in S^n then $F(*, s) = (x_0, e_{x_0})$ for all $s \in I$. Now using exactly the same trick we can build from F a homotopy from the two maps A and B. Indeed, writing $F(y, s) = (x_{y,s}, w_{y,s})$, our desired homotopy $H\colon B^{n+1} \times I \to X$ is defined by $H(t \cdot y, s) = w_{y,s}(t)$. This gives a pointed homotopy of pairs $H\colon A \simeq B$.

3. In this step we prove that Θ is a bijection. Suppose we are given a continuous pointed map of pairs $A\colon (B^{n+1}, S^n) \to (X, X')$. Assume for simplicity that $A(0) = x_0$ (note: $0 \in B^{n+1}$ is *not* the basepoint, since that is $* \in S^n$.) Given $y \in S^n$, let $w_y\colon I \to X$ by the path $w_y(t) = A(t \cdot y)$. Then define $a\colon S^n \to M\imath$ by $a(y) = (A(y), w_y)$. This construction $A \mapsto a$ is then an inverse to Θ.

Of course, this only worked because $A(0) = x_0$. But it is easy to see that—up to changing A through pointed pair homotopies—this can always be achieved.

Finally, the claim that the diagram commutes is left as an exercise in unravelling the definitions. ■

Corollary 44.18. *Let (X, X') be a pointed pair. Then $\pi_n(X, X')$ is a group for $n \geq 2$ and an abelian group for $n \geq 3$.*

Proof. As discussed in the proof of Theorem 44.17, one has $[S^0, \Omega^n(M\imath)]_* = [S^n, M\imath]_* = \pi_n(M\imath)$. Thus using the bijection Θ we can *define* a group structure on $\pi_{n+1}(X, X')$ for all $n \geq 1$. Since $\pi_n(M\imath)$ is abelian by Corollary 43.5 for $n \geq 2$, we see that $\pi_n(X, X')$ is abelian for $n \geq 3$. ■

We conclude this lecture by ticking off another axiom:

Theorem 44.19 (The long exact sequence axiom). *If (X, X') is a pointed pair, then there is an exact sequence*

$$\cdots \pi_{n+1}(X') \to \pi_{n+1}(X) \to \pi_{n+1}(X, X') \xrightarrow{\delta} \pi_n(X') \to \cdots$$

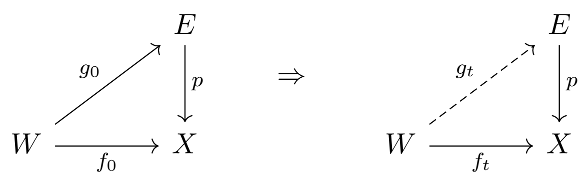
where all the maps are induced by inclusions apart from the connecting homomorphism δ, which was defined in (44.1).

Proof. Combine Corollary 44.13 and Theorem 44.17. ■

LECTURE 45

Fibrations and weak fibrations

In this lecture we return to the study of *fibrations*. Let $p\colon E \to X$ be a continuous map between two topological spaces, and let W be a topological space. Recall from Definition 33.13 that we say that p has the **homotopy lifting property** with respect to W if for any homotopy $f_t\colon W \to X$ (for $t \in [0,1]$) and any continuous map $g_0\colon W \to E$ such that $p \circ g_0 = f_0$, there exists a homotopy $g_t\colon W \to X$ such that $p \circ g_t = f_t$:



The homotopy g_t is called a **covering homotopy** of f_t.

Definition 45.1. We say that $p\colon E \to X$ is a **fibration** if p has the homotopy lifting property with respect to every space W. Given $x \in X$, we denote by $F := p^{-1}(x)$ a **fibre** of p.

It is *not* in general true that any two fibres are homeomorphic (contrast this to fibre bundles). However as we will shortly see, any two fibres have the same homotopy type when the base space X is path connected (and thus the notation "F" is unambiguous as far as homotopy type is concerned.)

Example 45.2. Let X and F be any topological spaces, set $E := X \times F$ and let $p\colon E \to X$ denote projection onto the first factor. Then p is a fibration. Indeed, if $f_t\colon W \to X$ is any homotopy and $g_0\colon W \to E$ satisfies $p \circ g_0 = f_0$ then writing $g_0(w) = (f_0(w), g_0'(w)) \in X \times F$, the homotopy g_t defined by

$$g_t(w) := (f_t(w), g_0'(w))$$

is a continuous map with $p \circ g_t = f_t$.

Example 45.2 tells us that any trivial fibre bundle is a fibration. We shall come back to this in the proof of Corollary 45.14. It is often convenient to regard a fibration $p\colon E \to X$ as a pointed map; to do so we simply pick any $x_0 \in X$ and let y_0 denote any point in the fibre $p^{-1}(x_0)$.

Proposition 45.3. *Let $p\colon (E, y_0) \to (X, x_0)$ be a fibration with fibre $F = p^{-1}(x_0)$. Then F and the mapping fibre Mp have the same homotopy type.*

Proof. We have the following commutative diagram, where $p_1\colon Mp \to E$ is the map $p_1(y, w) = y$, $q\colon Mp \to X^I$ is the map $q(y, w) = w$, and $\mathrm{ev}_1\colon X^I \to X$ is the map $\mathrm{ev}_1(w) = w(1)$:

$$\begin{array}{ccc} Mp & \xrightarrow{p_1} & E \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X^I & \xrightarrow[\mathrm{ev}_1]{} & X \end{array}$$

Let e_0 denote the constant path at x_0. Then $(y, e_0) \in Mp$ if $y \in F$ and hence we define $h\colon F \to Mp$ by $h(y) = (y, e_0)$. To complete the proof we will define a homotopy inverse to h. Consider the continuous map

$$f_t\colon Mp \to X, \qquad f_t(y, w) := w(1 - t)$$

(this is continuous as it is the composition of continuous maps). Since p is a fibration, we can find a homotopy $g_t\colon Mp \to E$ such that $p \circ g_t = f_t$ and $g_0 = p_1$. Since $p \circ g_1(y, w) = w(0) = x_0$, the function g_1 takes values in the fibre F. Thus there is a well-defined map $k\colon Mp \to F$ given by $k(y, w) = g_1(y, w)$. We claim that k is a homotopy inverse to h.

Indeed, since g_1 takes values in F, the function $(y, t) \mapsto g_t(h(y))$ is a homotopy from id_F to $k \circ h$. For the converse, we first define $j_t\colon Mp \to X^I$ by requiring that $j_t(y, w)$ is the path $s \mapsto w(s(1 - t))$. Then $j_0 = q$ and $j_1(y, w) = e_0$ since every element $(y, w) \in Mp$ has $w(0) = x_0$. Now consider the function

$$l_t\colon Mp \to Mp, \qquad l_t(y, w) := (g_t(y, w), j_t(y, w)).$$

To check that l_t is well-defined, we need $p \circ g_t(y, w) = j_t(y, w)(1)$. This is true, since both sides are equal to $w(1 - t)$. One has $l_0(y, w) = (g_0(y, w), j_0(y, w)) = (y, w)$ so that $l_0 = \mathrm{id}_{Mp}$. Finally $l_1(y, w) = h \circ k(y, w)$. Thus l_t is a homotopy from id_{Mp} to $k \circ h$. This completes the proof. ■

We can now use this to give a simple proof that all fibres of a fibration have the same homotopy type when the base is path connected.

Corollary 45.4. *Let $p\colon E \to X$ be a fibration over a path connected space X, and let $x_0, x_1 \in X$. Then the two fibres $p^{-1}(x_0)$ and $p^{-1}(x_1)$ have the same homotopy type.*

Proof. Let $M_i p$ denote the mapping fibre of p with respect to the basepoint x_i. It suffices to show that $M_0 p$ has the same homotopy type as $M_1 p$. But this is easy: if $u\colon I \to X$ is a path from x_1 to x_0 (which exists as X is path connected) then the map $M_0 p \to M_1 p$ given by $(y, w) \mapsto (y, u * w)$ is a homotopy equivalence (the inverse is given by $(y, w) \mapsto (y, \bar{u} * w)$.) ■

We can also use the Puppe Sequence (Theorem 44.11) to give a quick proof of the following long exact sequence.

Theorem 45.5 (The Homotopy Sequence of a Fibration). *Let $p\colon E \to X$ be a fibration with fibre F. Then there is a long exact sequence*

$$\cdots \pi_n(E) \xrightarrow{\pi_n(p)} \pi_n(X) \to \pi_{n-1}(F) \to \pi_{n-1}(E) \to \cdots$$

Proof. Proposition 45.3 tells us that Mp and F have the same homotopy type. Thus Proposition 43.16 tells us that $[S^n, Mp]_* \cong [S^n, F]_*$ for all $n \geq 0$. The result thus follows from applying $[S^0, \Box]_*$ to the Puppe Sequence (Theorem 44.11) to p, as in Corollary 44.13, and using adjointness of (Σ, Ω) to obtain $[S^0, \Omega^n(Mp)]_* \cong [S^n, Mp]_* \cong [S^n, F]_* = \pi_n(F)$. ∎

Let us now introduce a generalisation of the fundamental groupoid that involves two spaces. We will use this to state (and sketch the proof of) a powerful result which we call the *Homotopy Theorem for Fibrations*.

Definition 45.6. Let X and Y be topological spaces. We define a category $\Pi(X, Y)$ as follows: the objects of $\Pi(X, Y)$ are the continuous maps $f\colon X \to Y$. A morphism from f to g is an equivalence class of homotopies $F\colon f \simeq g$, where F and G define the same morphism if F is homotopic to G relative to $X \times \partial I$, that is, if there exists $\Phi\colon X \times I \times I \to Y$ such that $\Phi(x, 0, t) = f(x)$ is independent of t, $\Phi(x, 1, t) = g(x)$ is independent of t, and $\Phi_t(x, s) := \Phi(x, s, t)$ is a homotopy from $\Phi_0 = F$ and $\Phi_1 = G$. Informally: a morphism in $\Pi(X, Y)$ is a **homotopy of homotopies**.

The fundamental groupoid $\Pi(X)$ is the special case $\Pi(*, X)$. I will leave it up to you as a wholesome exercise to verify that $\Pi(X, Y)$ is a groupoid category (cf. Definition 3.19), i.e. that every morphism in $\Pi(X, Y)$ is an isomorphism.

Remark 45.7. The dependence of the groupoid category on X and Y leads to the notion of a **2-category**. However, higher category theory goes beyond the scope of this course, so I won't go into the details.

Let us go back to fibrations. Suppose $p\colon E \to X$ is a fibration, and $f\colon Z \to X$ is a continuous map. The **pullback** (cf. Example 40.6, compare also Definition 34.1) in Top is the topological space

$$Q = \{(z, y) \in Z \times E \mid f(z) = p(y)\},$$

equipped with the projection $q\colon Q \to Z$ given by $q(z, y) = z$. We denote by $\varphi\colon Q \to E$ the map $\varphi(z, y) = y$, so that the following commutes:

$$\begin{array}{ccc} Q & \xrightarrow{\varphi} & E \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f]{} & X \end{array} \tag{45.1}$$

Lemma 45.8. *The map q is a fibration.*

Proof. Let $h_t\colon W \to Z$ be a homotopy and $g_0\colon W \to Q$ be a continuous map such that $q \circ g_0 = h_0$. We want to find a homotopy $g_t\colon W \to Q$ such that $q \circ g_t = h_t$. Consider $f \circ h_t\colon W \to X$ and $\varphi \circ g_0\colon W \to E$. Then $p \circ \varphi \circ g_0 = f \circ q \circ g_0 = f \circ h_0$. Since p is a fibration, there exists $k_t\colon W \to E$ such that $p \circ k_t = f \circ h_t$ and $k_0 = \varphi \circ g_0$. We then extend g_0 to a homotopy $g_t\colon W \to Q$ via

$$g_t(w) = (h_t(w), k_t(w)).$$

This is a well-defined element of Q by construction and satisfies $q \circ g_t = h_t$. ■

Now suppose $F\colon f_0 \simeq f_1$ is homotopy from f_0 to f_1. Let $f_t = F(\cdot, t)$ as usual. Let $q_0\colon Q_0 \to Z$ and $q_1\colon Q_1 \to Z$ denote the two pullback fibrations with corresponding maps $\varphi_0\colon Q_0 \to E$ and $\varphi_1\colon Q_1 \to E$.

$$\begin{array}{ccc} Q_0 & \xrightarrow{\varphi_0} & E \\ {\scriptstyle q_0}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f_0]{} & X \end{array} \qquad\qquad \begin{array}{ccc} Q_1 & \xrightarrow{\varphi_1} & E \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow[f_1]{} & X \end{array}$$

Since p is a fibration, there exists a homotopy $\psi_t\colon Q_0 \to E$ such that $\psi_0 = \varphi_0$ and $p \circ \psi_t = f_t \circ q_0$. Since the right square is a pullback in Top, there exists a map $\theta\colon Q_0 \to Q_1$ such that $\varphi_1 \circ \theta = \psi_1$ and $q_1 \circ \theta = q_0$.

Suppose now that $F'\colon f_0 \simeq f_1$ is another homotopy from f_0 to f_1 such that F and F' define the same morphism in $\Pi(Z, X)$. This means that there exists a homotopy $\Phi\colon Z \times I \times I \to X$ relative to $Z \times \partial I$ from F to F'. By applying the construction above to F', we obtain another map $\theta'\colon Q_0 \to Q_1$. By lifting the homotopy $\Phi \circ (q_0 \times \mathrm{id}_I \times \mathrm{id}_I)$ one can show that there exists a homotopy $\theta_t\colon Q_0 \to Q_1$ such that $q_1 \circ \theta_t = q_0$ for all t.

Denote by hFib_Z the category of all fibrations over Z. An object of this category is a fibration $q\colon Q \to Z$, and a morphism θ from two objects $q_0\colon Q_0 \to Z$ and $q_1\colon Q_1 \to Z$ is a homotopy class $[\theta_t]$ of maps $Q_0 \to Q_1$ such that $q_1 \circ \theta_t = q_0$. The construction above tells us the following.

Theorem 45.9 (The Homotopy Theorem for Fibrations). *Let $p\colon E \to X$ be a fibration. There is a functor $T_p\colon \Pi(Z, X) \to \mathsf{hFib}_Z$ that assigns to a continuous map $f\colon Z \to X$ the fibration $q\colon Q \to Z$ defined above, and to a morphism $[f_t]\colon f_0 \to f_1$ it assigns the morphism $[\theta_t]$ from above.*

Proof. It remains to check that T is a functor. This is left as an exercise. ■

Since $\Pi(Z, X)$ is a groupoid category and T_p is a functor, the class $T_p[f_t]$ is *always an isomorphism*. We obtain the following formal consequence.

Corollary 45.10. *Let $p\colon E \to X$ be a fibration and let $f\colon Z \to X$ be a homotopy equivalence. Then the map $\varphi\colon Q \to E$ from (45.1) is a homotopy equivalence.*

Remark 45.11. Theorem 45.9 is a massive generalisation of Corollary 45.4, which (roughly speaking) corresponds to the case where we take Z to be the unit interval.

Let us now return to less abstract material. There is a slightly weaker notion of a fibration, which is (unimaginatively) called a **weak fibration**. This is a map $p\colon E \to X$ that has the homotopy lifting property with respect to any cell complex W. In fact, having the homotopy lifting property with respect to any cell complex is equivalent to having the homotopy lifting property with respect to all cubes I^n for $n \geq 0$ (where $I^0 = *$). This is the content of the following result.

PROPOSITION 45.12. *Let $p\colon E \to X$ have the homotopy lifting property with respect to any cube I^n. Then p is a weak fibration, that is, it has the homotopy lifting property with respect to any cell complex.*

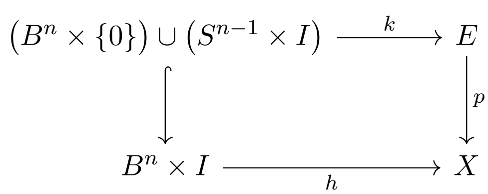
The proof will use the following observation that we will repeatedly make use of throughout the rest of this lecture. There is a homeomorphism of pairs

$$r\colon (I^{n+1}, I^n \times \{0\}) \to \big(B^n \times I, \big(B^n \times \{0\}\big) \cup \big(S^{n-1} \times I\big)\big)$$

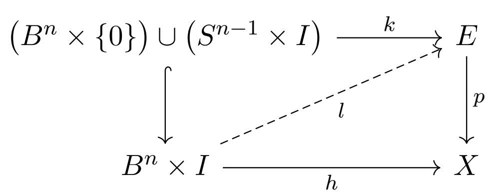
Thus if we are given maps

$$h\colon B^n \times I \to X, \qquad k\colon \big(B^n \times \{0\}\big) \cup \big(S^{n-1} \times I\big) \to E,$$

such that the following diagram commutes:



then the homotopy lifting property for cubes implies we can find a map $l\colon B^n \times I \to E$ that fits along the dotted arrow:



Proof of Proposition 45.12. Let W be a cell complex and let W^n denote the nth skeleton of W. Assume we are given $f_t\colon W \to X$ and a lift g_0 of f_0. Since W carries the colimit topology with respect to its skeleton filtration, it suffices to show that for every $n \geq 0$ there exists a homotopy $g_t^n\colon W^n \to E$ that lifts $f_t^n := f_t|_{X^n}$ and satisfies $g_0^n = g_0|_{X^n}$.

We prove this by induction on n. The case $n = 0$ is clear, since W^0 is a discrete space (a discrete collection of I^0s). For the inductive step, assume we have constructed g_t^{n-1}. Let $a\colon (B^n, S^{n-1}) \to (C \cup W^{n-1}, W^{n-1})$ denote the characteristic map of an n-cell[1] C. We build from a a map

$$k\colon \big(B^n \times \{0\}\big) \cup \big(S^{n-1} \times I\big) \to E$$

[1] I would normally call an n-cell E and the characteristic map g, but alas, both of those letters are taken...

by setting $k = g_0 \circ a$ on $B^n \times \{0\} \cong B^n$ and $k(x,t) = g_t^{n-1}(a(x))$ on $S^{n-1} \times I$ (this is well-defined because they both agree on the overlap $S^{n-1} \times \{0\}$.) This gives us the following commutative diagram:

$$\begin{array}{ccc}
\big(B^n \times \{0\}\big) \cup \big(S^{n-1} \times I\big) & \xrightarrow{\quad k \quad} & E \\
\big\downarrow & \nearrow & \big\downarrow p \\
B^n \times I & \xrightarrow[(x,t)\mapsto f_t(a(x))]{} & X
\end{array}$$

Thus we obtain a lifting over the cell E via the dotted arrow. Doing this for all the n-cells gives us the desired extension $g_t^n \colon W^n \to E$. The fact that g_t^n is continuous is due to the fact that g_t^n is continuous on each n-cell, together with part (4) of Proposition 18.17. ■

A similar argument allows us to prove the following result.

Proposition 45.13. *Let $p \colon E \to X$ be a continuous map. Let $\mathfrak{U}$ be a collection of subsets of X such $\{U^\circ \mid U \in \mathfrak{U}\}$ forms an open cover of X. Assume that $p_U := p|_U \colon p^{-1}(U) \to U$ is a weak fibration for every $U \in \mathfrak{U}$. Then p is a weak fibration.*

Proof. It suffices by the previous result to verify the homotopy lifting property for a cube I^n. Equivalently, it suffices to show that given h, k such that the following diagram commutes, we can always find an l to fit on the dotted line (this is the same observation as was used just before the proof of Proposition 45.12, only with (B^n, S^{n-1}) replaced with $(I^n, \partial I^n)$):

$$\begin{array}{ccc}
\big(I^n \times \{0\}\big) \cup \big(\partial I^n \times I\big) & \xrightarrow{\quad k \quad} & E \\
\big\downarrow & \overset{l}{\nearrow} & \big\downarrow p \\
I^n \times I & \xrightarrow[\quad h \quad]{} & X
\end{array}$$

By the Lebesgue Number Lemma (Lemma 6.7), there exists $N \in \mathbb{N}$ such that if we chop up I^n into N^n smaller cubes $I_1, I_2, \ldots, I_{N^n}$ of equal size, and I into N intervals $J_1, \ldots, J_N$ of equal size, then for every i, j, $h(I_i \times J_j)$ is contained in U° for some $U \in \mathfrak{U}$.

Let K^m denote the union of the m-dimensional faces of all the cubes I_i, for $m = 0, 1, \ldots, n$. Set $K^{-1} = \emptyset$ and $l_{-1} := k$. By induction on m, we claim we can find l_m to fit along the dotted arrow:

$$\begin{array}{ccc}
\big(I^n \times \{0\}\big) \cup \big(K^{m-1} \times J_1\big) & \xrightarrow{\quad l_{m-1} \quad} & E \\
\big\downarrow & \overset{l_m}{\nearrow} & \big\downarrow p \\
\big(I^n \times \{0\}\big) \cup \big(K^m \times J_1\big) & \xrightarrow[\quad h \quad]{} & X
\end{array}$$

Indeed, it suffices to do this one cube at a time. If L is an m-dimensional cube and ∂L the union of its $(m-1)$-dimensional faces, then since p_U is a weak fibration, for an appropriate U we can find l_L to fit along the dotted arrow:

$$\begin{array}{ccc} \big(L \times \{0\}\big) \cup \big(\partial L \times J_1\big) & \xrightarrow{\;l_{m-1}\;} & p^{-1}(U) \\ \big\downarrow & \overset{l_L}{\nearrow} & \big\downarrow p \\ L \times J_1 & \xrightarrow[\;h\;]{} & U \end{array}$$

Now the maps l_L as L ranges over the m-dimensional faces combine together to give us the desired l_m. Then for $m = n$ we obtain an extension of h to a map on $I^n \times J_1$. Now we repeat to extend over $I^n \times J_2$, and continue. Eventually we are done. ■

This gives us a proof of the first half of Theorem 33.17:

Corollary 45.14. *Let $F \to E \xrightarrow{p} X$ be a fibre bundle. Then p is a weak fibration.*

Proof. By Example 45.2, any fibre bundle satisfies the hypotheses of Proposition 45.13. ■

Remark 45.15. A harder version of Proposition 45.13 states that if X is paracompact and p_U is a fibration for every U then so is p. Combining this result with Example 45.2 shows that every fibre bundle over a paracompact base is a fibration (this is the second half of Theorem 33.17.)

We would like to combine Theorem 45.5 and Corollary 45.14 to obtain a long exact sequence for fibre bundles. Unfortunately, since not every fibre bundle is a fibration (as the base might not be paracompact[2]), this does not follow directly. Thus we need another argument. Here is the last result of today's lecture.

Theorem 45.16 (Serre's Theorem). *Let $p \colon E \to X$ be a weak fibration. Let $x_0 \in X$ and set $F := p^{-1}(x_0)$. Let $y_0 \in F$ and let $\imath \colon (E, y_0) \hookrightarrow (E, F)$ denote the inclusion, and let $p' \colon (E, F) \to (X, x_0)$ be as in the picture:*

$$\begin{array}{ccc} & & (E, y_0) \\ & \overset{\imath}{\swarrow} & \big\downarrow p \\ (E, F) & \xrightarrow[\;p'\;]{} & (X, x_0) \end{array}$$

Then $\pi_n(p') \colon \pi_n(E, F) \to \pi_n(X, x_0)$ is a bijection for all $n \geq 1$.

Remark 45.17. For $n \geq 2$, $\pi_n(p')$ is an isomorphism since $\pi_n(p')$ is a homomorphism. However for $n = 1$ it is just a bijection, since $\pi_1(E, F)$ does not have a group structure.

[2] Also we didn't prove fibre bundles over paracompact spaces are fibrations!

Proof of Theorem 45.16. First, since p is a weak fibration, by arguing by induction on n and using that $I^{n+1} = I^n \times I$, we see that given a commutative square we can always fill in the dashed map:

$$\begin{array}{ccc} I^0 & \longrightarrow & E \\ \big\downarrow & \nearrow & \big\downarrow \\ I^n & \longrightarrow & X \end{array}$$

That is, in order to lift a map $I^n \to X$ to E, it suffices to specify the lift of a single point.

We first prove that $\pi_n(p')$ is surjective. Let $u\colon (I^n, \partial I^n) \to (X, x_0)$ represent an element $[u] \in \pi_n(X, x_0)$. Define $g\colon I^0 \to E$ by $g(*) = y_0$. Then by the diagram above, we can find $v\colon I^n \to E$ such that $p \circ v = u$ and $v(0) = y_0$. Since $u(\partial I^n) = \{x_0\}$, we have $v(\partial I^n) \subset F$ and thus v is a map of pairs $(I^n, \partial I^n) \to (E, F)$. Thus $[v] \in \pi_n(E, F)$ and $\pi_n(p')[v] = [u]$. This proves surjectivity.

Now suppose we are given $u\colon (B^n, S^{n-1}) \to (E, F)$ such that the map $\overline{p \circ u}\colon B^n\big/S^{n-1} \to X$ induced by $p \circ u\colon (B^n, S^{n-1}) \to (X, x_0)$ is nullhomotopic. We claim that u itself is nullhomotopic. There is a homotopy of pointed pairs

$$U\colon (B^n \times I, S^{n-1} \times I) \to (X, x_0), \qquad U(z, 0) = p \circ u(z), \qquad U(z, 1) = x_0, \qquad \forall\, z \in B^n.$$

Define $k\colon \big(B^n \times \{0\}\big) \cup \big(S^{n-1} \times I\big) \to E$ by setting $k(z, t) = u(z)$. Then k is well-defined and continuous and the following diagram commutes:

$$\begin{array}{ccc} \big(B^n \times \{0\}\big) \cup \big(S^{n-1} \times I\big) & \xrightarrow{\ k\ } & E \\ \big\downarrow & & \big\downarrow p \\ B^n \times I & \xrightarrow[\ U\ ]{} & X \end{array}$$

The argument we have used many times now gives us a map $V\colon B^n \times I \to E$ such that both triangles commute:

$$\begin{array}{ccc} \big(B^n \times \{0\}\big) \cup \big(S^{n-1} \times I\big) & \xrightarrow{\ k\ } & E \\ \big\downarrow & \overset{V}{\nearrow} & \big\downarrow p \\ B^n \times I & \xrightarrow[\ U\ ]{} & X \end{array}$$

We can view V as a homotopy of pointed pairs from u to a new map $v := V(\cdot, 1)$. The map $v\colon B^n \to F$ has image in F, and the existence of V shows that $[u] = [v]$ is in the image of the map $\pi_n(F, F) \to \pi_n(E, F)$ induced by inclusion. But $\pi_n(F, F) = 0$ (this is true for any space, as I invite you to verify), and hence $[u] = 0$.

If $n \geq 2$ then $\pi_n(p')$ is a homomorphism. We have just shown that $\ker \pi_n(p')$ is trivial, and hence $\pi_n(p')$ is injective, and thus bijective. For $n = 1$ this argument does not work, so let us argue directly. Suppose $[u_0], [u_1] \in \pi_1(E, F)$ are such that $[p \circ u_0] = [p \circ u_1] \in \pi_1(X, x_0)$. The u_i are paths $(I, \partial I) \to (E, F)$ with $u_i(0) = y_0$. If $w := \bar{u}_0 * u_1$ then $p \circ w$ is nullhomotopic, and hence so is w. Since $u_1 = u_0 * w$, it follows $[u_0] = [u_1]$ as required. ■

Corollary 45.18 (Homotopy Sequence of a Weak Fibration). *Let $p\colon E \to X$ be a weak fibration. Choose basepoints $y_0 \in E$ and $x_0 = p(y_0) \in X$. Let $F := p^{-1}(x_0)$. Then there is an exact sequence*

$$\cdots \pi_n(E) \xrightarrow{\pi_n(p)} \pi_n(X) \to \pi_{n-1}(F) \to \pi_{n-1}(E) \to \cdots$$

Proof. We consider the long exact sequence of homotopy groups for the pair (E, F) (Theorem 44.17), and use Serre's Theorem to replace $\pi_n(E, F)$ with $\pi_n(X)$. The map $\pi_n(E) \to \pi_n(X)$ is then the composition $\pi_n(p') \circ \pi_n(\imath)$, where p' and $\imath$ are as in the statement of Serre's Theorem. Since this composition is just $\pi_n(p)$, the result follows. ■

Note that Corollary 45.18 is what we needed to finish the proof of part (1) of Theorem 43.18. Further applications to the homotopy groups of spheres are on Problem Sheet R.

LECTURE 46

Epilogue

In this final lecture I will discuss (without proof) several important topics that unfortunately we ran out of time to cover properly. Everything in this lecture is non-examinable.

We begin by stating three key theorems in homotopy theory. The first is the higher dimensional analogue of Theorem 9.7. If $u\colon (S^n, *) \to (X, x_0)$, let $u'\colon \Delta^n \to X$ denote the same map, precomposed with a homeomorphism of pairs $(\Delta^n, \partial\Delta^n) \to (S^n, *)$. This is the same notational convention we used in Remark 9.1. Arguing as in Proposition 9.2, there is a well-defined map

$$h\colon \pi_n(X, x_0) \to \tilde{H}_n(X), \qquad [u] \mapsto \langle u' \rangle.$$

Moreover as in Proposition 21.7, this is actually a natural transformation $h\colon \pi_n \to \tilde{H}_n$. The Hurewicz Theorem we proved in Lecture 9 required us to abelianise $\pi_1(X, x_0)$ in order to get an isomorphism. For the higher homotopy groups, they are already abelian, so this is unnecessary. On the other hand, the theorem is only true for π_n if all the lower homotopy groups vanish.

THEOREM 46.1 (The Hurewicz Theorem). *Let X be a topological space and let $n \geq 2$. Assume that $\pi_i(X) = 0$ for all $i < n$. Then also $\tilde{H}_i(X) = 0$ for $i < n$ and the Hurewicz map is an isomorphism $\pi_n(X) \to \tilde{H}_n(X)$.*

An immediate corollary of Theorem 46.1 and part (2) of Theorem 43.18 is:

COROLLARY 46.2. *For any $n \geq 1$, $\pi_n(S^n) \cong \mathbb{Z}$.*

Another corollary is the following higher-dimensional version of Proposition 15.5:

COROLLARY 46.3. *Two maps $f, g\colon S^n \to S^n$ are homotopic if and only if they have the same degree.*

It is convenient to give the hypotheses of Theorem 46.1 a name.

DEFINITION 46.4. A topological space X is said to be n-**connected** if $\pi_i(X) = 0$ for all $i \leq n$. Similarly a pair (X, X') is n-**connected** if $\pi_i(X, X') = 0$ for all $i \leq n$.

Thus a sphere is $(n-1)$-connected, and the Hurewicz Theorem in degree n holds for $(n-1)$-connected spaces. We now present a partial version of the excision axiom for homotopy groups.

Theorem 46.5 (The Blakers-Massey Theorem). *Let X_1, X_2 be subspaces of X and assume that $X = X_1^\circ \cup X_2^\circ$. Set $X_0 = X_1 \cap X_2$. Assume that (X_1, X_0) is $(n-1)$-connected and (X_2, X_0) is $(m-1)$-connected. Let $\jmath\colon (X_1, X_0) \hookrightarrow (X, X_2)$ be the inclusion. Then $\pi_i(\jmath)\colon \pi_i(X_1, X_0) \to \pi_i(X, X_2)$ is an isomorphism for $i < m+n-2$ and a surjection for $i = m+n-2$.*

Now let us return to *weak homotopy equivalences*, which were mentioned at the end of Lecture 18.

Definition 46.6. A continuous map $f\colon X \to Y$ is called a **weak homotopy equivalence** if the induced map $\pi_n(f)\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all $n \geq 0$ and all $x \in X$.

A natural question is: when is a weak homotopy equivalence a genuine homotopy equivalence.

Theorem 46.7 (Whitehead's Theorem). *Let X and Y be path connected cell complexes. Then a weak homotopy equivalence $f\colon X \to Y$ is a genuine homotopy equivalence.*

Whitehead's Theorem is *not* saying that two cell complexes all of whose homotopy groups are isomorphic are necessarily homotopy equivalent: indeed, there has to be a map $f\colon X \to Y$ realising the given the isomorphisms. As a challenging example, I encourage you to verify that $\mathbb{R}P^2 \times S^3$ and $\mathbb{R}P^3 \times S^2$ have the isomorphic homotopy groups in every degree, but are *not* homotopy equivalent (thus by Whitehead's Theorem there is no map $\mathbb{R}P^2 \times S^3 \to \mathbb{R}P^3 \times S^2$ that induces an isomorphism on each homotopy group.)

Corollary 46.8. *A cell complex X is contractible if and only if all its homotopy groups vanish.*

Proof. If all the homotopy groups vanish then the constant map $X \to *$ is a weak homotopy equivalence. ■

The next result tells us that weak homotopy equivalences behave nicely with respect to singular homology.

Theorem 46.9. *A weak homotopy equivalence $f\colon X \to Y$ induces an isomorphism*

$$H_n(f)\colon H_n(X; A) \to H_n(Y; A)$$

and

$$H^n(f)\colon H^n(Y; A) \to H^n(X; A)$$

for any $n \geq 0$ and any abelian group A.

Remark 46.10. Theorem 46.9 tells us that singular homology satisfies the weak equivalence axiom in Definition 21.9, and hence we finally see that singular homology really is (!) a homology theory.

As a sample corollary, let us prove the following result, which was used in Problem Sheet O.

Corollary 46.11. *Let $F \to E \xrightarrow{p} X$ be a fibre bundle and assume that the fibre F is contractible. Then for all $n \geq 0$ and any abelian group A, the map $H^n(p)\colon H^n(X;A) \to H^n(E;A)$ is an isomorphism.*

Proof. By Corollary 43.17, Corollary 45.14 and Corollary 45.18, we see that $\pi_n(E) \cong \pi_n(X)$ for all $n \geq 0$, and moreover, this isomorphism is realised by the projection p. Thus p is a weak homotopy equivalence, and thus the claim follows from Theorem 46.9. ■

Now let us define a special class of maps between cell complexes.

Definition 46.12. Suppose X and Y are cell complexes with skeleton filtrations (X^n) and (Y^n) respectively. A continuous map $f\colon X \to Y$ is said to be **cellular** if $f(X^n) \subset Y^n$ for all $n \geq 0$.

In fact, every map can be made cellular.

Theorem 46.13 (The Cellular Approximation Theorem for Maps). *Every continuous map $f\colon X \to Y$ between cell complexes ia homotopic to a cellular map. If f is already cellular on a subcomplex $X' \subset X$ then the homotopy can be chosen to be constant on X'.*

This allows us to finish the proof of part (2) of Theorem 43.18.

Corollary 46.14. *If $k < n$ then any continuous map $f\colon (S^k, x) \to (S^n, y)$ is null-homotopic rel x.*

Proof. Give S^k and S^n their standard cell structure (Example 18.7) consisting of one 0-cell (which we choose to be the two basepoints x and y respectively) and one k-cell (resp. one n-cell). By the Cellular Approximation Theorem we can homotope f rel x to a map that carries S^k into the k-skeleton of S^n. But this is just $\{y\}$. ■

We now state a more precise version of Theorem 18.19.

Theorem 46.15 (The Cellular Approximation Theorem for Spaces). *For any space X, there is a cell complex ΓX and a weak homotopy equivalence $\gamma(X)\colon \Gamma X \to X$. If $f\colon X \to Y$ is a continuous map then there is a continuous map $\Gamma(f)\colon \Gamma X \to \Gamma Y$ which is unique up to homotopy such that the following diagram commutes:*

$$\begin{array}{ccc} \Gamma X & \xrightarrow{\Gamma(f)} & \Gamma Y \\ \gamma\downarrow & & \downarrow\gamma \\ X & \xrightarrow[f]{} & Y \end{array}$$

In other words: there is a **cellular approximation functor** $\Gamma\colon \mathsf{Top} \to \mathsf{Cell}$ *and a natural transformation $\gamma\colon I \circ \Gamma \to \mathrm{id}_{\mathsf{Top}}$, where $I\colon \mathsf{Cell} \to \mathsf{Top}$ is the inclusion functor.*

Remark 46.16. Combining Theorem 46.15 and Theorem 46.9 tells us that from the point of view of (co)homology, we may always assume our spaces are cell complexes. We used this several times during the course, for instance in the proof of Step 5 of the Leray-Hirsch Theorem 34.7.

Finally, let us sketch the proof of the "adult" existence and uniqueness result for homology theories (this was stated previously as Theorem 21.11—we proved the "baby" version in Theorem 21.12.)

Theorem 46.17 (Existence and uniqueness of a homology theory). *Singular homology is a homology theory. Moreover if* $(\mathcal{H}_\bullet, \delta)$ *is any homology theory then* $(\mathcal{H}_\bullet, \delta)$ *is naturally isomorphic to singular homology.*

Proof (Sketch). The first statement was covered by Remark 46.10 above. Now suppose $(\mathcal{H}_\bullet, \delta)$ is any homology theory. The aim is to show that $\mathcal{H}_\bullet$ agrees with cellular homology when we feed it a cell complex. Since a homology theory necessarily vanishes on a contractible space, by considering the exact sequence $S^{n-1} \to B^n \to B^n/S^{n-1} \cong S^n$ and using induction, we see that any homology theory agrees with singular homology on a sphere S^n:

$$\mathcal{H}_i(S^n) \cong H_i(S^n), \qquad \forall\, i \geq 0,\ n \geq 0.$$

We now attempt to copy the proof of Theorem 20.5, replacing singular homology with our given homology theory $\mathcal{H}_\bullet$. One sees that if X is a cell complex, then the corresponding reduced homology theory $\tilde{\mathcal{H}}_\bullet$ has the property that $\tilde{\mathcal{H}}_n(X)$ agrees with the nth homology of the the chain complex

$$\cdots \to \tilde{\mathcal{H}}_n(X^n/X^{n-1}) \to \tilde{\mathcal{H}}_{n-1}(X^{n-1}/X^{n-2}) \to \cdots$$

where each term is free on the n-cells of X. Here we are using the fact that the axioms of a homology theory imply that $\mathcal{H}_\bullet$ commutes with colimits to deal with the case when X is infinite dimensional. This complex agrees with the cellular complex $C_\bullet^{\mathrm{cell}}(X)$, but the two boundary maps could be very different. Thus the key step is to show that the analogue of the Cellular Boundary Formula (Theorem 20.11) holds for $\tilde{\mathcal{H}}_\bullet$. This essentially comes down to showing that any map $f\colon S^n \to S^n$ of degree k induces multiplication by k on $\tilde{\mathcal{H}}_n(S^n)$. By Corollary 46.3 it suffices to construct a single map $f\colon S^n \to S^n$ of degree k for which $\mathcal{H}_n(f)$ is multiplication by k. For this, one considers a composition $S^n \to \bigvee_{i=1}^k S^n \to S^n$ where we factor S^n into the k-fold wedge of S^n. That this map induces multiplication by k is not to hard to see: one argues as in Lemma 20.12, and uses the fact that the additivity axiom implies that $\tilde{\mathcal{H}}_\bullet$ is additive on wedge sums.

Thus our arbitrary homology theory agrees with cellular homology (and hence also singular homology) on a cell complex. The weak equivalence axiom then shows that $\mathcal{H}_\bullet$ agrees with singular homology on any space. This completes the proof. ■

And this completes the course! Thank you everyone for coming.

Problem Sheet A

This Problem Sheet is based on Lecture 1 and Lecture 2. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

Problem A.1. Let X be a topological space. Assume X can be written as an arbitrary union

$$X = \bigcup_i X_i,$$

where each X_i is an open subspace of X. Assume we given a topological space Y and continuous functions

$$f_i \colon X_i \to Y,$$

with the property that

$$f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}, \qquad \forall\, i, j \text{ such that } X_i \cap X_j \neq \emptyset.$$

Then there exists a unique continuous function $f \colon X \to Y$ such that

$$f|_{X_i} = f_i, \qquad \forall\, i \in \mathbb{N}.$$

Problem A.2 (†). Let C and D be categories and $T \colon \mathsf{C} \to \mathsf{D}$ a functor. Suppose f is an isomorphism in C. Prove that $T(f)$ is an isomorphism in D.

Problem A.3 (†). Let C and D be categories. Suppose $\sim$ is a congruence on C and $T \colon \mathsf{C} \to \mathsf{D}$ is a functor. Assume that whenever $f \sim g$ one has $T(f) = T(g)$. Prove that T induces a functor $T' \colon \mathsf{C}' \to \mathsf{D}$, where C' denotes the quotient category.

Problem A.4 (†). Show that a topological space X has the same homotopy type as a point if and only if X is contractible.

Problem A.5. Let X a topological space. Define an equivalence relation on $X \times I$ by $(x, t) \sim (x', t')$ if $t = t' = 1$. Let CX denote the quotient space $(X \times I)/\sim$. We call CX the **cone** on X. Prove that CX is always contractible, and deduce that any topological space can be embedded inside a contractible one.

Solutions to Problem Sheet A

This Problem Sheet is based on Lecture 1 and Lecture 2. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem A.1. Let X be a topological space. Assume X can be written as an arbitrary union

$$X = \bigcup_i X_i,$$

where each X_i is an open subspace of X. Assume we given a topological space Y and continuous functions

$$f_i \colon X_i \to Y,$$

with the property that

$$f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}, \qquad \forall\, i, j \text{ such that } X_i \cap X_j \neq \emptyset.$$

Then there exists a unique continuous function $f \colon X \to Y$ such that

$$f|_{X_i} = f_i, \qquad \forall\, i \in \mathbb{N}.$$

Solution. First we prove the existence. For any $x \in X$ there exists an i such that $x \in X_i$. Set $f(x) = f_i(x)$. Clearly, f is well-defined, since for $j \neq i$ with $x \in X_j$ we have by assumption that $f_i(x) = f_j(x)$. Since x is arbitrary it suffices to prove continuity of f at x. Note that X_i is open and $f(x) = f|_{X_i}(x) = f_i(x)$. Since f_i is continuous at x and $x \in X_i = \operatorname{int}(X_i)$ it follows that also f is continuous at x. Now suppose that g is another such map with the same properties. Then for every $x \in X$ we have $f(x) = f_i(x) = g(x)$. Hence $f = g$ which proves uniqueness.

Problem A.2 (†). Let C and D be categories and $T \colon \mathsf{C} \to \mathsf{D}$ a functor. Suppose f is an isomorphism in C. Prove that $T(f)$ is an isomorphism in D.

Solution. Let A and B be objects of the category C such that f is a morphism between them. By assumption $f \colon A \to B$ is an isomorphism in C, hence there exists a morphism $g \colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Since T is a functor we see that $\mathrm{id}_{T(A)} = T(\mathrm{id}_A) = T(g \circ f) = T(g) \circ T(f)$ and similarly $\mathrm{id}_{T(B)} = T(f) \circ T(g)$. This proves that $T(f)$ is an isomorphism in D.

Problem A.3 (†). Let C and D be categories. Suppose $\sim$ is a congruence on C and $T \colon \mathsf{C} \to \mathsf{D}$ is a functor. Assume that whenever $f \sim g$ one has $T(f) = T(g)$. Prove that T induces a functor $T' \colon \mathsf{C}' \to \mathsf{D}$, where C' denotes the quotient category.

Solution. On objects of the category C the functor T' is equal to T. On morphisms we define $T'([f]) \colon = T(f)$. This is well-defined, indeed for $[f] = [g]$ we have that $T(f) = T(g)$ by assumption. We need to show that T' satisfies the properties of a functor. Clearly $T'([g] \circ [f]) = T'([g \circ f]) = T(g \circ f) = T(g) \circ T(f) = T'([g]) \circ T'([f])$ since T is a functor, and for any object $A \in \mathsf{C}$, $T'([\mathrm{id}_A]) = T(\mathrm{id}_A) = \mathrm{id}_{T(A)} = \mathrm{id}_{T'(A)}$.

Problem A.4 (†). Show that a topological space X has the same homotopy type as a point if and only if X is contractible.

Solution.

"$\Rightarrow$" There exists $f\colon X \to \{*\}$ and $g\colon \{*\} \to X$ continuous such that $g \circ f \simeq \mathrm{id}_X$. But $g \circ f\colon X \to \{*\} \to X$ is necessarily a constant map. Hence id_X is homotopic to a constant map, which proves that X is contractible.

"$\Leftarrow$" Let $c\colon X \to X$ be the constant map sending every point $x \in X$ to a fixed point $q \in X$ and assume $\mathrm{id}_X \simeq c$. Define $f\colon X \to \{*\}$ the constant map and $g\colon \{*\} \to X$ the constant map sending $*$ to q. Clearly $g \circ f = c \simeq \mathrm{id}_X$ and $f \circ g = \mathrm{id}_{\{*\}}$. This shows that X has the homotopy type of a point.

Problem A.5. Let X a topological space. Define an equivalence relation on $X \times I$ by $(x,t) \sim (x',t')$ if $t = t' = 1$. Let CX denote the quotient space $(X \times I)/\sim$. We call CX the **cone** on X. Prove that CX is always contractible, and deduce that any topological space can be embedded inside a contractible one.

Solution. Let $c\colon CX \to CX$ denote the constant map sending every point to the equivalence class $[x,1]$. (Note that $[x,1] = [x',1]$ for any two points x and x' in X.) We define the homotopy $H\colon CX \times [0,1] \to CX$ by $H([x,t],s) := [x, s + (1-s)t]$. One can see that H is a homotopy between id_{CX} and c, which proves that CX is contractible. Moreover, every topological space X can be embedded into the contractible space CX via the map $i\colon X \to CX$ given by $x \mapsto [x,0]$.

Problem Sheet B

This Problem Sheet is based on Lecture 3 and Lecture 4. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem B.1 (†). Let X and Y be topological spaces. Prove that any two maps $X \to Y$ are homotopic (in fact, nullhomotopic) if **either**

- Y is contractible *or*
- X is contractible and Y is path connected.

Problem B.2 (†). Let X, Y and Z be topological spaces. Suppose $A \subset X$ and $B \subset Y$. Assume we are given two continuous maps $f_0, f_1 \colon X \to Y$ such that $f_0|_A = f_1|_A$ and such that $f_i(A) \subset B$ for $i = 0, 1$, and also two continuous maps $g_0, g_1 \colon Y \to Z$ such that $g_0|_B = g_1|B$. Assume that $f_0 \simeq f_1$ rel A and $g_0 \simeq g_1$ rel B. Prove that $g_0 \circ f_0 \simeq g_1 \circ f_1$ rel A.

Problem B.3 (†). If (X, p) and (Y, q) are pointed spaces, prove that

$$\pi_1(X \times Y, (p, q)) \cong \pi_1(X, p) \times \pi_1(Y, q).$$

Problem B.4. If $f \colon X \to Y$ is freely nullhomotopic, prove that for any $p \in X$ the group homomorphism $\pi_1(f) \colon \pi_1(X, p) \to \pi_1(Y, f(p))$ is trivial[1].

Problem B.5 (†). Take $(1, 0) \in \mathbb{R}^2$ as the basepoint of S^1. Prove that for any pointed topological space, there is an isomorphism of groups:

$$\pi_1(X, p) \cong [(S^1, 1), (X, p)],$$

where $[(S^1, 1), (X, p)]$ denotes the morphism space in hTop_* (cf. Remark 4.8.)

[1]Recall a group homomorphism $\phi \colon G \to H$ between two groups G and H is called *trivial* if $\phi(g) = 1$ for all $g \in G$, where 1 is the identity element in H.

Solutions to Problem Sheet B

This Problem Sheet is based on Lecture 3 and Lecture 4. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem B.1 (†). Let X and Y be topological spaces. Prove that any two maps $X \to Y$ are homotopic (in fact, nullhomotopic) if **either**

- Y is contractible *or*
- X is contractible and Y is path connected.

Solution. First suppose that Y is contractible. Let $F\colon Y \times I \to Y$ be a homotopy between id_Y and the constant map $c(y) \equiv p$ for a fixed point $p \in Y$. I.e. $F(\cdot, 0) = \mathrm{id}_Y$ and $F(\cdot, 1) = c$. Let $f\colon X \to Y$ be a continuous map. We want to show that f is homotopic to the constant map $\tilde{c}\colon X \to Y$ with $\tilde{c}(x) \equiv p$. We define the homotopy

$$\begin{array}{rcl} H\colon X \times I & \to & Y \\ (x,t) & \mapsto & F(f(x),t). \end{array}$$

Clearly, $H(x,0) = F(f(x),0) = \mathrm{id}_Y(f(x)) = f(x)$ and $H(x,1) = F(f(x),1) = c(f(x)) = \tilde{c}(x)$. Hence, f is homotopic to a constant map and hence null homotopic.

Now assume that X is contractible and Y is path connected. Since X is contractible there exists a homotopy

$$F\colon X \times I \to X,$$

such that $F(\cdot, 0) = \mathrm{id}_X$ and $F(\cdot, 1) = c$ with $c(x) \equiv q$ for every $x \in X$ and for a fixed $q \in Y$. Let $f\colon X \to Y$ be a continuous map. We want to show that f is nullhomotopic. We define

$$\begin{array}{rcl} H\colon X \times I & \to & Y \\ (x,t) & \mapsto & f(F(x,t)). \end{array}$$

Then we have $H(x,0) = f(F(x,0)) = f(x)$ and $H(x,1) = f(F(x,1)) \equiv f(q)$ a constant map. Note that two constant maps are homotopic if their image (a point) lies in the same path component. Indeed, let $c_0\colon X \to Y$ and $c_1\colon X \to Y$ be two constant maps with $c_i(x) \equiv q_i$ for $i = 1,2$. Let $\gamma\colon I \to Y$ be a path in Y such that $\gamma(0) = q_0$ and $\gamma(1) = q_1$. Then the map $H(x,t) := \gamma(t)$ defines a homotopy between c_0 and c_1. (In fact, two constant maps are homotopic if and only if their image lies in the same path component.)

Since Y is path connected any two constant maps are homotopic. Moreover, since homotopy is an equivalence relation, it follows that any two null homotopic maps are also homotopic.

Will J. Merry and Berit Singer. Last modified: Sept 01, 2018.

Problem B.2 (†). Let X, Y and Z be topological spaces. Suppose $A \subset X$ and $B \subset Y$. Assume we are given two continuous maps $f_0, f_1 \colon X \to Y$ such that $f_0|_A = f_1|_A$ and such that $f_i(A) \subset B$ for $i = 0, 1$, and also two continuous maps $g_0, g_1 \colon Y \to Z$ such that $g_0|_B = g_1|B$. Assume that $f_0 \simeq f_1$ rel A and $g_0 \simeq g_1$ rel B. Prove that $g_0 \circ f_0 \simeq g_1 \circ f_1$ rel A.

Solution. Let $F \colon X \times I \to Y$ be a homotopy rel A between f_0 and f_1. Let $G \colon Y \times I \to Z$ be a homotopy rel B between g_0 and g_1. Then the map

$$\begin{array}{rcl} H \colon X \times I & \to & Z \\ (x,t) & \mapsto & G(F(x,t),t) \end{array}$$

is a homotopy between $g_0 \circ f_0$ and $g_1 \circ f_1$. Moreover, for a point $a \in A$ we have by assumption that $F(a,t) = f_0(a) = f_1(a) \in B$ for and for every $b \in B$ we have that $G(b,t) = g_0(b) = g_1(b)$. Hence $H(a,t) = G(F(a,t),t) = g_0(f_0(a)) = g_1(f_1(a))$.

Problem B.3 (†). If (X, p) and (Y, q) are pointed spaces, prove that

$$\pi_1(X \times Y, (p,q)) \cong \pi_1(X,p) \times \pi_1(Y,q).$$

Solution. Define $\phi \colon \pi_1(X,p) \times \pi_1(Y,q) \to \pi_1(X \times Y, (p,q))$ as follows. Let u and v be representatives of the classes $[u] \in \pi_1(X,p)$ and $[v] \in \pi_1(Y,q)$. Put $\phi([u],[v]) := [w]$, where

$$\begin{array}{rcl} w \colon I & \to & X \times Y \\ t & \mapsto & (u(t), v(t)). \end{array}$$

Now we define a map $\psi \colon \pi_1(X \times Y, (p,q)) \to \pi_1(X,p) \times \pi_1(Y,q)$. Let $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ denote the two obvious projections. For a map $w \colon I \to X \times Y$ representing a class $[w] \in \pi_1(X \times Y, (p,q))$ we define $\psi([w]) := ([p_1 \circ w], [p_2 \circ w]) \in \pi_1(X,p) \times \pi_1(Y,q)$. Clearly $\phi \circ \psi = \mathrm{id}_{\pi_1(X \times Y, (p,q))}$ and $\psi \circ \phi = \mathrm{id}_{\pi_1(X,p) \times \pi_1(Y,q)}$.

Problem B.4. If $f \colon X \to Y$ is freely nullhomotopic, prove that for any $p \in X$ the group homomorphism $\pi_1(f) \colon \pi_1(X,p) \to \pi_1(Y, f(p))$ is trivial[1].

Solution. Here are two different proofs:

1. First consider the special case where f is actually the constant map $c(x) = q$. Then the result is obvious: the induced homomorphism $\pi_1(c) : \pi_1(X,p) \to \pi_1(Y,q)$ is trivial, since if u is any loop based at p then $c \circ u$ is the constant loop e_q.
   Now for the general case: suppose $F \colon c \simeq f$ where c is the constant map $c(x) = q$. Fix $p \in X$, and let $w(s) := F(p,s)$, so that w is a path from q to $f(p)$. Then by Proposition 4.13, we can write

   $$\pi_1(f) = \lambda_w \circ \pi_1(c)$$

   as maps $\pi_1(X,p) \to \pi_1(Y, f(p))$. Thus for any $[u] \in \pi_1(X,p)$,

   $$\pi_1(f)[u] = \lambda_w(\pi_1(c)[u]) = \lambda_w[q] = [f(p)],$$

   and hence $\pi_1(f)$ is trivial.

[1]Recall a group homomorphism $\phi \colon G \to H$ between two groups G and H is called *trivial* if $\phi(g) = 1$ for all $g \in G$, where 1 is the identity element in H.

2. Now for a different approach. Let $F\colon X \times I \to Y$ be a free homotopy between f and the constant map $c(x) \equiv q$. For a class $[u] \in \pi_1(X,p)$ the map

$$\begin{aligned} G : S^1 \times I &\to Y \\ (s,t) &\mapsto F(u(s),t) \end{aligned}$$

defines a free homotopy between $f \circ u$ and $c \circ u$. In other words, the map $f \circ u : S^1 \to Y$ is freely nullhomotopic. Hence, by Proposition 2.15 the map $f \circ u$ extends to a map $g : D^2 \to Y$, where D^2 is the disk with boundary S^1, so have $g|_{S^1} = f \circ u$. Because D^2 is contractible, we know that there exists a homotopy relative 1

$$K\colon D^2 \times I \to D^2$$

such that $K(x,0) = \mathrm{id}_{D^2}$, $K(x,1) \equiv 1\,\forall x$ and $K(1,t) = 1\,\forall t$. This gives rise to a homotopy relative $f(p)$ between the loops $f \circ u$ and the constant loop $f(p)$. Indeed, define

$$\begin{array}{rcc} H : S^1 \times I & \to & Y \\ (s,t) & \mapsto & g(K|_{S^1}(s,t)). \end{array}$$

Therefor for every $[u] \in \pi_1(X,p)$ we have $\pi_1(f)[u] = [f \circ u] \cong [f(p)]$, which is the identity in $\pi_1(Y, f(p))$.

Problem B.5 (†). Take $(1,0) \in \mathbb{R}^2$ as the basepoint of S^1. Prove that for any pointed topological space, there is an isomorphism of groups:

$$\pi_1(X,p) \cong [(S^1,1),(X,p)],$$

where $[(S^1,1),(X,p)]$ denotes the morphism space in hTop_* (cf. Remark 4.8.)

Solution. Consider the continuous map $\omega\colon I \to S^1$ defined by $\omega(s) := \exp|_I$, where exp was defined in Definition 5.1. If u is a loop in X, then $\hat{u} := u \circ \omega^{-1}$ is a well-defined continuous map $\hat{u}\colon S^1 \to X$. Moreover if u and v are loops with $u \simeq v$ rel ∂I via a homotopy $u_t(s)$ then $\hat{u} \simeq \hat{v}$ rel 1 via the homotopy $\hat{u}_t(x) = u_t \circ \omega^{-1}(x)$. Similarly if u_0, u_1, v_0, v_1 are four loops such that $u_0(0) = v_0(0)$, $u_0 \simeq u_1$ rel ∂I and $v_0 \simeq v_1$ rel ∂I, then $(u_0 * v_0) \circ \omega^{-1} \simeq (u_1 * v_1) \circ \omega^{-1}$ rel 1.

This means that there is a well defined function $\pi_1(X,p) \to [(S^1,1),(X,p)]$ that sends $[u]$ to $[u \circ \omega^{-1}]$. This function is in fact a bijection, as the inverse is given by $[\hat{u}] \mapsto [\hat{u} \circ \omega]$. Moreover, we can use this bijection to give $[(S^1,1),(X,p)]$ a group structure: given loops u, v set $w = u * v$ and *define*

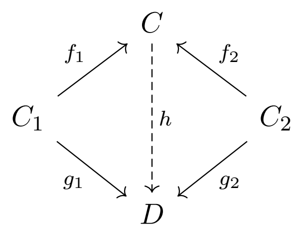
$$[\hat{u}] * [\hat{v}] := [\hat{w}].$$

With this definition the RHS has a group structure and our bijection is in fact an isomorphism of groups.

Problem Sheet C

This Problem Sheet is based on Lecture 5 and Lecture 6. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem C.1. Let C be a category, and let C_1, C_2 be two objects in C. The **coproduct** of C_1 and C_2 is triple (C, f_1, f_2), where C is another object in C, and $f_1 \colon C_1 \to C$ and $f_2 \colon C_2 \to C$ are morphisms which satisfy the following universal property: if D is any other object in C and $g_1 \colon C_1 \to D$ and $g_2 \colon C_2 \to D$ are any two morphisms, then there exists a unique morphism $h \colon C \to D$ such that the following diagram commutes:



1. Prove that if the coproduct exists then it is unique up to isomorphism.
2. Prove that the coproduct exists in Groups. *Hint:* Show that the free product $G * H$ is the coproduct of G and H.

Problem C.2. Given two pointed spaces (X, p) and (Y, q), we define their **wedge product** $X \vee Y$ as the subset of $X \times Y$:

$$X \vee Y := \{(x, y) \in X \times Y \mid x = p \text{ or } y = q\}.$$

We can view $X \vee Y$ as a pointed space with basepoint (p, q). Inductively, one can also define the k-fold wedge $\bigvee_{i=1}^{k} X_i$ of k pointed spaces (X_i, p_i). Compute the fundamental group of the k-fold wedge[1] of the circle S^1.

Problem C.3 (†). Consider the square $I \times I$ and identify the edges as indicated in Figure C.1. This gives us three different topological spaces: the **torus** T^2, the **real projective plane** $\mathbb{R}P^2$ and the **Klein bottle**. Compute the three fundamental groups.

Problem C.4 (⋆). Consider the complex plane $\mathbb{C}$. Given $R > 0$, let Σ_R denote the set

$$\Sigma_R := \{z \in \mathbb{C} \mid |z| = R\}.$$

1. Show that Σ_R has the same homotopy type as $\mathbb{C} \setminus \{0\}$ for each $R > 0$.
2. Let $P_R^n \colon \Sigma_R \to \mathbb{C} \setminus \{0\}$ denote the restriction to Σ_R of the map $z \mapsto z^n$. Show that P_R^n is never freely nullhomotopic.

[1] This space is often called a “bouquet of circles”.

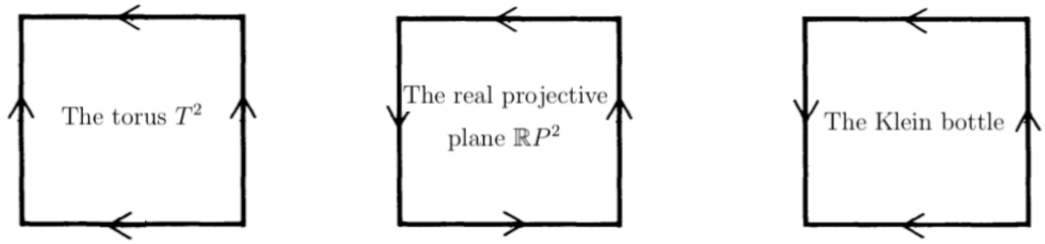


Figure C.1: Three ways to identify edges of $I \times I$.

3. Deduce the *Fundamental Theorem of Algebra*: that every nonconstant complex polynomial has a complex root.

PROBLEM C.5 ($\star$). Let (X, p) be a pointed space. Assume there is a pointed continuous map $m\colon (X \times X, (p, p)) \to (X, p)$ with the property that the pointed maps

$$m_1\colon (X, p) \to (X, p), \qquad m_1(x) := m(x, p)$$

and

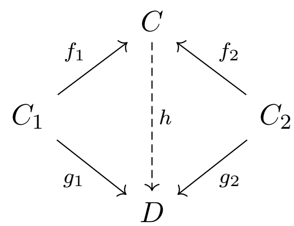
$$m_2\colon (X, p) \to (X, p), \qquad m_2(x) := m(p, x)$$

are both homotopic to id_X rel p. Prove that $\pi_1(X, p)$ is abelian.

Solutions to Problem Sheet C

This Problem Sheet is based on Lecture 5 and Lecture 6. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

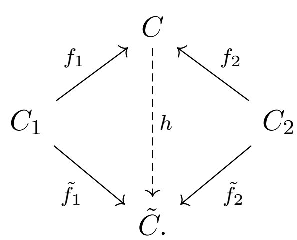
Problem C.1. Let C be a category, and let C_1, C_2 be two objects in C. The **coproduct** of C_1 and C_2 is triple (C, f_1, f_2), where C is another object in C, and $f_1 \in \mathrm{Hom}(C_1, C)$ and $f_2 \in \mathrm{Hom}(C_2, C)$, which satisfies the following universal property: if D is any other object in C and $g_1 \in \mathrm{Hom}(C_1, D)$ and $g_2 \in \mathrm{Hom}(C_2, D)$ are any two morphisms, then there exists a unique morphism $h \in \mathrm{Hom}(C, D)$ such that the following diagram commutes:



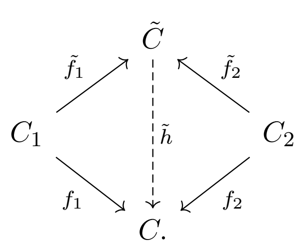
1. Prove that if the coproduct exists then it is unique up to isomorphism.
2. Prove that the coproduct exists in Groups. *Hint:* Show that the free product $G * H$ is the coproduct of G and H.

Solution.

1. Suppose (C, f_1, f_2) and $(\tilde{C}, \tilde{f}_1, \tilde{f}_2)$ are two coproducts. By the universal property of (C, f_1, f_2) there exists a unique $h : C \to \tilde{C}$ such that the following diagram commutes.



Similarly there exists a unique $\tilde{h}$ such that the next diagram also commutes.



Will J. Merry and Berit Singer.
Last modified: Sept 01, 2018.

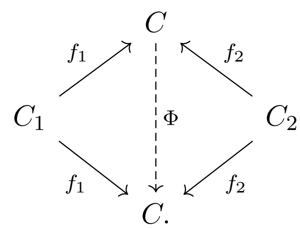
We need to show that $h \circ \tilde{h} = \mathrm{id}_{\tilde{C}}$ and $\tilde{h} \circ h = \mathrm{id}_C$. Notice that

$$\begin{aligned} \tilde{h} \circ h \circ f_1 &= \tilde{h} \circ \tilde{f}_1 &= f_1 \\ \tilde{h} \circ h \circ f_2 &= \tilde{h} \circ \tilde{f}_2 &= f_2 \end{aligned}$$

and

$$\begin{aligned} h \circ \tilde{h} \circ \tilde{f}_1 &= h \circ f_1 &= \tilde{f}_1 \\ h \circ \tilde{h} \circ \tilde{f}_2 &= h \circ f_2 &= \tilde{f}_2. \end{aligned}$$

Consider the diagram



Again by the universal property of the coproduct (C, f_1, f_2) there exists a unique ϕ such that the above diagram commutes. The identity id_C fulfils this property and hence by the uniqueness of the map ϕ we have that $\phi = \mathrm{id}_C$. But also $\tilde{h} \circ h$ fulfils this property. Hence $\mathrm{id}_C = \phi = \tilde{h} \circ h$. It follows from a similar argument that $\mathrm{id}_{\tilde{C}} = h \circ \tilde{h}$.

2. Let $G * H$ denote the free product of G and H. Let $i_1 : G \to G * H$ and $i_2 : H \to G * H$ be the inclusions. We show that $(G * H, i_1, i_2)$ is the coproduct of G abd H. Given a third group K and two morphisms $\phi_1 : G \to K$ and $\phi_2 : H \to K$. Then we define the map

$$\begin{aligned} \eta \colon G * H &\to K \\ g_1 h_1 \cdots g_r h_r &\mapsto \phi_1(g_1)\phi_2(h_2) \cdots \phi_1(g_r)\phi_2(h_r). \end{aligned}$$

We can see that $\eta \circ i_1 = \phi_1$ and $\eta \circ i_2 = \phi_2$. The uniqueness of η follows from the fact that η has to be a group homomorphism and thus must satisfy $\eta(g_1 h_1 \cdots g_r h_r) = \eta(g_1)\eta(h_1) \cdots \eta(g_r)\eta(h_r) = \phi_1(g_1)\phi_2(h_2) \cdots \phi_1(g_r)\phi_2(h_r)$.

PROBLEM C.2. Given two pointed spaces (X, p) and (Y, q), we define their **wedge product** $X \vee Y$ as the subset of $X \times Y$:

$$X \vee Y := \{(x, y) \in X \times Y \mid x = p \text{ or } y = q\}.$$

We can view $X \vee Y$ as a pointed space with basepoint (p, q). Inductively, one can also define the k-fold wedge $\bigvee_{i=1}^k X_i$ of k pointed spaces (X_i, p_i). Compute the fundamental group of the k-fold wedge[1] of the circle S^1.

SOLUTION. We use the Seifert-van Kampen theorem to compute the fundamental group. First we note that this theorem can be applied recursively and that the free product with amalgamation is associative in the sense that $(A * B) * C = A * (B * C)$ for groups A, B and C. Hence the Seifert-van Kampen theorem can be generalised

[1]This space is often called a "bouquet of circles"!

to the case of a topological space $X = X_1 \cup \ldots \cup X_k$, where the X_i are open subsets and $X_1 \cap \ldots \cap X_k$ is non-empty and path connected. We assume that the base point of the k-fold wedge product of S^1 is the point $(1, \ldots, 1)$. Then $\bigvee_{i=1}^k S^1$ looks like the union of k copies of S^1 with one point in common. The common point is $(1, \ldots, 1)$. Let U_ϵ be a small neighbourhood of 1 inside the circle S^1, such that U_ϵ is contractible.

For $i = 1, \ldots, k$ let X_i be the set $U_\epsilon \vee \ldots U_\epsilon \vee S^1 \vee U_\epsilon \vee \ldots \vee U_\epsilon$, where the S^1 is at the ith position. The sets X_i are open and $X_1 \cap \ldots \cap X_k$ is non-empty and path connected. Moreover $X_1 \cap \ldots \cap X_k = U_\epsilon \vee \ldots \vee U_\epsilon$ it is contractible and hence simply connected. We know from the lecture that $\pi_1(S^1, 1) \cong \mathbb{Z}$. Thus is follows from Corollary 6.6 that

$$\pi_1(\bigvee_{i=1}^k S^1, (1, \ldots, 1)) \cong \mathbb{Z}^{*k},$$

where $\mathbb{Z}^{*k}$ denotes the k-fold free product of $\mathbb{Z}$.

PROBLEM C.3 (†). Consider the square $I \times I$ and identify the edges as indicated in Figure C.1. This gives us three different topological spaces: the **torus** T^2, the **real projective plane** $\mathbb{R}P^2$ and the **Klein bottle**. Use the Seifert-van Kampen theorem

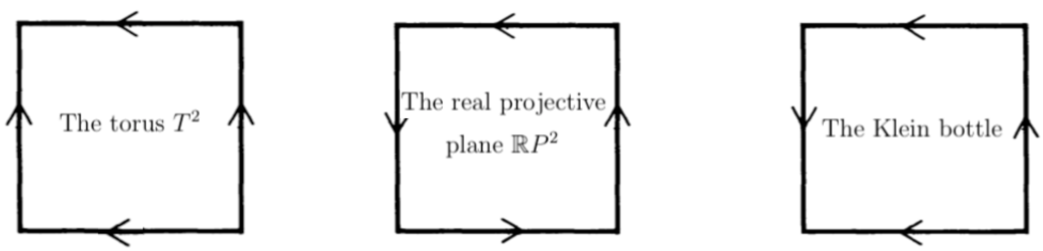


Figure C.1: Three ways to identify edges of $I \times I$.

to compute the three fundamental groups.

SOLUTION.

1. We choose the first open subset X_1 of the torus to be an open ball in the interior of the square. The second open subset X_2 is chosen to be a open subset containing the complement of X_1. (See figure C.2.) The set X_1 is contractible and thus simply connected. Moreover, X_2 is homotopy equivalent to the boundary of the rectangle, where the edges are identified as indicated in the picture. Thus X_2 has the homotopy type of $S^1 \vee S^1$. (See Figure C.2.) Let a and b denote the two generators of $\pi_1(X_2, p) \cong \pi_1(S^1 \vee S^1, p) \cong \mathbb{Z} * \mathbb{Z}$. Cleary, $X_1 \cap X_2$ has the homotopy type of S^1. Note that $\pi_1(\imath_1)\big(\pi_1(X_1 \cap X_2), p)\big) \subset \pi_1(X_1, p)$ is trivial, since $\pi_1(X_1, p) = 1$. We need to compute the subgroup $\pi_1(\imath_2)\big(\pi_1(X_1 \cap X_2), p)\big) \subset \pi_1(X_2, p)$. The map $\pi_1(\imath_2)$ sends the generator of $\pi_1(X_1 \cap X_2, p)$ to the loop corresponding to the word $aba^{-1}b^{-1}$. Therefore $\pi_1(X, p)$ is the quotient of the free product of $\pi_1(X_1, p)$ and $\pi_1(X_2, p)$ by the normal subgroup generated by elements of the form $aba^{-1}b^{-1} = 1$. This means

that a and b commute. By Corollary 6.6 we conclude that

$$\pi_1(T^2, p) \cong \pi_1(X_2, p)/\pi_1(\imath_2)\big(\pi_1(X_1 \cap X_2), p)\big) \cong \langle a, b | ab = ba \rangle \cong \mathbb{Z} \oplus \mathbb{Z}.$$

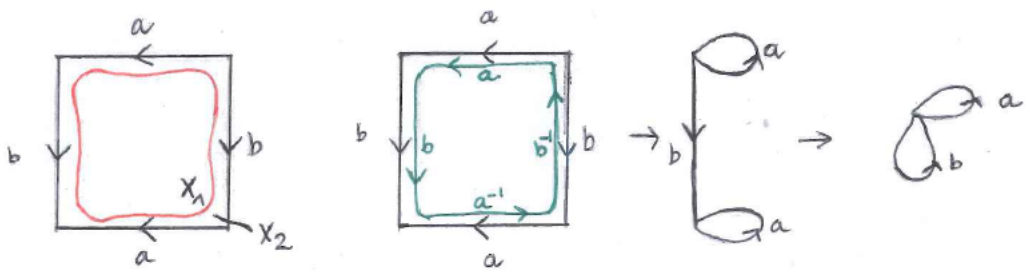


Figure C.2: The torus T^2

2. As before, let X_1 be an open ball in the interior of the square and X_2 a small open neighbourhood of the complement of X_1. (See Figure C.3.) Then X_1 is again simply connected and X_2 has the homotopy type of a circle S^1. Note that $c := ab$ is a generator for $\pi_1(X_2, p)$. The map $\pi_1(\imath_2)$ sends the generator of $\pi_1(X_1 \cap X_2, p)$ to the loop $abab$ in $\pi_1(X_2, p)$. Moreover $\pi_1(\imath_1)(\pi_1(X_1 \cap X_2, p) = 1$ since X_1 is simply connected. Therefore

$$\pi_1(\mathbb{R}P^2, p) \cong \pi_1(X_2, p)/\pi_1(\imath_2)\big(\pi_1(X_1 \cap X_2), p)\big) \cong \langle c | cc = 1 \rangle \cong \mathbb{Z}_2.$$

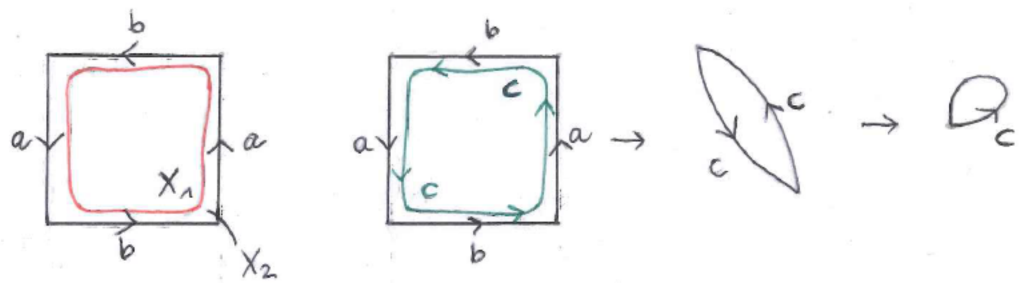


Figure C.3: The projective plane $\mathbb{R}P^2$

3. Let K denote the Klein bottle. Let X_1 be an open ball in the interior of the square and X_2 a small open neighbourhood of the complement of X_1. (See Figure C.4.) The set X_1 is simply connected and X_2 has the homotopy type of $S^1 \vee S^1$. Let a and b denote the edges of the square. Then a and b are generators for $\pi_1(X_2, p) \cong \pi_1(S^1 \vee S^1, p)$. The loop corresponding to the word $aba^{-1}b$ is contractible in X_1. Therefore we have

$$\pi_1(X_2, p)/\pi_1(\imath_2)\big(\pi_1(X_1 \cap X_2), p)\big) \cong \langle a, b | aba^{-1}b = 1 \rangle.$$

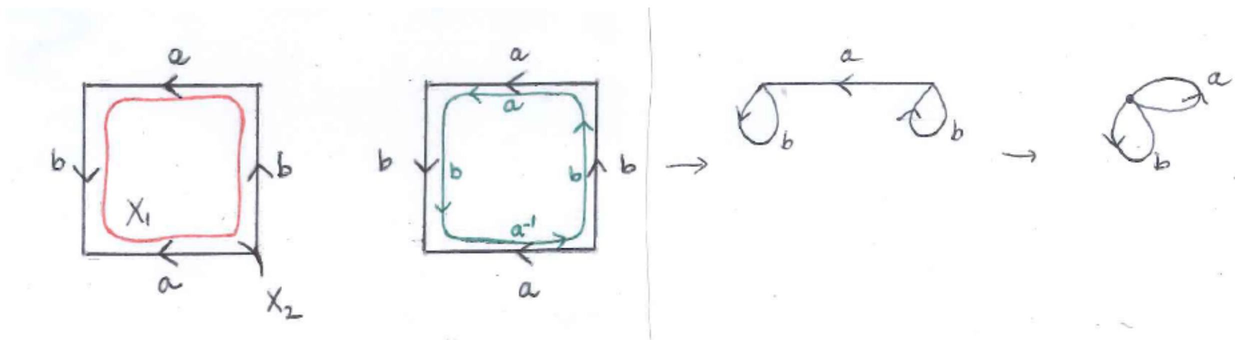


Figure C.4: The Klein bottle

4.

PROBLEM C.4 (⋆). Consider the complex plane $\mathbb{C}$. Given $R > 0$, let Σ_R denote the set

$$\Sigma_R := \{z \in \mathbb{C} \mid |z| = R\}.$$

1. Show that Σ_R has the same homotopy type as $\mathbb{C} \setminus \{0\}$ for each $R > 0$.
2. Let $P_R^n \colon \Sigma_R \to \mathbb{C} \setminus \{0\}$ denote the restriction to Σ_R of the map $z \mapsto z^n$. Show that P_R^n is never freely nullhomotopic for $n > 0$.
3. Deduce the *Fundamental Theorem of Algebra*: that every nonconstant complex polynomial has a complex root.

SOLUTION.

1. We define the map

$$\begin{array}{rcl} r_R : \mathbb{C} \setminus \{0\} & \to & \Sigma_R \\ z & \mapsto & \frac{z}{|z|} R \end{array}$$

and the inclusion $i : \Sigma_R \hookrightarrow \mathbb{C} \setminus \{0\}$. Then $r_R \circ i = \mathrm{id}_{\Sigma_R}$. Moreover, the map

$$\begin{array}{rcl} H : \mathbb{C} \setminus \{0\} \times I & \to & \mathbb{C} \setminus \{0\} \\ (z, t) & \mapsto & z(1-t) + \frac{z}{|z|} Rt \end{array}$$

is a homotopy between $i \circ r_R = H(\cdot, 1)$ and $H(\cdot, 0) = \mathrm{id}_{\mathbb{C} \setminus \{0\}}$.

2. Suppose by contradiction that $P_R^n \colon \Sigma_R \to \mathbb{C} \setminus \{0\}$ is freely nullhomotopic. Let $F : P_R^n \simeq c$ be a homotopy to the constant map $c(z) \equiv p$. Then P_R^n extends to the map

$$\begin{array}{rcl} f : D_R & \to & \mathbb{C} \setminus \{0\} \\ z & \mapsto & \begin{cases} p, & 0 \le |x| \le \frac{R}{2}, \\ F(\frac{z}{|z|} R, 2 - \frac{2}{R}|z|), & \frac{R}{2} \le |x| \le R, \end{cases} \end{array}$$

where D_R is the disk with radius R and centre 0. The set D_R is contractible and f is continuous. For simplicity we assume w.l.o.g. that $R = 1$, so that D_R is the unit ball B^2. We have group homomorphisms

$$\pi_1(B^2, 1) \xrightarrow{\pi_1(f)} \pi_1(\mathbb{C} \setminus \{0\}, 1) \xrightarrow{\pi_1(r_1)} \pi_1(S^1, 1).$$

But $\pi_1(B^2, 1) = \{1\}$ and $\pi_1(r_1) \circ \pi_1(f)$ maps the contractible loop $t \to e^{it2\pi}$ in B^2 to n times the generator in $\pi_1(S^1, 1)$. This is a contradiction.

3. Consider the polynomial with complex coefficients:

$$g(z) = z^n + a_{n-1}z^{n-1} + \dots a_1 z + a_0.$$

First we show that $g|_{\Sigma_R}$ is homotopic in $\mathbb{C} \setminus \{0\}$ to the polyominal $P_R^n|_{\Sigma_R}$. Choose $R > \max\{1, \sum_{i=0}^{n-1} |a_i|\}$. Define the map

$$\begin{array}{rcc} H : \Sigma_R \times I & \to & \mathbb{C} \\ (z,t) & \mapsto & z^n + \sum_{i=0}^{n-1}(1-t)a_i z^i. \end{array}$$

If we can show that $H(z,t)$ is never zero then the image of H lies in $\mathbb{C}\setminus\{0\}$ and H is a homotopy between $g(z)|_{\Sigma_R}$ and $P_R^n(z)$ in $\mathbb{C} \setminus \{0\}$. Suppose by contradiction that there exists a $t \in I$ and a z with $|z| = R$ such that $H(z,t) = 0$. Then we have $z^n = \sum_{i=0}^{n-1}(1-t)a_i z^i$. The triangle inequality gives

$$R^n \le \sum_{i=0}^{n-1}(1-t)|a_i|R^i \le \sum_{i=0}^{n-1}|a_i|R^i \le \left(\sum_{i=0}^{n-1}|a_i|\right) R^{n-1}.$$

If $R > 1$ we have $R \le \sum_{i=0}^{n-1} |a_i|$, which contradicts the choice of R.

Assume now that g has no complex roots. Define $G : \Sigma_R \times I \to \mathbb{C} \setminus \{0\}$ by $G(z,t) = g((1-t)z)$. (Since g has no root the values of G do lie in $\mathbb{C} \setminus \{0\}$.) Visibly, G is a homotopy of $g|_{\Sigma_R}$ in $\mathbb{C}\setminus\{0\}$ to the constant map a_0. Therefore g is nullhomotopic and, by transitivity, P_R^n is nullhomotopic, contradicting part (2).

Problem C.5 ($\star$). Let (X,p) be a pointed space. Assume there is a pointed map $m\colon (X \times X, (p,p)) \to (X,p)$ with the property that the pointed maps

$$m_1\colon (X,p) \to (X,p), \qquad m_1(x) := m(x,p)$$

and

$$m_2\colon (X,p) \to (X,p), \qquad m_2(x) := m(p,x)$$

are both homotopic to id_X rel p. Prove that $\pi_1(X,p)$ is abelian.

Solution. In Problem B.3, we proved that $\Phi : \pi_1(X,p) \times \pi_1(X,p) \to \pi_1(X \times X, (p,p))$, defined by $([u],[v]) \to [(u,v)]$, is an isomorphism, where (u,v) is the path in $X \times X$ given by $s \mapsto (u(s), v(s))$. Choose $[u],[v] \in \pi_1(X,p)$. Let $c : X \to X$ be the constant path $c(x) \equiv p$. Note that $m_1 = m \circ (\mathrm{id}_X, c)$ and $m_2 = m \circ (c, \mathrm{id}_X)$. Now

$$\begin{array}{rcll} [v] & = & \pi_1(\mathrm{id}_X)[v] & \\ & = & \pi_1(m_2)[v], & \text{as } m_2 \simeq \mathrm{id}_X \\ & = & \pi_1(m \circ (c, \mathrm{id}_X))[v] & \\ & = & \pi_1(m)([c \circ v, v]) & \\ & = & \pi_1(m) \circ \Phi([c \circ v],[v]), & \\ & = & \pi_1(m) \circ \Phi([p],[v]), & \text{as } c \circ v = e_p \end{array}$$

where $[p]$ is the identity element in $\pi_1(X,p)$. Similarly,

$$[u] = \pi_1(m) \circ \Phi([u],[p]).$$

Now observe that as elements of $\pi_1(X,p) \times \pi_1(X,p)$, one has

$$([u],[v]) = ([p],[v]) \cdot ([u],[p]).$$

Since $\pi_1(m) \circ \Phi : \pi_1(X,p) \times \pi_1(X,p) \to \pi_1(X,p)$ is a homomorphism, we have

$$\begin{aligned}
\pi_1(m) \circ \Phi([u],[v]) &= \pi_1(m) \circ \Phi\big(([p],[v]) \cdot ([u],[p])\big) \\
&= \pi_1(m) \circ \Phi([p],[v]) \cdot \pi_1(m) \circ \Phi([u],[p]) \\
&= [v][u].
\end{aligned}$$

If instead one factors $([u],[v]) = ([u],[p]) \cdot ([p],[v])$, the same reasoning shows that $\pi_1(m) \circ \Phi([u],[v]) = [u][v]$. We conclude that $[v][u] = [u][v]$, hence $\pi_1(X,p)$ is abelian.

Problem Sheet D

This Problem Sheet is based on Lecture 7 and Lecture 8. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

PROBLEM D.1 (†). Some questions on (free) abelian groups:

1. Prove that every abelian group G is isomorphic to a quotient group of the form F/R, where F is a free abelian group.
2. Prove that every any two bases of a free abelian group have the same cardinality (and thus the notion of the *rank* of a free abelian group is well-defined).
3. Prove that two free abelian groups are isomorphic if and only if they have the same rank.
4. Prove that if G is an arbitrary abelian group then there exists a free abelian subgroup F of G such that G/F is torsion.

PROBLEM D.2 (†). Let $P = [z_0, z_1, \dots, z_n]$ denote an n-simplex, where $n \geq 1$. Construct an explicit homeomorphism $(P, \partial P) \to (B^n, S^{n-1})$.

PROBLEM D.3 (†). Let X be a one-point space $\{p\}$. Prove[1] that $H_n(X) = 0$ for all $n > 0$.

PROBLEM D.4 (†). Let X be a topological space. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ denote the path components of X. Prove that for every $n \geq 0$ one has[2]

$$H_n(X) \cong \bigoplus_{\lambda \in \Lambda} H_n(X_\lambda).$$

[1] This is called the **dimension axiom**.

[2] In general, if $\{G_\lambda \mid \lambda \in \Lambda\}$ is a collection of groups, an element of $\bigoplus_{\lambda \in \Lambda} G_\lambda$ is a tuple (g_λ) where all but finitely many of the g_λ are equal to the identity.

Solutions to Problem Sheet D

This Problem Sheet is based on Lecture 7 and Lecture 8. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem D.1 (†). Some questions on (free) abelian groups:

1. Prove that every abelian group G is isomorphic to a quotient group of the form F/R, where F is a free abelian group.
2. Prove that any two bases of a free abelian group have the same cardinality (and thus the notion of the *rank* of a free abelian group is well-defined).
3. Prove that two free abelian groups are isomorphic if and only if they have the same rank.
4. Prove that if G is an arbitrary abelian group then there exists a free abelian subgroup F of G such that G/F is torsion.

Solution.

1. For an abelian group G let $F(G)$ be the free abelian group generated by the set G. Let B be a basis of $F(G)$. Since G generates $F(G)$ we can assume that $B \subset G$. By Lemma 7.2 there exists a unique group homomorphsim $\phi\colon F(G) \to G$ such that $\phi(b) = b$ for every $b \in B$. Clearly, $\phi|_G = Id_G$ and hence ϕ is surjective. Let $R := \ker \phi$ then $G \cong F(G)/R$.
2. Let p be prime. A free abelian group is a $\mathbb{Z}$-module. Then G/pG is a module over $\mathbb{Z}/p\mathbb{Z}$. But $\mathbb{Z}/p\mathbb{Z}$ is a field and thus G/pG is a vector space over $\mathbb{Z}/p\mathbb{Z}$. A basis B of G induces a basis of G/pG as a vector space over $\mathbb{Z}/p\mathbb{Z}$ and they have the same cardinality. (The basis of G/pG is given by $\{b + p\mathbb{Z} \mid b \in B\}$.) Any two bases of a vector space have the same cardinality.
3. Here are two proofs:

   Two free abelian groups F and G are isomorphic if and only if the vector spaces F/pF and G/pG are isomorphic. Hence, they are isomorphic if and only if their bases have the same cardinality.

   Let F and G be two isomorphic free abelian groups and ϕ an isomorphism between them. Let B be a basis of F. Since ϕ is bijective and a group homomorphism it follows that $\phi(B)$ is a basis of G and $|B| = |\phi(B)|$. For the other direction, let B be a basis of F and B' a basis of G and suppose that $|B| = |B'|$. Let $\phi\colon B \to B'$ be a bijection and $\psi\colon B' \to B$ its inverse. By Lemma 7.2 there exists a unique group homomorphism $\tilde{\phi}\colon F \to G$ such that $\tilde{\phi}(b) = \phi(b)$ for every $b \in B$. Similarly there exists a unique $\tilde{\psi}\colon G \to F$ such that $\tilde{\psi}(b') = \psi(b')$ for every $b' \in B'$. Since B and B' generate F and G and $\tilde{\phi}$ and $\tilde{\psi}$ are group homomorphisms it follows that $\tilde{\phi}\colon F \to G$ is a group isomorphism with inverse $\tilde{\psi}\colon G \to F$.

4. Define a subset A of G to be *independent* if $\forall a_i \in A$ and $\forall z_i \in \mathbb{Z}$ if $\sum_i z_i a_i = 0$ then necessarily $z_i = 0\ \forall i$. By Zorn's Lemma a maximal independent set exists, call it B. Let F be the free abelian subgroup of G generated by B. Then G/F is torsion by maximality of B.

   Here is another proof which is more constructive (i.e. it does not need Zorn's Lemma), but only[1] works for *finitely generated groups*. Let $T := \{g \in G \mid g \text{ is torsion}\}$ be the torsion subgroup. This is a normal subgroup of G and G/T is free. Then there exists a short exact sequence

$$0 \longrightarrow T \xrightarrow{\ i\ } G \xrightarrow{\ p\ } G/T \longrightarrow 0.$$

   (Short exact sequences will soon be defined in Lecture 12.) Since G/T is free the sequence is split, i.e. there exists a map $s\colon G/T \to G$, such that $p \circ s = \mathrm{id}_{G/T}$ and $r\colon G \to T$ such that $r \circ i = \mathrm{id}_T$ (you will prove this on Problem Sheet F.) The group homomorphism s is injective and $s(G/T) < G$ is a free subgroup. Moreover, $G/s(G/T) \cong T$ is torsion.

Problem D.2 (†). Let $P = [z_0, z_1, \ldots, z_n]$ denote an n-simplex, where $n \geq 1$. Construct an explicit homeomorphism $(P, \partial P) \to (B^n, S^{n-1})$.

Solution. Let b denote the barycentre of the n-simplex P. Since P is convex, for a point $x \in P \setminus \{b\}$ the line spanned by $\vec{x} - \vec{b}$ intersects ∂P at a unique point, which we call $s(x)$. Define

$$\begin{array}{rcl} \phi\colon (P, \partial P) & \to & (B^n, \partial B^n) \\ x & \mapsto & \frac{\vec{x}-\vec{b}}{|s(\vec{x})-\vec{b}|} \\ b & \mapsto & 0. \end{array}$$

This is a bijective continuous map between compact Hausdorff spaces, and hence is a homeomorphism.

Problem D.3 (†). Let X be a one-point space $\{*\}$. Prove[2] that $H_n(X) = 0$ for all $n > 0$.

Solution. For every $n \geq 0$ the group of n-chains $C_n(X)$ has exactly one generator, namely the constant n-chain $\kappa_n\colon \Delta^n \to \{*\}$. The boundary of the constant n-chain κ_n is precisely

$$\partial \kappa_n = \sum_{i=0}^{n} (-1)^i \kappa_n \circ \varepsilon_i = \sum_{i=0}^{n} (-1)^i \kappa_{n-1} = \begin{cases} \kappa_{n-1}, & n \text{ even and } \geq 2, \\ 0, & n \text{ odd and } \geq 1, \\ 0, & n = 0, \end{cases}$$

(Recall that the boundary of a 0-chain is by definition zero.) Hence, the group $Z_n(X)$ of singular n-chains is

$$Z_n(X) = \begin{cases} 0, & n \text{ even and } \geq 0, \\ C_n(X) = \mathbb{Z} \cdot \kappa_n, & n \text{ odd and } \geq 1, \\ C_n(X) = \mathbb{Z} \cdot \kappa_n, & n = 0, \end{cases},$$

[1] Thanks to "asdf" for pointing this out! The group $(\mathbb{Q}, +)$ (which is not finitely generated) is a counterexample since it has zero torsion subgroup but is not free.

[2] This is called the **dimension axiom**.

and

$$B_n(X) = \begin{cases} 0, & n \text{ even and } \geq 0, \\ C_n(X) = \mathbb{Z} \cdot \kappa_n, & n \text{ odd and } \geq 1, \end{cases}.$$

In both cases we conclude that $H_n(X) = Z_n(X)/B_n(X) = 0$ for every $n \geq 1$. In dimension zero $Z_0(X) = \mathbb{Z} \cdot \kappa_0$ and $B_0(X) = 0$ which shows that $H_0(X) \cong \mathbb{Z}$.

PROBLEM D.4 (†). Let X be a topological space. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ denote the path components of X. Prove that for every $n \geq 0$ one has[3]

$$H_n(X) \cong \bigoplus_{\lambda \in \Lambda} H_n(X_\lambda).$$

SOLUTION. Since an n-chain is continuous and an n-simplex is path connected, its image must lie inside one path component. Hence $C_n(X) \cong \bigoplus_{\lambda \in \Lambda} C_n(X_\lambda)$. Moreover the boundary of an n-chain must lie in the same path component, i.e. the boundary operator can be written as a sum of operators $\sum \partial_\lambda \colon \bigoplus_{\lambda \in \Lambda} C_n(X_\lambda) \to \bigoplus_{\lambda \in \Lambda} C_{n-1}(X_\lambda)$. Therefore we have for every λ: $B_n(X_\lambda) \subseteq Z_n(X_\lambda) \subseteq C_n(X_\lambda)$, which implies

$$H_n(X) \cong \bigoplus_{\lambda \in \Lambda} H_n(X_\lambda).$$

[3]In general, if $\{G_\lambda \mid \lambda \in \Lambda\}$ is a collection of groups, an element of $\bigoplus_{\lambda \in \Lambda} G_\lambda$ is a tuple (g_λ) where all but finitely many of the g_λ are equal to the identity.

Problem Sheet E

This Problem Sheet is based on Lecture 9 and Lecture 10. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

Problem E.1 (†). Let G be a group. Prove that there exists at most one abelian group A and a group homomorphism $p : G \to A$ satisfying the following *universal property*: if A' is any other abelian group and $\varphi : G \to A'$ is a homomorphism, then φ induces a *unique* homomorphism $\tilde{\varphi} : A \to A'$:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & A' \\ {\scriptstyle p}\downarrow & \nearrow_{\tilde{\varphi}} & \\ A & & \end{array}$$

Prove that taking $A = G^{\mathrm{ab}} := G/[G,G]$ and p the quotient map solves this universal property.

Problem E.2 (†). Prove that the Hurewicz map is natural[1]. That is, suppose $f : (X,p) \to (Y,q)$ is a pointed map. Prove that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,p) & \xrightarrow{\pi_1(f)} & \pi_1(Y,q) \\ {\scriptstyle h_p}\downarrow & & \downarrow{\scriptstyle h_q} \\ H_1(X) & \xrightarrow[H_1(f)]{} & H_1(Y) \end{array}$$

Problem E.3 (†). Let $u \colon I \to X$ be a (not necessarily closed) path. Prove that:

1. The singular 1-chain $u' + \bar{u}'$ belongs to $Z_1(X)$, where we are using the notation convention from Remark 9.1, and hence $\langle u' + \bar{u}' \rangle$ is a well defined homology class.
2. Prove that $\langle u' + \bar{u}' \rangle = 0 \in H_1(X)$.

Problem E.4 (†). Suppose u, v, w are three not necessarily closed paths in X. Assume that $u(1) = v(0)$ and $v(1) = w(0)$ and $w(1) = u(0)$, so the concatenation $u * v * w$ is well defined and is a loop. Prove that:

1. The singular 1-chain $u' + v' + w'$ belongs to $Z_1(X)$ and hence defines a homology class $\langle u' + v' + w' \rangle$.
2. Since $u * v * w$ is a loop, we know from Proposition 9.2 that $(u * v * w)'$ belongs to $Z_1(X)$ and hence determines a homology class $\langle (u * v * w)' \rangle$. Prove that actually,

$$\langle (u * v * w)' \rangle = \langle u' + v' + w' \rangle.$$

[1]The precise meaning of the word "natural" and why this exercises means h is natural will be explained at the end of the course when we study **natural transformations**.

Problem E.5 (†). Prove that a sequence $A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$ is exact in Comp if and only if $A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$ is exact in Ab for every $n \in \mathbb{Z}$.

Problem E.6 (†). Suppose $\{(C_\bullet^\lambda, \partial^\lambda) \mid \lambda \in \Lambda\}$ is a family of complexes indexed by a set Λ. Recall their **direct sum** is the complex $\bigoplus_\lambda C_\bullet^\lambda$ equipped with the boundary operator $\sum_\lambda \partial^\lambda$. Prove that for all $n \geq 0$,

$$H_n\left(\bigoplus_\lambda C_\bullet^\lambda\right) = \bigoplus_\lambda H_n(C_\bullet^\lambda).$$

Solutions to Problem Sheet E

This Problem Sheet is based on Lecture 9 and Lecture 10. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

Problem E.1 (†). Let G be a group. Prove that there exists at most one abelian group A and a group homomorphism $p : G \to A$ satisfying the following *universal property*: if A' is any other abelian group and $\varphi : G \to A'$ is a homomorphism, then φ induces a *unique* homomorphism $\tilde{\varphi} : A \to A'$:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & A' \\ {\scriptstyle p}\downarrow & \nearrow_{\tilde{\varphi}} & \\ A & & \end{array}$$

Prove that taking $A = G^{\mathrm{ab}} := G\big/[G,G]$ and p the quotient map solves this universal property.

Solution. The fact that there exists at most once such pair (A, p) is the "usual" universal property argument. If (A, p) and (A', p') were two pairs then we would get unique homomorphisms $\varphi\colon A \to A'$ and $\varphi'\colon A' \to A$ such that $\varphi \circ p = p'$ and $\varphi' \circ p' = p$. But then $\varphi' \circ \varphi\colon A \to A$ satisfies $(\varphi' \circ \varphi) \circ p = p$. By uniqueness, $\varphi' \circ \varphi = \mathrm{id}_A$. Similarly $\varphi \circ \varphi' = \mathrm{id}_{A'}$.

Let $A = G^{\mathrm{ab}} := G\big/[G,G]$ and $p : G \to G\big/[G,G]$ the quotient map. We want to show that (G, p) fulfills the universal property. Let A' be an abelian group and $\phi : G \to A'$ a group homomorphism. Define $\tilde{\phi} : A \to A'$ by $\tilde{\phi}([a]) := \phi(a)$, where $[a] \in A$ with representative $a \in G$. This is well-defined since $\phi(gh - hg) = \phi(g)\phi(h) - \phi(h)\phi(g) = 0$ for every $g, h \in G$ and hence ϕ descends to the quotient $G\big/[G,G]$. It is easy to see that $\tilde{\phi} \circ p = \phi$. Uniqueness of the map $\tilde{\phi}$ also follows directly from its definition. Suppose $\tilde{\psi}$ is another such homomorphism. Then $\tilde{\psi}([g]) = \phi(g) = \tilde{\phi}([g])$, which shows that $\tilde{\psi} = \tilde{\phi}$.

Problem E.2 (†). Prove that the Hurewicz map is natural[1]. That is, suppose $f : (X, p) \to (Y, q)$ is a pointed map. Prove that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,p) & \xrightarrow{\pi_1(f)} & \pi_1(Y,q) \\ {\scriptstyle h_p}\downarrow & & \downarrow{\scriptstyle h_q} \\ H_1(X) & \xrightarrow[H_1(f)]{} & H_1(Y) \end{array}$$

Will J. Merry and Berit Singer. Last modified: Sept 01, 2018.

[1]The precise meaning of the word "natural" and why this exercises means h is natural will be explained at the end of the course when we study **natural transformations**.

Solution. Recall that $\pi_1(f)([u]) := [f \circ u]$ and $H_1(f)([\sum_i a_i \sigma_i]) := [\sum_i a_i f \circ \sigma_i]$. Since all maps involved are group homomorphism is suffices to show that the diagram commutes for a generator $[u] \in \pi_1(X,p)$. Then $h_q \circ \pi_1(f)([u]) = \langle (f \circ u)' \rangle$ and $H_1(f) \circ h_p([u]) = \langle f \circ (u)' \rangle)$. But $(f \circ u)'(s_0, s_1) = f \circ u(s_1) = f \circ u'(s_0, s_1)$, which proves the claim.

Problem E.3 (†). Let $u \colon I \to X$ be a (not necessarily closed) path. Prove that:

1. The singular 1-chain $u' + \bar{u}'$ belongs to $Z_1(X)$, where we are using the notation convention from Remark 9.1, and hence $\langle u' + \bar{u}' \rangle$ is a well defined homology class.
2. Prove that $\langle u' + \bar{u}' \rangle = 0 \in H_1(X)$.

Solution.

1. We check $\partial(u' + \bar{u}') = u(1) - u(0) + \bar{u}(1) - \bar{u}(0) = 0$, since $\bar{u}(1) = u(o)$ and $\bar{u}(0) = u(1)$.
2. Let $\sigma : \Delta^2 \to X$ be the simplex defined by $\sigma(s_0, s_1, s_2) := u(s_1)$. Then $\sigma \circ \varepsilon_2 = u'$, $\sigma \circ \varepsilon_0 = \bar{u}'$ and $\sigma \circ \varepsilon_1 = \kappa_1$, where $\kappa_1 : \Delta^1 \to X$ denotes the constant 1-chain with image $u(0)$. We calculate the boundary $\partial\sigma = u' + \bar{u}' - \kappa_1$. Now κ_1 itself is a boundary. Indeed, let $\kappa_2 : \Delta^2 \to X$ denote the constant 2-chain with image $u(0)$. Then $\partial\kappa_2 = \kappa_1$ (cf. the solution to Problem D.3.) Hence $\langle u' + \bar{u}' \rangle = \langle u' + \bar{u}' - \kappa_1 \rangle = 0$.

Problem E.4 (†). Suppose u, v, w are three not necessarily closed paths in X. Assume that $u(1) = v(0)$ and $v(1) = w(0)$ and $w(1) = u(0)$, so the concatenation $u * v * w$ is well defined and is a loop. Prove that:

1. The singular 1-chain $u' + v' + w'$ belongs to $Z_1(X)$ and hence defines a homology class $\langle u' + v' + w' \rangle$.
2. Since $u * v * w$ is a loop, we know from Proposition 9.2 that $(u * v * w)'$ belongs to $Z_1(X)$ and hence determines a homology class $\langle (u * v * w)' \rangle$. Prove that actually,
$$\langle (u * v * w)' \rangle = \langle u' + v' + w' \rangle.$$

Solution.

1. $\partial(u' + v' + w') = u(1) - u(0) + v(1) - v(0) + w(1) - w(0) = 0$, where the last equality follows from the fact that $u(1) = v(0)$ and $v(1) = w(0)$ and $w(1) = u(0)$.
2. Define two 2-chains σ_1 and σ_2 such that $\partial\sigma_1 = v' - (u * v)' + u'$ and $\partial\sigma_2 = w' - ((u * v) * w)' + (u * v)'$. They are given for example by $\sigma_1(s_0, s_1, s_2) := (u * v)'(s_0 + \frac{s_1}{2}, \frac{s_1}{2} + s_2)$ and $\sigma_2(s_0, s_1, s_2) := ((u * v) * w)'(s_0 + \frac{s_1}{2}, \frac{s_1}{2} + s_2)$. Then $\sigma_1 + \sigma_2 \in C_2(X)$ and their boundary is $\partial(\sigma_1 + \sigma_2) = v' - (u * v)' + u' + w' - ((u * v) * w)' + (u * v)' = v' + u' + w' - ((u * v) * w)'$. This proves that $\langle v' + u' + w' \rangle = \langle ((u * v) * w)' \rangle = \langle (u * v * w)' \rangle$.

Problem E.5 (†). Prove that a sequence $A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$ is exact in Comp if and only if $A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$ is exact in Ab for every $n \in \mathbb{Z}$.

Solution. Saying that $A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$ is exact for all n is saying that $\operatorname{im} f_n = \ker g_n$ for all n. A priori, that claim that the two complexes $(\operatorname{im} g)_\bullet$ and $(\ker f)_\bullet$ should agree is a stronger statement (since it also requires the boundary operators to coincide). But in this case since the boundary operator is simply given by restriction (as these are both subcomplexes of $B_\bullet$), and hence this latter condition is automatic.

Problem E.6 (†). Suppose $\{(C^\lambda_\bullet, \partial^\lambda) \mid \lambda \in \Lambda\}$ is a family of complexes indexed by a set Λ. Recall their **direct sum** is the complex $\bigoplus_\lambda C^\lambda_\bullet$ equipped with the boundary operator $\sum_\lambda \partial^\lambda$. Prove that for all $n \geq 0$,

$$H_n\left(\bigoplus_\lambda C^\lambda_\bullet\right) = \bigoplus_\lambda H_n(C^\lambda_\bullet).$$

Solution. Fix $n \in \mathbb{Z}$. For every λ we have $\partial^\lambda_n \colon C^\lambda_n \to C^\lambda_{n-1}$ and $\partial^\lambda_n(C^\lambda_n) \subseteq C^\lambda_{n-1}$. This implies $\operatorname{im}(\partial^\lambda_{n+1}) \subseteq \ker(\partial^\lambda_n) \subseteq C^\lambda_n$ and thus

$$H_n(\bigoplus_\lambda C^\lambda_\bullet) = \frac{\ker \sum_\lambda \partial^\lambda_n}{\operatorname{im} \sum_\lambda \partial^\lambda_{n+1}} \cong \bigoplus_\lambda \frac{\ker \partial^\lambda_n}{\operatorname{im} \partial^\lambda_{n+1}} \cong \bigoplus_\lambda H_n(C^\lambda_\bullet).$$

The isomorphism can be written explicitly as follows. An element of $\bigoplus_\lambda C^\lambda_n$ is a tuple $(c_\lambda)_{\lambda\in\Lambda}$ where all but finitely many of the $c_\lambda \in C^\lambda_n$ are zero. The boundary operator is given by $(\sum_\lambda \partial^\lambda)(c_\lambda)_{\lambda\in\Lambda} = (\partial^\lambda c_\lambda)_{\lambda\in\Lambda}$. We define a map

$$f \colon H_n(\bigoplus_\lambda C^\lambda_\bullet) \to \bigoplus_\lambda H_n(C^\lambda_\bullet)$$

by

$$\langle (c_\lambda)_{\lambda\in\Lambda} \rangle \mapsto \big(\langle c_\lambda \rangle\big)_{\lambda\in\Lambda}.$$

The element on the right-hand side does indeed belong to $\bigoplus_\lambda H_n(C^\lambda_\bullet)$, since as at most finitely many of the c_λ are non-zero, also at most finitely many of the $\langle c_\lambda \rangle$ are non-zero. The map f is well defined as $\langle (c_\lambda)_{\lambda\in\Lambda} \rangle = \langle (c'_\lambda)_{\lambda\in\Lambda} \rangle$ if and only if $0 = \langle (c_\lambda - c'_\lambda)_{\lambda\in\Lambda} \rangle$. The latter if true if and only if there exists $(b_\lambda)_{\lambda\in\Lambda} \in \bigoplus_\lambda C^\lambda_{n+1}$ with the property that $(\sum_\lambda \partial^\lambda)(b_\lambda)_{\lambda\in\Lambda} = (c_\lambda - c'_\lambda)_{\lambda\in\Lambda}$. Then $\partial^\lambda b_\lambda = c_\lambda - c'_\lambda$ for each λ, and hence also $\langle c_\lambda \rangle = \langle c'_\lambda \rangle$ for each λ.

Next, for any element $\big(\langle c_\lambda \rangle\big)_{\lambda\in\Lambda} \in \bigoplus_\lambda H_n(C^\lambda_\bullet)$, there always exists another element $\big(\langle \tilde{c}_\lambda \rangle\big)_{\lambda\in\Lambda}$ with the property that

$$\big(\langle c_\lambda \rangle\big)_{\lambda\in\Lambda} = \big(\langle \tilde{c}_\lambda \rangle\big)_{\lambda\in\Lambda}$$

and at most finitely many of the $\tilde{c}_\lambda$ are non-zero. Indeed, if $\langle c_\lambda \rangle = 0$, set $\tilde{c}_\lambda := 0$, and if $\langle c_\lambda \rangle \neq 0$, set $\tilde{c}_\lambda := c_\lambda$.

This implies that if $\big(\langle c_\lambda \rangle\big)_{\lambda\in\Lambda} \in \bigoplus_\lambda H_n(C^\lambda_\bullet)$ then $\langle (c_\lambda)_{\lambda\in\Lambda} \rangle$ does indeed belong to $H_n(\bigoplus_\lambda C^\lambda_\bullet)$. Then the same argument as above shows that the map

$$g \colon \bigoplus_\lambda H_n(C^\lambda_\bullet) \to H_n(\bigoplus_\lambda C^\lambda_\bullet).$$

by

$$\big(\langle c_\lambda \rangle\big)_{\lambda \in \Lambda} \mapsto \langle (c_\lambda)_{\lambda \in \Lambda} \rangle$$

is well defined.

Finally, both f and g are homomorphisms by construction, and it is clear they are mutually inverse. Thus in particular f is an isomorphism, and the proof is complete.

Problem Sheet F

This Problem Sheet is based on Lecture 11 and Lecture 12. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem F.1 (†). Prove the **Five Lemma**: Suppose we have a commutative diagram of abelian groups, where the two rows are exact:

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle k} & & \big\downarrow{\scriptstyle l} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Prove that

1. If g and k are injectice and f is surjective, h is injective.
2. If g and k are surjective and l is injective, h is surjective.
3. If f, g, k, l are all isomorphisms then so is h.

Problem F.2 (†). Prove the **Barratt-Whitehead Lemma**: Suppose we have the following commutative diagram of abelian groups, where the two rows are exact:

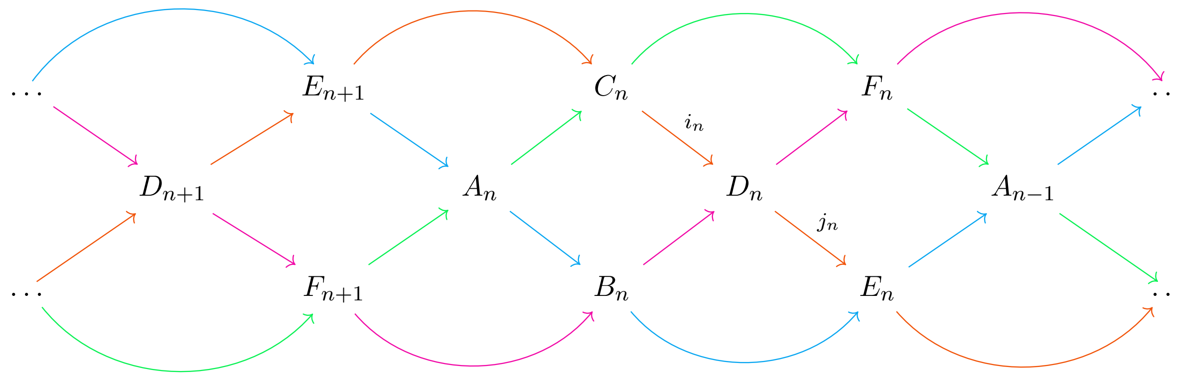
$$\begin{array}{ccccccccccc} \dots & \longrightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{j_n} & C_n & \xrightarrow{k_n} & A_{n-1} & \longrightarrow & \dots \\ & & {\scriptstyle f_n}\big\downarrow & & {\scriptstyle g_n}\big\downarrow & & \big\downarrow{\scriptstyle h_n} & & \big\downarrow{\scriptstyle f_{n-1}} & & \\ \dots & \longrightarrow & A'_n & \xrightarrow[i'_n]{} & B'_n & \xrightarrow[j'_n]{} & C'_n & \xrightarrow[k'_n]{} & A'_{n-1} & \longrightarrow & \dots \end{array}$$

Assume each map $h_n \colon C_n \to C'_n$ is an isomorphism. Then there is a long exact sequence:

$$\dots \to A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - i'_n} B'_n \xrightarrow{k_n h_n^{-1} j'_n} A_{n-1} \to \dots,$$

where $(i_n, f_n) \colon A_n \to B_n \oplus A'_n$ is given by $a \mapsto (i_n(a), f_n(a))$ and $g_n - i'_n \colon B_n \oplus A'_n \to B'_n$ is given by $(b, a') \mapsto g_n(b) - i'_n(a')$.

Problem F.3 ($\star$). Consider the following **commutative braid** of abelian groups:



Assume that three of the "strands" form long exact sequences:

$$\dots \longrightarrow E_{n+1} \longrightarrow A_n \longrightarrow B_n \longrightarrow E_n \longrightarrow A_{n-1} \longrightarrow \dots$$

$$\dots \longrightarrow D_{n+1} \longrightarrow F_{n+1} \longrightarrow B_n \longrightarrow D_n \longrightarrow F_n \longrightarrow \dots$$

$$\dots \longrightarrow F_{n+1} \longrightarrow A_n \longrightarrow C_n \longrightarrow F_n \longrightarrow A_{n-1} \longrightarrow \dots$$

Assume in addition that

$$\operatorname{im} i_n \subseteq \ker j_n \qquad \text{or} \qquad \ker j_n \subseteq \operatorname{im} i_n, \qquad \forall\, n \in \mathbb{Z}.$$

Prove that the fourth strand

$$\dots \longrightarrow D_{n+1} \longrightarrow E_{n+1} \longrightarrow C_n \xrightarrow{i_n} D_n \xrightarrow{j_n} E_n \longrightarrow \dots$$

is also a long exact sequence.

Problem F.4 ($\dagger$). Let $X'' \subseteq X' \subseteq X$ be subspaces. Prove there is a long exact sequence

$$\dots H_n(X', X'') \to H_n(X, X'') \to H_n(X, X') \xrightarrow{\delta} H_{n-1}(X', X'') \to \dots$$

We call this the **long exact sequence of the triple** (X, X', X''). *Hint:* Use the previous problem!

Suppose we are given two triples (X, X', X'') and (Y, Y', Y''), together with a continuous map $f \colon X \to Y$ such that $f(X') \subseteq Y'$ and $f(X'') \subseteq Y''$. Prove there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & H_n(X', X'') & \longrightarrow & H_n(X, X'') & \longrightarrow & H_n(X, X') & \longrightarrow & H_{n-1}(X', X'') & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & H_n(Y', Y'') & \longrightarrow & H_n(Y, Y'') & \longrightarrow & H_n(Y, Y') & \longrightarrow & H_{n-1}(Y', Y'') & \longrightarrow & \dots
\end{array}$$

where all the vertical maps are induced by f.

Problem F.5 (†). Let $\emptyset \neq X' \subseteq X$. Prove there is a long exact sequence

$$\cdots \to \tilde{H}_n(X') \to \tilde{H}_n(X) \to H_n(X, X') \to \tilde{H}_{n-1}(X') \to \dots$$

which ends with $\tilde{H}_0(X') \to \tilde{H}_0(X) \to H_0(X, X') \to 0$.

Problem F.6 (†). Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of abelian groups.

1. Prove that the sequence splits if and only if there exists a map $k\colon B \to A$ such that $kf = \mathrm{id}_A$.
2. Give an example of such a short exact sequence where $B \cong A \oplus C$ but such that the sequence does not split.
3. Prove that if C is free abelian then the sequence always splits.

Solutions to Problem Sheet F

This Problem Sheet is based on Lecture 11 and Lecture 12. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem F.1 (†). Prove the **Five Lemma**: Suppose we have a commutative diagram of abelian groups, where the two rows are exact:

$$\begin{array}{ccccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & D & \xrightarrow{\delta} & E \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \xrightarrow{\gamma'} & D' & \xrightarrow{\delta'} & E' \end{array}$$

Prove that

1. If g and k are injective and f is surjective, h is injective.
2. If g and k are surjective and l is injective, h is surjective.
3. If f, g, k, l are all isomorphisms then so is h.

Solution.

1. Let $c \in C$ such that $h(c) = 0$. Then $k \circ \gamma(c) = \gamma' \circ h(c) = 0$ and thus $\gamma(c) = 0$ since k is injective. Then $\beta(b) = c$ for some $b \in B$ and $\beta' \circ g(b) = h \circ \beta(b) = h(c) = 0$. So $g(b) \in \ker(\beta')$ and $\alpha'(a') = g(b)$ for some $a' \in A'$. Since f is surjective $f(a) = a'$ for some $a \in A$ and $\alpha'(f(a)) = g(b)$ by commutativity. Then $g(b - \alpha(a)) = g(b) - g \circ \alpha(a) = g(b) - \alpha' \circ f(a) = 0$. Since g is injective $b - \alpha(a) = 0$. Hence $c = \beta(b) = \beta \circ \alpha(a) = 0$ which shows that h is injective.
2. Let $c' \in C'$. Since k is surjective $k(d) = \gamma'(c')$ for some $d \in D$. From exactness of the lower row $\delta' \circ \gamma'(c') = 0$ and thus also $0 = \delta' \circ k(d) = l \circ \delta(d)$. Now $\delta(d) = 0$ since l is injective and thus $\gamma(c) = d$ for some $c \in C$ by exactness of the upper row. Then $\gamma'(h(c) - c') = 0$ since $\gamma' \circ h(c) = k \circ \gamma(c) = k(d) = \gamma'(c')$. So $\beta'(b') = c' - h(c)$ for some $b' \in B'$ by exactness of the lower row. Since g is surjective $g(b) = b'$ for some $b \in B$ and $h \circ \beta(b) = \beta' \circ g(b) = \beta'(b') = c' - h(c)$. Now $h(\beta(b) + c) = h \circ \beta(b) + h(c) = \beta' \circ g(b) + h(c) = c' - h(c) + h(c) = c'$, which proves that h is surjective.
3. Follows directly from part (1) and part (2).

Problem F.2 (†). Prove the **Barratt-Whitehead Lemma**: Suppose we have the following commutative diagram of abelian groups, where the two rows are exact:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{j_n} & C_n & \xrightarrow{k_n} & A_{n-1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \dots & \longrightarrow & A'_n & \xrightarrow[i'_n]{} & B'_n & \xrightarrow[j'_n]{} & C'_n & \xrightarrow[k'_n]{} & A'_{n-1} & \longrightarrow & \dots \end{array}$$

Assume each map $h_n\colon C_n \to C'_n$ is an isomorphism. Then there is a long exact sequence:

$$\cdots \to A_n \xrightarrow{(i_n,f_n)} B_n \oplus A'_n \xrightarrow{g_n - i'_n} B'_n \xrightarrow{k_n h_n^{-1} j'_n} A_{n-1} \to \dots,$$

where $(i_n, f_n)\colon A_n \to B_n \oplus A'_n$ is given by $a \mapsto (i_n(a), f_n(a))$ and $g_n - i'_n\colon B_n \oplus A'_n \to B'_n$ is given by $(b, a') \mapsto g_n(b) - i'_n(a')$.

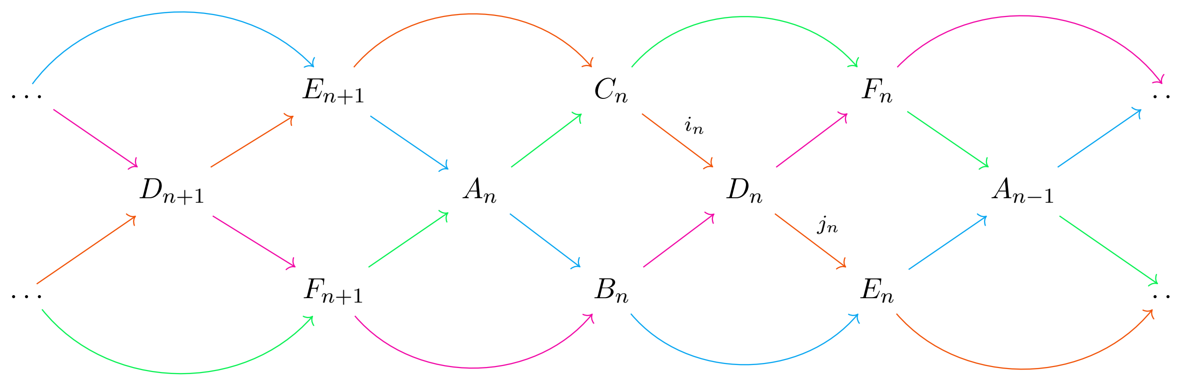
Solution. We need to prove the exactness at every place.

1. $\operatorname{im}(i_n, f_n) \subseteq \ker(g_n - i'_n)$: $(g_n - i'_n) \circ (i_n, f_n)(a) = g_n \circ i_n(a) - i'_n \circ f_n(a) = 0$ by exactness.
2. $\ker(g_n - i'_n) \subseteq \operatorname{im}(i_n, f_n)$: Let (b, a') such that $g_n(b) - i'_n(a') = 0$. Then $0 = j'_n(g_n(b) - i'_n(a')) = j'_n \circ g_n(b) - j'_n \circ i'_n(a') = j'_n \circ g_n(b)$ since $j'_n \circ i'_n(a') = 0$ by exactness of the lower row. Hence $0 = j'_n \circ g_n(b) = h_n \circ j_n(b)$ and since h_n is an isomorphism $j_n(b) = 0$. Then $i_n(a) = b$ for some $a \in A_n$ by exactness and $i'_n \circ f_n(a) = g_n \circ i_n(a) = g_n(b) = i'_n(a')$. So $i'_n(f_n(a) - a') = 0$ and by exactness $k'_{n+1}(c') = f_n(a) - a'$ for some $c' \in C'_{n+1}$. Now $f_n(a - k_{n+1} \circ h_{n+1}^{-1}(c')) = f_n(a) - k'_{n+1}(c') = a'$. Moreover, as $i_n \circ k_{n+1} = 0$ by exactness we also have $i_n(a - k_{n+1} \circ h_{n+1}^{-1}(c')) = i_n(a) = b$. Hence $(i_n, f_n)(a - k_{n+1} \circ h_{n+1}^{-1}(c')) = (b, a')$.
3. $\operatorname{im}(g_n - i'_n) \subseteq \ker(k_n h_n^{-1} j'_n)$: $k_n h_n^{-1} j'_n(g_n - i'_n(b, a')) = k_n h_n^{-1} j'_n g_n(b)$ since $j'_n i'_n = 0$ by exactness. But $k_n h_n^{-1} j'_n g_n(b) = k_n j_n(b)$ by commutativity and hence by exactness $k_n j_n(b) = 0$. Thus $k_n h_n^{-1} j'_n(g_n - i'_n(b, a')) = 0$.
4. $\ker(k_n h_n^{-1} j'_n) \subseteq \operatorname{im}(g_n - i'_n)$: Suppose $k_n h_n^{-1} j'_n(b') = 0$. Then by exactness of the upper row $j_n(b) = h_n^{-1} j'_n(b')$ for some $b \in B$. Moreover $j'_n g_n(b) = h_n j_n(b) = j'_n(b')$ again by commutativity. Then $i'_n(a') = g_n(b) - b'$ and therefor $b' = g_n(b) - i'_n(a') \in \operatorname{im}(g_n - i'_n)$.
5. $\operatorname{im}(k_n h_n^{-1} j'_n) \subseteq \ker(i_{n-1}, f_{n-1})$: For this note

   $$(i_{n-1}, f_{n-1} \circ k_n h_n^{-1} j'_n(b') = (i_{n-1} k_n h_n^{-1} j'_n(b'), f_{n-1} k_n h_n^{-1} j'_n(b') = (0, k'_n j'_n(b')) = 0,$$

   where the first equality follows from exactness of the upper row and commutativity. The second equality follows from exactness of the lower row.
6. $\ker(i_{n-1}, f_{n-1}) \subseteq \operatorname{im}(k_n h_n^{-1} j'_n)$: Suppose $(i_{n-1}(a, f_{n-1}(a)) = 0$. Then by exactness $k_n(c) = a$ for some $c \in C_n$. By commutativity $k'_n h_n(c) = f_{n-1}(a) = 0$ and thus $j'_n(b') = h_n(c)$ for some $b' \in B'_n$. Hence $k_n h_n^{-1} j'_n(b') = k_n(c) = a$.

Problem F.3 ($\star$). Consider the following **commutative braid** of abelian groups:



Assume that three of the "strands" form long exact sequences:

$$\dots \longrightarrow E_{n+1} \longrightarrow A_n \longrightarrow B_n \longrightarrow E_n \longrightarrow A_{n-1} \longrightarrow \dots$$

$$\dots \longrightarrow D_{n+1} \longrightarrow F_{n+1} \longrightarrow B_n \longrightarrow D_n \longrightarrow F_n \longrightarrow \dots$$

$$\dots \longrightarrow F_{n+1} \longrightarrow A_n \longrightarrow C_n \longrightarrow F_n \longrightarrow A_{n-1} \longrightarrow \dots$$

Assume in addition that

$$\operatorname{im} i_n \subseteq \ker j_n \qquad \text{or} \qquad \ker j_n \subseteq \operatorname{im} i_n, \qquad \forall\, n \in \mathbb{Z}.$$

Prove that the fourth strand

$$\dots \longrightarrow D_{n+1} \longrightarrow E_{n+1} \longrightarrow C_n \xrightarrow{i_n} D_n \xrightarrow{j_n} E_n \longrightarrow \dots$$

is also a long exact sequence.

Solution. For simplicity in doing the chase we shall introduce some special notation for it. Elements of, for example, A_n will be denoted by $a_n, a'_n, \dots$ etc. To indicate that an element a_n goes to an an element $b_n \in B_n$ we simply write $a_n \to b_n$. To indicate that there exists an element $e_{n+1} \in E_{n+1}$, not yet defined, that goes to $a_n \in A_n$ we write $\exists\, e_{n+1} \to a_n$. The zero in A_n is denoted by 0_{A_n} and similar for the other groups.

The point where we need the condition that $\operatorname{im} i_n \subseteq \ker j_n$ or $\ker j_n \subseteq \operatorname{im} i_n$ is for the exactness at $C_n \to D_n \to E_n$. The exactness at the other places of the sequence follow without this assumption. Let us start with the exactness at $C_n \to D_n \to E_n$.

1. Exactness at $C_n \to D_n \to E_n$:

   (a) Assume first that $\operatorname{im} i_n \subseteq \ker j_n$. Let $d_n \to 0_{E_n}$. Then $d_n \to f_n \to 0_{A_{n-1}}$ for some $f_n \in F_n$, since 0_{E_n} and by commutativity. Hence $\exists\, c_n \to f_n$. Then $c_n \to d'_n \to f_n$ for some $d'_n \in D_n$ and by commutativity. Now by assumption and since $c_n \to d'_n$ it must follow that $d'_n \to 0_{E_n}$. Then $d_n - d'_n \to 0_{F_n}$ and $\exists\, b_n \to d_n - d'_n$ by exactness. Moreover, $b_n \to 0_{E_n}$ by commutativity and thus $\exists\, a_n \to b_n$. Let $a_n \to c'_n$, then $c'_n \to d_n - d'_n$ by commutativity. Finally, we see $c'_n + c_n \to d_n$ and hence $\ker j_n \subseteq \operatorname{im} i_n$.

(b) Now assume that $\ker j_n \subseteq \operatorname{im} i_n$. Let $c_n \to d_n$. Let $d_n \to f_n$, then $c_n \to f_n \to 0_{A_{n-1}}$ by exactness. Then we also have $d_n \to e_n \to 0_{A_{n-1}}$. By exactness $\exists\, b_n \to e_n$ and moreover $b_n \to d'_n \to e_n$ by commutativity. For exactness it also follows $d'_n \to 0_{F_n}$. Now $d_n \to e_n$ and $d'_n \to e_n$ implies $d_n - d'_n \to 0_{E_n}$. By assumption $\exists\, c'_n \to d_n - d'_n$. By commutativity and since $d_n + d'_n \to f_n$ it follows $c'_n \to f_n$. Then $c_n - c'_n \to d'_n$ and $c_n - c'_n \to 0_{F_n}$ by commutativity. Hence $\exists\, a_n \to c_n - c'_n$. Let $a_n \to b'_n \to d'_n$. By exactness $a_n \to b'_n \to 0_{E_n}$, and thus by commutativity $d'_n \to 0_{E_n}$. Since $d_n - d'_n \to 0_{E_n}$ we thus have $d_n \to 0_{E_n}$.

2. Exactness at $E_{n+1} \to C_n \to D_n$:

   (a) ker $\subseteq$ im: Suppose $c_n \to 0_{D_n}$. Then $0_{D_n} \to 0_{F_n}$ and hence also $c_n \to 0_{F_n}$, which implies $\exists\, a_n \to c_n$ by exactness. Now $a_n \to b_n \to 0_{D_n}$ and hence $\exists\, f_{n+1} \to b_n$. Let $f_{n+1} \to a'_n$, then $a'_n \to b_n$ by commutativity and it follows $a_n - a'_n \to 0_{B_n}$. From exactness $\exists\, e_{n+1} \to a_n - a'_n$ and since $a'_n \to 0_{C_n}$, which also follows from exactness, we have $a_n - a'_n \to c_n$. Then $e_{n+1} \to c_n$ by commutativity and thus $c_n \in \operatorname{im}(E_{n+1} \to C_n)$.

   (b) im $\subseteq$ ker: Let $e_{n+1} \to c_n$. Then $e_{n+1} \to a_n \to 0_{B_n} \to 0_{D_n}$ by commutativity and exactness. So $a_n \to c_n$ and $a_n \to 0_{D_n}$ by commutativity which proves that $a_n \in \ker(E_{n+1} \to C_n)$.

3. Exactness at $D_n \to E_n \to C_{n-1}$:

   (a) ker $\subseteq$ im: Suppose $e_n \to 0_{C_{n-1}}$, then $e_n \to a_n \to 0_{C_{n-1}}$ and $a_n \to 0_{B_n}$. It follows from exactness and commutativity that $\exists\, f_n \to a_n$ with $f_n \to 0_{B_n}$. Again from exactness $\exists\, d_n \to f_n$. Then $d_n \to e'_n \to a_n$ and $e'_n - e_n \to 0_{A_n}$. Thus $\exists\, b_n \to e'_n - e_n$ and $b_n \to d'_n \to e'_n - e_n$. We conclude $d_n - d'_n \to e_n$, which shows that $e_n \in \operatorname{im}(D_n \to E_n)$.

   (b) im $\subseteq$ ker: Let $d_n \to e_n$ then $d_n \to f_n \to a_n \to 0_{C_{n-1}}$. By exactness $\exists\, e_n \to a_n \to 0_{C_{n-1}}$ and hence $e_n \to 0_{C_{n-1}}$.

Problem F.4 (†). Let $X'' \subseteq X' \subseteq X$ be subspaces. Prove there is a long exact sequence

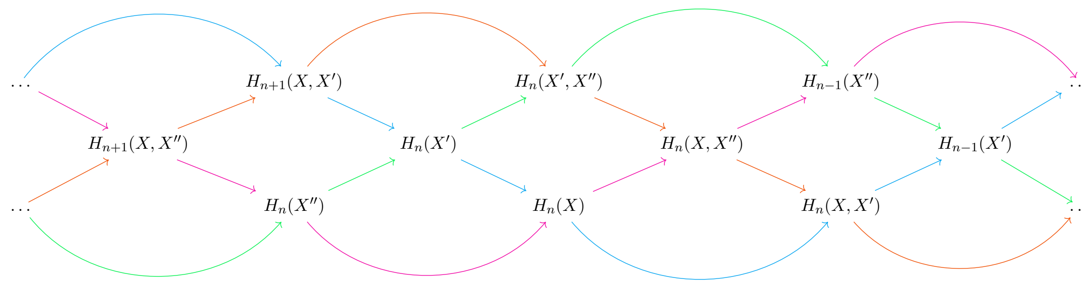
$$\dots H_n(X', X'') \to H_n(X, X'') \to H_n(X, X') \xrightarrow{\delta} H_{n-1}(X', X'') \to \dots$$

We call this the **long exact sequence of the triple** (X, X', X''). Moreover if we are given two triples (X, X', X'') and $(Y, Y', Y")$, together with a continuous map $f\colon X \to Y$ such that $f(X') \subseteq Y'$ and $f(X'') \subseteq Y''$, prove

$$\begin{array}{ccccccccc}
\dots \longrightarrow & H_n(X', X'') & \longrightarrow & H_n(X, X'') & \longrightarrow & H_n(X, X') & \longrightarrow & H_{n-1}(X', X'') & \longrightarrow \dots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\dots \longrightarrow & H_n(Y', Y'') & \longrightarrow & H_n(Y, Y'') & \longrightarrow & H_n(Y, Y') & \longrightarrow & H_{n-1}(Y', Y'') & \longrightarrow \dots
\end{array} \tag{F.1}$$

where all the vertical maps are induced by f.

SOLUTION. There exists a commutative braid



Three blue strand is the long exact sequence of the pair (X', X). The green strand is the long exact sequence of the pair (X'', X'). The pink strand is the long exact sequence of the pair (X'', X). The diagram commutes because homology is a functor and because of naturality of the connecting homomorphism. To apply Problem F.3 we check that the composition

$$H_n(X', X'') \to H_n(X, X'') \to H_n(X, X')$$

is zero. But this is clear since it factors through $H_n(X', X') = 0$. Thus by Problem F.3 the fourth strand is also exact.

To show naturality, i.e. that the diagram (F.1) commutes, it suffices to show that that the right-most square commutes, since the others obviously do as H_n is a functor. Reading off from the braid, the connecting homomorphism $H_n(X, X') \to H_{n-1}(X', X'')$ is the composition of the maps $H_n(X, X') \to H_{n-1}(X') \to H_{n-1}(X', X'')$, where the first map is the connecting homomorphism of the pair (X, X') and the second map is induced from the inclusion $(X', \emptyset) \hookrightarrow (X', X'')$. We already know that the connecting homomorphism $H_n(X, X') \to H_{n-1}(X')$ is a natural transformation, i.e. the diagram

$$\begin{array}{ccc} H_n(X, X') & \longrightarrow & H_{n-1}(X') \\ \downarrow & & \downarrow \\ H_n(Y, Y') & \longrightarrow & H_{n-1}(Y') \end{array}$$

commutes, where the vertical maps are induced by f, and hence it follows immediately that the right-most square also commutes.

Remark: It is also possible to prove this by using the short exact sequence of chain complexes

$$0 \to C_\bullet(X')/C_\bullet(X'') \to C_\bullet(X)/C_\bullet(X'') \to C_\bullet(X)/C_\bullet(X') \to 0$$

(this is a short exact sequence of chain complexes by Problem G.7). However the proof using the commutative braid is "better", since this does not use any properties of singular homology other than the long exact sequence (which is one of the axioms of a homology theory, as we will see at the end of the course.) Meanwhile this second proof uses the fact that singular homology is the homology of a chain complex (which is not true for an arbitrary homology theory.)

Problem F.5 (†). Let $\emptyset \neq X' \subseteq X$. Prove there is a long exact sequence

$$\cdots \to \tilde{H}_n(X') \to \tilde{H}_n(X) \to H_n(X, X') \to \tilde{H}_{n-1}(X') \to \dots$$

which ends with $\tilde{H}_0(X') \to \tilde{H}_0(X) \to H_0(X, X') \to 0$.

Solution. Let $X'' := \{p\}$, where p is a point in X'. Then the claim follows immediately from Problem F.4 and Corollary 12.22.

Problem F.6 (†). Suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence of abelian groups.

1. Prove that the sequence splits if and only if there exists a map $k \colon B \to A$ such that $kf = \mathrm{id}_A$.
2. Give an example of such a short exact sequence where $B \cong A \oplus C$ but such that the sequence does not split.
3. Prove that if C is free abelian then the sequence always splits.

Solution. Recall that by definition the sequence splits if and only if there exists a map $h : C \to B$ such that $gh = \mathrm{id}_C$.

1. (a) $\Rightarrow$: Define $k : B \to A$ by $k(b) : f^{-1}(b - hg(b))$. This is well-defined since $b - hg(b) \in \ker(g) = \mathrm{im}(f)$. Indeed $g(b - hg(b)) = g(b) - ghg(b) = g(b) - g(b) = 0$, since $gh = \mathrm{id}_C$. Moreover, $kf(a) = f^{-1}(f(a) - hgf(a)) = a$, since $gf = 0$.

   (b) $\Leftarrow$: For a $c \in C$, let b be any preimage of g. Define $h : C \to B$ by $h(c) = b - fk(b)$. This map is independent of the choice of the preimage b. Indeed, let b' be another preimage. Then $b - b' \in \ker(g) = \mathrm{im}(f)$ and hence it exists $a \in A$ such that $f(a) = b - b'$. Then $fk(b - b') = fkf(a) = f(a) = b - b'$, since $kf = \mathrm{id}_A$. Therefor $b - fk(b) - (b' - fk(b')) = b - b' - fk(b - b') = 0$. We verify: $gh(c) = g(b - fk(b)) = c$, since b is a preimage of c under g and $gf = 0$.
2. Let $A := \langle a | a^2 = 1 \rangle$ bet he cyclic group of order 2 and $B := \langle b | b^4 = 1 \rangle$. Consider the short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} A \to 0$, where $f(a) := 2b$ and $g(b) = a$. This sequence does not split, since B is not isomorphic to the direct product $A \oplus A$. Let $M := \bigoplus_{i=1}^{\infty} A \oplus B$. Clearly $A \oplus M \cong M \cong B \oplus M$ and we have a exact sequence $0 \to A \xrightarrow{\tilde{f}} B \oplus M \xrightarrow{\tilde{g}} A \oplus M \to 0$, where $\tilde{f}(a) := (f(a), 0)$ and $\tilde{g}(b, m) := (g(b), m)$. This sequence is exact and $B \oplus M \cong A \oplus (A \oplus M)$, but it is not split.
3. Choose a basis $\mathcal{C}$ of the free abelian group C. For every $c \in \mathcal{C}$ choose a preimage $b \in B$ of the surjective map g and define $h(c) = b$. Extend this by linearity (Lemma 7.2). Clearly $gh = \mathrm{id}_C$. Notice however that the map h is not unique.

Problem Sheet G

This Problem Sheet is based on Lecture 13 and Lecture 14. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem G.1. Give an explicit formula for $\mathrm{Sd}_n(\sigma)$ for $\sigma\colon \Delta^n \to X$ in the case $n = 0, 1, 2$.

Problem G.2 (†). Prove that if D is a convex bounded subset of some Euclidean space then Definition 13.7 agrees with Definition 13.6:

$$\mathrm{Sd}_n(\sigma) = \mathrm{Sd}_n^{\mathrm{cv}}(\sigma), \qquad \forall\, \sigma\colon \Delta^n \to D \text{ affine}.$$

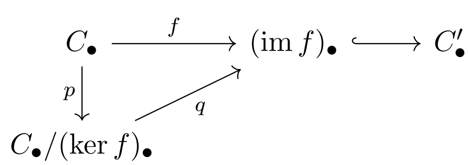
Problem G.3 (†). Show that the two forms of the excision axiom (Theorems 14.7 and Theorem 14.8) are equivalent.

Problem G.4. Let $C_\bullet$ be a subcomplex of $C'_\bullet$. Define a map $p_n\colon C'_n \to C'_n/C_n$ by $c \mapsto c + C_n$ (the coset). Show that the p_n's form a chain map $p\colon C'_\bullet \to C'_\bullet/C_\bullet$ with $(\ker p)_\bullet = C_\bullet$.

Problem G.5. Prove that the **first isomorphism theorem** holds in Comp. If $f\colon C_\bullet \to C'_\bullet$ is a chain map, prove there is an isomorphism of chain complexes:

$$q\colon C_\bullet/(\ker f)_\bullet \to (\operatorname{im} f)_\bullet$$

such that the following diagram commutes:



Here p is the map from Problem G.4.

Problem G.6 (†). Prove that the **second isomorphism theorem** holds in Comp: if $A_\bullet$ and $B_\bullet$ are subcomplexes of a chain complex $C_\bullet$ then as chain complexes, one has

$$\frac{A_\bullet}{A_\bullet \cap B_\bullet} \cong \frac{A_\bullet + B_\bullet}{B_\bullet}$$

Problem G.7 (†). Prove that the **third isomorphism theorem** holds in Comp: if $A_\bullet \subseteq B_\bullet \subseteq C_\bullet$ are subcomplexes, prove there is a short exact sequence of chain complexes

$$0 \to B_\bullet/A_\bullet \xrightarrow{i} C_\bullet/A_\bullet \xrightarrow{p} C_\bullet/B_\bullet \to 0$$

where $i_n\colon b_n \mapsto b_n + A_n$ is the inclusion and p_n is the map $c_n + A_n \mapsto c_n + B_n$.

Solutions to Problem Sheet G

This Problem Sheet is based on Lecture 13 and Lecture 14. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem G.1. Give an explicit formula for $\mathrm{Sd}_n(\sigma)$ for $\sigma\colon \Delta^n \to X$ in the case $n = 0, 1, 2$.

Solution. Let us first give an explicit formula for the affine barycentric subdivision.

Let D be a convex set and let $v_0, \dots, v_n$ be elements of D. Let $\sigma\colon \Delta^n \to D$ denote the affine singular n-simplex defined by $\sigma(e_i) = v_i$. By a slight abuse of notation we write $\sigma = [v_0, v_1 \dots, v_n]$. Note however that the v_i may not be affinely independent and hence $[v_0, v_1, \dots, v_n]$ may not be a genuine n-simplex.

1. Suppose $n = 0$. Then by definition $\mathrm{Sd}_0^{\mathrm{cv}}(\sigma) = \sigma$.
2. Suppose $n = 1$. Let $v_{01} := \frac{1}{2}(v_0 + v_1)$. Then
$$\mathrm{Sd}_1^{\mathrm{cv}}([v_0, v_1]) = [v_{01}, v_1] - [v_{01}, v_0].$$
3. Suppose $n = 2$. For $i < j$, let $v_{ij} := \frac{1}{2}(v_i + v_j)$ and let $v_{012} = \frac{1}{3}(v_0 + v_1 + v_2)$. Then:
$$\begin{aligned}\mathrm{Sd}_2^{\mathrm{cv}}([v_0, v_1, v_2] =& [v_{012}, v_{12}, v_2] - [v_{012}, v_{12}, v_1] - [v_{012}, v_{02}, v_2] \\ &+ [v_{012}, v_{02}, v_0] + [v_{012}, v_{01}, v_1] - [v_{012}, v_{01}, v_0]\end{aligned}$$
4. General case: Given a permutation g of $\{0, 1, \dots, n\}$, let v_i^g denote the barycentre of the $(n-i)$-simplex $[v_{g(i)}, \dots, v_{g(n)}]$:
$$v_i^g := b\big([v_{g(i)}, \dots, v_{g(n)}]\big)$$
(where by definition the barycentre of a 0-simplex $[v_i]$ is just v_i again.) Then by induction one can show that
$$\mathrm{Sd}_n^{\mathrm{cv}}(\sigma) = \sum_{g \in \mathfrak{S}(n+1)} \mathrm{sgn}(g)[v_0^g, v_1^g, \dots, v_n^g],$$
where $\mathfrak{S}(n+1)$ denotes the group of all permutations of $\{0, 1, \dots, n\}$ and $\mathrm{sgn}(g) \in \{\pm 1\}$ denotes the signature of the permutation.

Finally for arbitrary singular simplices σ we just apply the above with $v_i = e_i$ and compose with σ:

$$\begin{aligned}
\mathrm{Sd}_1(\sigma) =& \sigma \circ [e_{01}, e_1] - \sigma \circ [e_{01}, e_0], && \sigma : \Delta^1 \to X,\\
\mathrm{Sd}_2(\sigma) =& \sigma \circ [e_{012}, e_{12}, e_2] - \sigma \circ \circ [e_{012}, e_{12}, e_1] - \sigma \circ [e_{012}, e_{02}, e_2] \\
&+ \sigma \circ [e_{012}, e_{02}, e_0] + \sigma \circ [e_{012}, e_{01}, e_1] - \sigma \circ [e_{012}, e_{01}, e_0], && \sigma : \Delta^2 \to X,\\
\mathrm{Sd}_n(\sigma) =& \sum_{g \in \mathfrak{S}(n+1)} \mathrm{sgn}(g) \sigma \circ [e_0^g, e_1^g, \dots, e_n^g], && \sigma : \Delta^n \to X.
\end{aligned}$$

PROBLEM G.2 (†). Prove that if D is a convex bounded subset of some Euclidean space then Definition 13.7 agrees with Definition 13.6:

$$\mathrm{Sd}_n(\sigma) = \mathrm{Sd}_n^{\mathrm{cv}}(\sigma), \qquad \forall\, \sigma\colon \Delta^n \to D.$$

SOLUTION. For an affine n-simplex σ let F_1 denote the set of faces of σ. Inductively we define F_i to be the set of faces of elements in F_{i-1}. Notice that $F_{n+1} = \emptyset$ as σ is a n-simplex. Let B_i denote the set of barycentres of elements in F_i and set $B := \bigcup B_i$. Let $\mathrm{Sd}_n^{\mathrm{cv}}(\sigma) = \sum m_i \tau_i$ for $\tau_i \in C_n(X)$ and $m_i \in \mathbb{Z}$. Clearly the faces of affine simplices are affine. From the recursive formula in Definition 13.6 it follows that the τ_i are in 1-1 correspondence with n-simplexes spanned by n points in B and one vertex of σ. Now the claim follows from the fact that the barycentre of an affine n-simplex σ is equal to $\sigma(b_n)$, where b_n is the barycentre of the standard simplex. In other words, chopping up an affine simplex σ into smaller simplices is the same as first chopping up the standard simplex and then applying σ.

Here is an alternative argument. Using the notation from the previous question, if $\sigma = [v_0, v_1 \ldots, v_n]$ is an affine singular n-simplex then since

$$\sigma\left(\sum_{i=0}^{n} s_i\, e_i\right) = \sum_{i=0}^{n} s_i\, v_i, \qquad \forall \sum_{i=0}^{n} s_i = 1,$$

we see that

$$\sigma \circ [e_0^g, e_1^g, \ldots, e_n^g] = [v_0^g, v_1^g, \ldots, v_n^g], \qquad \forall\, g \in \mathfrak{S}(n+1),$$

and hence $\mathrm{Sd}_n^{\mathrm{cv}}(\sigma) = \mathrm{Sd}_n(\sigma)$.

PROBLEM G.3 (†). Show that the two forms of the excision axiom (Theorems 14.7 and Theorem 14.8) are equivalent.

SOLUTION. Assume that Theorems 14.7 holds. With $X = X$, $X_1 = X'$ and $X_2 = X \setminus X''$ Theorem 14.8 follows, as $X_1 \cap X_2 = X' \cap (X \setminus X'') = X' \setminus X''$. Conversely, if Theorem 14.8 holds, set $X = X$, $X' = X_1$ and $X'' = X \setminus X_2$. Then $X \setminus X'' = X \setminus (X \setminus X_2) = X \cap X_2 = X_2$ and $X' \setminus X'' = X_1 \setminus (X \setminus X_2) = X_1 \cap X_2$ and Theorem 14.7 follows.

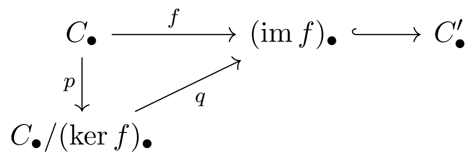
PROBLEM G.4. Let $C_\bullet$ be a subcomplex of $C'_\bullet$. Define a map $p_n\colon C'_n \to C'_n/C_n$ by $c \mapsto c + C_n$ (the coset). Show that the p_n's form a chain map $p\colon C'_\bullet \to C'_\bullet/C_\bullet$ with $(\ker p)_\bullet = C_\bullet$.

SOLUTION. The boundary operator of the complex $C'_\bullet/C_\bullet$ is given by $\partial'(c + C_n) = \partial'(c) + C_{n-1}$. Let ∂ denote the boundary operator of $C_\bullet$, which is just the restriction of the boundary operator ∂' of $C'_\bullet$. We need to check that p_n commutes with the boundary operators. Indeed, $p_{n-1} \circ \partial'(c) = \partial'(c) + C_{n-1} = \partial'(c + C_n) = \partial' \circ p_n$.

PROBLEM G.5. Prove that the **first isomorphism theorem** holds in Comp. If $f\colon C_\bullet \to C'_\bullet$ is a chain map, prove there is an isomorphism of chain complexes:

$$q\colon C_\bullet/(\ker f)_\bullet \to (\operatorname{im} f)_\bullet$$

such that the following diagram commutes:



Here p is the map from Problem G.4.

Solution. By the first Isomorphism Theorem for groups we have an isomorphsim

$$\begin{aligned} q : C_n \big/ \ker f_n &\to \mathrm{im}\, f_n \\ c + \ker f_n &\mapsto f(c) \end{aligned}$$

for every n. It is left to show that q commutes with the boundary operators. As f is a chain map we have $f(\partial(c)) = \partial'(f(c))$ and thus $q \circ \partial(c + \ker f_n) = q(\partial(c) + \ker f_{n-1}) = f(\partial(c)) = \partial'(f(c)) = \partial' \circ q(c + \ker f_n)$. Commutativity of the diagram follows directly from the definition of q.

Problem G.6 (†). Prove that the **second isomorphism theorem** holds in Comp: if $A_\bullet$ and $B_\bullet$ are subcomplexes of a chain complex $C_\bullet$ then as chain complexes, one has

$$\frac{A_\bullet}{A_\bullet \cap B_\bullet} \cong \frac{A_\bullet + B_\bullet}{B_\bullet}$$

Solution. Define

$$\begin{aligned} g : A_n + B_n &\to A_n / A_n \cap B_n \\ a + b &\to a + A_n \cap B_n. \end{aligned}$$

This map is well-defined as $a + b = a' + b$ if and only if $a - a' = b' - b \in A_n \cap B_n$. Moreover, g is surjective and $\ker g = B_n$ as $g(a + b) = a + A_n \cap B_n = 0 + A_n \cap B_n$ if and only if $a \in A_n \cap B_n$. It follows $a + b \in \ker g$ if and only if $a + b \in B_n$. The map g is a chain map, since $\partial \circ g(a + b) = \partial(a + A_n \cap B_n) = \partial(a) + A_{n-1} \cap B_{n-1} = g(\partial(a) + \partial(b)) = g(\partial(a + b))$, where we used the fact that for subcomplexes $D_\bullet \subseteq C_\bullet$ we have $\partial_{D_\bullet}(D_\bullet) \subseteq D_{\bullet-1}$

Problem G.7 (†). Prove that the **third isomorphism theorem** holds in Comp: if $A_\bullet \subseteq B_\bullet \subseteq C_\bullet$ are subcomplexes, prove there is a short exact sequence of chain complexes

$$0 \to B_\bullet / A_\bullet \xrightarrow{i} C_\bullet / A_\bullet \xrightarrow{p} C_\bullet / B_\bullet \to 0$$

where $i_n \colon b_n \mapsto b_n + A_n$ is the inclusion and p_n is the map $c_n + A_n \mapsto c_n + B_n$.

Solution. For every n, $p_n \circ i_n = p_n(b_n + A_n) = B_n$, hence $\mathrm{im}\, i_n \ker p_n$. Conversely, if $c_n + A_n \in \ker p_n$ then $c_n \in B_n$ and $i_n(c_n + A_n) = c_n + A_n$. The maps are chain maps, because all boundary operators are given by restriction of the boundary operator of $C_\bullet$.

Problem Sheet H

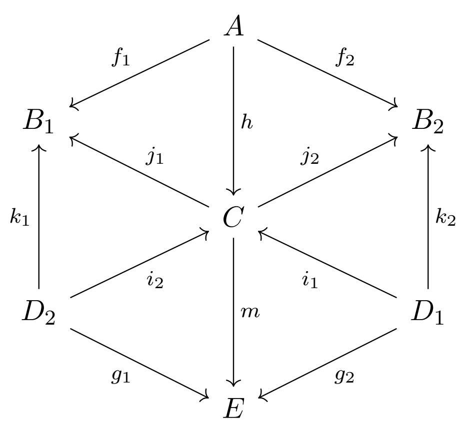
This Problem Sheet is based on Lecture 15 and Lecture 16. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem H.1 (†). Let X be path connected and let $\zeta\colon \pi_1(X,p) \to [S^1, X]$ be the function that sends a path class $[u]$ to the free homotopy class of the map $\hat{u}\colon S^1 \to X$ given by

$$\hat{u}(e^{2\pi i s}) := u(s), \qquad s \in I.$$

Prove this function is surjective. Moreover if $\zeta([u]) = \zeta([v])$, prove there exists $[w] \in \pi_1(X,p)$ such that $[u] = [w] * [v] * [w]^{-1}$. Thus if $\pi_1(X,p)$ is abelian then ζ is an isomorphism, and hence $\pi_1(X,p) \cong [S^1, X]$.

Problem H.2 (†). Prove the **Hexagon Lemma**: Suppose we have a commuting hexagon of abelian groups and group homomorphisms:



Assume that k_1 and k_2 are isomorphisms.

1. If $\operatorname{im} i_1 \subseteq \ker j_1$ and $\operatorname{im} i_2 = \ker j_2$, prove that the maps

$$\begin{aligned} D_2 \oplus D_1 \to C, \quad (x,y) &\mapsto i_2(x) + i_1(y), \\ C \mapsto B_1 \oplus B_2, \quad z &\mapsto (j_1(z), j_2(z)). \end{aligned}$$

   are both isomorphisms, and that in fact $\operatorname{im} i_1 = \ker j_1$.

2. Assume that $\operatorname{im} i_1 = \ker j_1$ and $\operatorname{im} i_2 = \ker j_2$ and that $\operatorname{im} h \subseteq \ker m$. Prove that $g_1 k_1^{-1} f_1 = -g_2 k_2^{-1} f_2$.

Problem H.3. Prove the **Borsuk-Ulam Theorem**: Let $f\colon S^n \to \mathbb{R}^n$ is continuous, there exists a point $x \in S^n$ such that $f(x) = f(-x)$. Deduce that S^n is not homeomorphic to any subspace[1] of $\mathbb{R}^n$.

[1]This result has several interesting physical connotations. Taking $n = 2$, this implies that a map of the earth cannot be drawn (homeomorphically) onto a page of an atlas. Alternatively, consider the function $f\colon \text{the earth} \to \mathbb{R}^2$ given by $f(\text{a place}) = (\text{temperature at that place}, \text{pressure at that place})$. Then there exists a pair of points on opposite sides of the world with the same temperature and pressure.

Problem H.4. Prove the **Lusternik-Schnirelmann Theorem**: Let $n \geq 1$ and assume we can write $S^n = A_1 \cup A_2 \cup \cdots \cup A_{n+1}$ where each A_i is either open or closed. Then at least one of the A_i contains a pair of antipodal points.

Problem H.5. Let J be a category with two objects and two non-identity morphisms:

$$\spadesuit \rightrightarrows \heartsuit$$

Let $T\colon \mathsf{J} \to \mathsf{Sets}$ be a functor. Prove that the colimit of T exists. This is type of colimit is called a **coequaliser**. Do the same with $T\colon \mathsf{J} \to \mathsf{Ab}$ and $T\colon \mathsf{J} \to \mathsf{Top}$.

Problem H.6 ($\star$). Prove that the coequaliser exists in the category of *compact Hausdorff* spaces.

Solutions to Problem Sheet H

This Problem Sheet is based on Lecture 15 and Lecture 16. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem H.1 (†). Let X be path connected and let $\zeta\colon \pi_1(X,p) \to [S^1, X]$ be the function that sends a path class $[u]$ to the free homotopy class of the map $\hat{u}\colon S^1 \to X$ given by

$$\hat{u}(e^{2\pi i s}) := u(s), \qquad s \in I.$$

Prove this function is surjective. Moreover if $\zeta([u]) = \zeta([v])$, prove there exists $[w] \in \pi_1(X,p)$ such that $[u] = [w] * [v] * [w]^{-1}$. Thus if $\pi_1(X,p)$ is abelian then ζ is an isomorphism, and hence $\pi_1(X,p) \cong [S^1, X]$.

Solution. We consistently use the hat notation to go back and forth between loops $u\colon (I, \partial I) \to (X,p)$ and maps $\hat{u}\colon S^1 \to X$.

First note that every map $\hat{u}\colon S^1 \to X$ is homotopic to a map $\hat{v}\colon S^1 \to X$ such that $\hat{v}(1) = p$. Indeed, if w is a path from $u(1)$ to p then the homotopy

$$U(s,t) = \begin{cases} w(t-3s), & 0 \le s \le \frac{t}{3}, \\ u\big(e^{2\pi i (\frac{3s-t}{3-2t})}\big), & \frac{t}{3} \le s \le \frac{3-t}{3}, \\ w(3s+t-3), & \frac{3-t}{3} \le s \le 1, \end{cases}$$

is such that $U(s,0) = u(e^{2\pi i s})$ and $U(s,1)$ is the product loop $\bar{w} * u * w$. Thus the homotopy $V\colon S^1 \times I \to X$ given by $V(e^{2\pi i s}, t) = U(s,t)$ starts at $\hat{u}$ and ends at a map $\hat{v}$ such that $\hat{v}(1) = w(1) = p$. This shows that ζ is surjective.

Let us now assume that $\zeta([u]) = \zeta([v])$; then we have a homotopy $V\colon S^1 \times I \to X$ such that $V(e^{2\pi i s}, 0) = u(s)$ and $V(e^{2\pi i s}, 1) = v(s)$. Thus the path $w\colon I \to X$ given by $w(t) = V(1,t)$ is a loop representing an element $[w] \in \pi_1(X,p)$. The homotopy

$$W(s,t) = \begin{cases} U(2(1-t)s, 2st), & 0 \le s \le \frac{1}{2}, \\ U(1+2t(s-1), t+(1-t)(2s-1)), & \frac{1}{2} \le s \le 1, \end{cases},$$

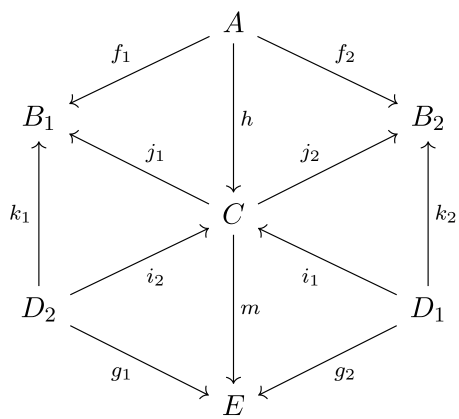
where $U(s,t) := V(e^{2\pi i s}, t)$, shows that $u * w \simeq w * v$.

Conversely, if $u * w = w * v$, then there exists a homotopy $U\colon u \simeq w * v * \bar{w}$ rel ∂I. So $V(e^{2\pi i s}, t) := U(s,t)$ is a well-defined homotopy from $\hat{u}$ to $\widehat{w * v * \bar{w}}$. Moreover the homotopy

$$Z(s,t) = \begin{cases} w(3s+t) & 0 \le s \le \frac{1-t}{3} \\ v(\frac{3s+t-1}{1+2t}) & \frac{1-t}{3} \le \frac{2+t}{3} \\ w(3-3s+t) & \frac{2+t}{3} \le s \le 1 \end{cases}$$

is such that $Z\colon w * v * \bar{w} \simeq v$ and $Z(0,t) = w(t) = Z(1,t)$; therefore, it defines a homotopy $Y\colon S^1 \times I \to X$ via $Y(e^{2\pi i s}, t) := Z(s,t)$ which starts at $\widehat{w * v * \bar{w}}$ and ends at $\hat{v}$. Now compose the homotopies U and Y to obtain one from $\hat{u}$ to $\hat{v}$. Thus $\zeta([u]) = \zeta([v])$. This completes the proof.

PROBLEM H.2 (†). Prove the **hexagon lemma**: Suppose we have a commuting hexagon of abelian groups and group homomorphisms:



Assume that k_1 and k_2 are isomorphisms.

1. If $\operatorname{im} i_1 \subseteq \ker j_1$ and $\operatorname{im} i_2 = \ker j_2$, prove that the maps

$$\begin{array}{lll} D_2 \oplus D_1 \to C, & (x,y) & \mapsto i_2(x) + i_1(y), \\ C \mapsto B_1 \oplus B_2, & z & \mapsto (j_1(z), j_2(z)). \end{array}$$

   are both isomorphisms, and that in fact $\operatorname{im} i_1 = \ker j_1$.

2. Assume that $\operatorname{im} i_1 = \ker j_1$ and $\operatorname{im} i_2 = \ker j_2$ and that $\operatorname{im} h \subseteq \ker m$. Prove that $g_1 k_1^{-1} f_1 = -g_2 k_2^{-1} f_2$.

SOLUTION.

1. As $j_1 i_2 = k_1$ is an isomorphism, $\ker(j_1) \cap \operatorname{im}(i_2) = \{0\}$. Since $\operatorname{im}(i_2) = \ker(j_2)$ is follows that

$$\ker(j_2) \cap \ker(j_1) = \{0\}. \tag{H.1}$$

   For a $c \in C$ let $\overline{c} = i_1 k_2^{-1} j_2(c) + i_2 k_1^{-1} j_1(c)$ then $j_1(\overline{c}) = j_1 i_2 k_1^{-1} j_1(c) = j_1(c)$ and similarly $j_2(\overline{c}) = j_2(c)$. Hence $c = \overline{c}$ by H.1. Therefore every $c \in C$ has a representation of the form $c = i_1(d_1) + i_2(d_2)$, where $d_1 \in D_1$ and $d_2 \in D_2$. It is left to show the uniqueness of this representation. For any representation of the form $c = i_1(d_1) + i_2(d_2)$ we have $j_2(c) = j_2 i_1(d_1) + j_2 i_2(d_2) = k_2(d_1)$ and hence $d_1 = k_2^{-1} j_2(c)$. Similarly $d_2 = k_1^{-1} j_1(c)$ and the representation is unique.

   To prove the second part take $b_1 \in B_1$ and $b_2 \in B_2$ and set $c = i_2 k_1^{-1}(b_1) + i_1 k_2^{-1}(b_2)$. Then $j_1(c) = j_1 i_2 k_1^{-1}(b_1) = k_1 k_1^{-1}(b_1) = b_1$ and $j_2(c) = b_2$. Hence for every (b_1, b_2) there exists a $c \in c$ such that $(j_1(c), j_2(c)) = (b_1, b_2)$. Uniqueness follows from H.1.

   Last but not least, if $c \in \ker(j_1)$ then $i_1 k_2^{-1} j_2(c) \in \ker j_1$ as $j_1 i_1 = 0$. Moreover $j_2 i_1 k_2^{-1} j_2(c) = k_2 k_2^{-1} j_2(c) = j_2(c)$. Hence $i_1 k_2^{-1} j_2(c) - c \in \ker(j_1) \cap \ker(j_2) = \{0\}$ by H.1 and $c =\in \operatorname{im}(i_1)$.

2. Let $a \in A$. By part 1 and commutativity of the diagram

$$\begin{aligned} h(a) &= i_2 k_1^{-1} j_1 h(a) + i_1 k_2^{-1} j_2 h(a) \\ &= i_2 k_1^{-1} f_1(a) + i_1 k_2^{-1} f_2(a). \end{aligned}$$

Applying m to both sides gives

$$\begin{aligned} 0 &= mh(a) \\ &= m i_2 k_1^{-1} f_1(a) + m i_1 k_2^{-1} f_2(a) \\ &= g_1 k_1^{-1} f_1(a) + g_2 k_2^{-1} f_2(a), \end{aligned}$$

where the last equality uses commutativity.

Problem H.3. Prove the **Borsuk-Ulam Theorem**: If $f\colon S^n \to \mathbb{R}^n$ is continuous, there exists a point $x \in S^n$ such that $f(x) = f(-x)$. Deduce that S^n is not homeomorphic to any subspace[1] of $\mathbb{R}^n$.

Solution. Assume that $f(-x) \neq f(x) \forall x$. Then the map

$$\begin{aligned} g\colon S^n &\to S^{n-1} \\ x &\mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|} \end{aligned}$$

is well-defined. Moreover, we can see that g is odd. Then $g|_{S^{n-1}}\colon S^{n-1} \to S^{n-1}$ is also odd and hence has odd degree by Theorem 15.12. But $g|_{S^{n-1}}$ extends to the upper hemisphere of S^n and thus it has degree zero by Proposition 2.15. But zero is an even degree, which is a contradiction.

Problem H.4. Prove the **Lusternik-Schnirelmann Theorem**: Assume we can write $S^n = A_1 \cup A_2 \cup \dots \cup A_{n+1}$ where each A_i is either open or closed. Then at least one of the A_i contains a pair of antipodal points.

Solution.

1. First we consider the case that all A_i are closed. Define the map

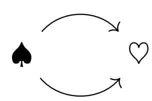
$$\begin{aligned} f\colon S^n &\to \mathbb{R}^n \\ x &\to (\operatorname{dist}(x, A_1), \dots, \operatorname{dist}(x, A_n)). \end{aligned}$$

By the Borsuk-Ulam Theorem there exists a point $x \in S^n$ with $f(x) = f(-x)$. If the i-th coordinate of x is zero, then both x and $-x$ are in A_i. If all coordinates are non-zero, then x and $-x$ lie in A_{n+1}.

[1]This result has several interesting physical connotations. Taking $n = 2$, this implies that a map of the earth cannot be drawn (homeomorphically) onto a page of an atlas. Alternatively, consider the function $f\colon$ the earth $\to \mathbb{R}^2$ given by $f(\text{a place}) = (\text{temperature at that place}, \text{pressure at that place})$. Then there exists a pair of points on opposite sides of the world with the same temperature and pressure.

2. If all A_i are open, then we find slighly smaller closed sets $B_i \subset A_i$ such that $S^n = A_1 \cup A_2 \cup \cdots \cup A_{n+1}$ and we apply the first case.
3. Suppose now that $A_1, \ldots, A_m$ are closed and $A_{m+1}, \ldots, A_{n+1}$ are open. Assume that none of the set A_i for $i = m+1, \ldots, n+1$ contains antipodal points. For $i = 1, \ldots, m$ define the open sets $C_{i,1/k} := \{x \in S^n \mid \operatorname{dist}(x, A_i) < 1/k\}$, where $k \in \mathbb{N}$. By the first case, there exists for every k an $i \in \{1, \ldots, m\}$ such that $C_{i,1/k}$ contains antipodal points. As $A_1, \ldots, A_m$ are all closed, we see that, by taking the limit for $k \to 0$, there exists an $i \in \{1, \ldots, m\}$ such that A_i contains antipodal points.

Problem H.5. Let J be a category with two objects and two non-identity morphisms:



Let $T\colon \mathsf{J} \to \mathsf{Sets}$ be a functor. Prove that the colimit of T exists. This is type of colimit is called a **coequaliser**. Do the same with $T\colon \mathsf{J} \to \mathsf{Ab}$ and $T\colon \mathsf{J} \to \mathsf{Top}$.

Solution. Before proving the problem, let us insert a definition:

Definition. Let X be a set and suppose R is an arbitrary binary relation on X (i.e. R is simply a subset of $X \times X$, and we think of two elements x, y in X being related if $(x, y) \in R$.) The **equivalence relation $\sim$ generated by** R is by definition the smallest equivalence relation on X that preserves all the relations specified by R. Explicitly, one defines $\sim$ by saying that $x \sim y$ if there exists a sequence $x = x_0, x_1, , \ldots, x_n = y$ such that for each $i = 1, \ldots, n$, either:

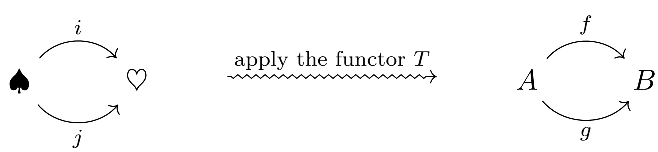
(i) $x_i = x_{i-1}$ or

(ii) $(x_{i-1}, x_i) \in R$ or

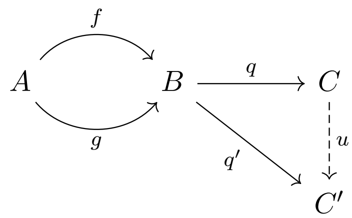
(iii) $(x_i, x_{i-1}) \in R$.

We now prove the problem.

1. Let the two morphisms in J be i and j, and let $A = T(\spadesuit)$, $B = T(\heartsuit)$. Write $f = T(i)$ and $g = T(j)$, so the picture is this:



Let $\sim$ be the equivalence relation on B generated by the relation R on B given by $R = \{(f(a), g(a)) \mid a \in A\}$. Let $q\colon B \to C$ denote the map that sends $b \to [b]$, where $[b]$ denotes the equivalence class. Then $q \circ f = q \circ g$ and C is a solution with $c_\heartsuit = q$ and $c_\spadesuit = q \circ f = q \circ g$. To check that C is the colimit, assume we have another space C' and another map $q'\colon B \to C'$ such that $q' \circ f = q' \circ g$. We need to find a unique map $u\colon C \to C'$ such that the following commutes:



Define $u([b]) := q'(b)$, where b is any element in the equivalence class $[b]$. In order for this to be well-defined, we need to know that if $b_1 \sim b_2$ then $q'(b_1) = q'(b_2)$. But this is clear from the definition of $\sim$ from R. The map u satisfies $u \circ q = q'$ and is unqiue by construction.

2. The proof for Ab is similar. With the notation as above, first consider the special case where g is the zero homomorphism. Then the coequaliser of f and 0 is simply the quotient of B by the subgroup $f(A)$. In the general case, the coequaliser of f and g is the coequaliser of $f - g$ and the zero homomorphism.

3. Finally to obtain the coequaliser in Top we simply take the coqualiser constructed in Sets and endow it with the quotient topology determined by q.

PROBLEM H.6 ($\star$). Prove that the coequaliser exists in the category of *compact Hausdorff* spaces.

SOLUTION. In the category of Hausdorff spaces the coequaliser of two continuous mappings $f, g\colon X \to Y$ is the quotient Z of Y by the *closure* $\overline{R} \subset Y \times Y$ of the equivalence relation R generated by the pairs $(f(x), g(x))$ for $x \in X$[2]. Elementary point-set topology tells us that the quotient of a compact Hausdorff space Y by a closed relation is another Hausdorff space. Moreover since Y is compact the quotient Z is also compact (being the continuous image of a compact set). Thus Z does indeed belong to the desired category (this is the most important thing to check!)

Now take another compact Hausdorff space W and suppose we are given a continuous map $h\colon Y \to W$ such that $h \circ f = h \circ g$. The diagonal $\Delta \subset W \times W$ is closed (again recall from the fact from point-set topology that a space is Hausdorff if and only if the diagonal is closed). Thus $(h \times h)^{-1}(\Delta)$ is a closed equivalence relation on Y containing all the pairs $(f(x), g(x))$ for $x \in X$. Thus in particular it contains R and so h factors through the quotient, giving us the desired (necessarily unique) continuous map $u\colon Z \to W$. Hence Z is indeed the colimit.

[2]Here $\overline{R}$ is the smallest closed equivalence relation containing R. This may *not* be equal to the actual topological closure of R, as the latter may not be transitive.

Problem Sheet I

This Problem Sheet is based on Lecture 17 and Lecture 18. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

Problem I.1 (†). Prove the **Invariance of Domain Theorem**: Suppose U and U' are two subsets of S^n and $f\colon U \to U'$ is a homeomorphism. If U is open then so is U'. *Hint:* Use the Jordan-Brouwer Separation Theorem.

Problem I.2. Prove that if $\mathbb{R}^n$ contains a subspace homeomorphic to $\mathbb{R}^m$ then $m \leq n$.

Problem I.3 (†). Recall the definition of a weakly Hausdorff space from Definition 17.1.

1. Prove that any weakly Hausdorff space is a T_1 space. Give an example of a T_1 space which is not weakly Hausdorff.
2. Prove that any Hausdorff space is weakly Hausdorff. Give an example of a weakly Hausdorff space which is not Hausdorff.
3. Let X be a weakly Hausdorff space and let K be a compact Hausdorff space. Assume $f\colon K \to X$ is continuous. Prove that $f(K)$ is a compact Hausdorff subspace of X with respect to the subspace topology.

Problem I.4 (†). Let X and Y be topological spaces, and let $X' \subseteq X$ be a closed subspace. Let $f\colon X' \to Y$ be continuous, and let $X \cup_f Y$ denote the adjunction space. The canonical inclusions $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$ induce maps $g\colon X \to X \cup_f Y$ and $j\colon Y \to X \cup_f Y$. Prove that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \imath}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow[\ g\ ]{} & X \cup_f Y \end{array}$$

is a pushout in Top. Then prove:

1. The map j is a closed embedding,
2. The restriction of g to $X \setminus X'$ is an open embedding.
3. If X and Y are T_1 spaces then so is $X \cup_f Y$.
4. The quotient map $X \sqcup Y \to X \cup_f Y$ is closed if and only if f is closed.
5. If X and Y are Hausdorff and $X' \subseteq X$ is compact then $X \cup_f Y$ is Hausdorff.
6. If X is compact and $X \cup_f Y$ is Hausdorff then $X \mapsto g(X)$ is a quotient map.

Problem I.5 (†). Compute the homology of $\mathbb{C}P^n$. Why doesn't this work for $\mathbb{R}P^n$?[1]

Problem I.6 (†). Prove that for any $m, n \geq 0$, the space $S^m \times S^n$ can be obtained from $S^m \vee S^n$ by attaching a $(m+n)$-cell. Use this to compute the homology $H_\bullet(S^n \times S^m)$ for all $n, m \geq 0$.

[1] We will eventually be able to compute the homology of $\mathbb{R}P^n$, but not until Lecture 21.

Solutions to Problem Sheet I

This Problem Sheet is based on Lecture 17 and Lecture 18. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

PROBLEM I.1 (†). Prove the **Invariance of Domain Theorem**: Suppose U and U' are two subsets of S^n and $f\colon U \to U'$ is a homeomorphism. If U is open then so is U'. *Hint:* Use the Jordan-Brouwer Separation Theorem.

SOLUTION. Let $p \in f(U) = U'$. Then there exists an $x \in U$ such that $f(x) = p$. Let $B_\epsilon(x) \subset U$ be an open ball such that $\overline{B_\epsilon(x)} \subset U$. Consider $S := f(\partial \overline{B_\epsilon(x)}) \subset S^n$ an embedded $n-1$-sphere. By Theorem 17.11 $S^n \setminus S$ has two components, say X and Y, and S is the boundary of both X and Y. As $f(B_\epsilon(x)) \cap S = \emptyset$ and $f(B_\epsilon(x))$ is connected we have w.l.o.g. that $f(B_\epsilon(x)) \subset X$. Then $f(\overline{B_\epsilon(x)}) \subset \overline{X} = Y^c$, which is equivalent to saying that $Y \subset S^n \setminus f(\overline{B_\epsilon(x)})$. But Y is a path component of $S^n \setminus S = S \setminus f(\partial \overline{B_\epsilon(x)})$ hence $Y = S^n \setminus f(\overline{B_\epsilon(x)})$ and $S^n \setminus \overline{X} \subset S^n \setminus f(\overline{B_\epsilon(x)}) = Y$. This implies that $\overline{X} = f(\overline{B_\epsilon(x)})$ and $X = f(B_\epsilon(x))$. Therefore $p \in f(B_\epsilon(x)) = X \subset U'$, where X is an open neighborhood of p in U'.

PROBLEM I.2. Prove that if $\mathbb{R}^n$ contains a subspace homeomorphic to $\mathbb{R}^m$ then $m \leq n$.

SOLUTION. Suppose there exists a homeomorphism $f\colon \mathbb{R}^m \to U$, where U is open in $\mathbb{R}^n$ and $m > n$. The inclusion

$$\begin{aligned} i\colon \mathbb{R}^n &\to \mathbb{R}^m \\ (x_1, \dots, x_n) &\to (x_1, \dots, x_n, 0, \dots, 0) \end{aligned}$$

is continuous and injective. Then the composition $f \circ i\colon \mathbb{R}^n \to \mathbb{R}^n$ is also continuous and injective. But $\operatorname{im}(f \circ i)$ is not open in $\mathbb{R}^n$ since the image of i is contained in the hyperplane $\mathbb{R}^n \times 0 \subset \mathbb{R}^m$. The one point compactification of $\mathbb{R}^n$ is the sphere S^n and $U := \mathbb{R}^n$ is an open subset of S^n. Then $f \circ i\colon U \to f \circ i(U)$ is a homeomorphism. But U is open in S^n and $\operatorname{im}(f \circ i) \in S^n$ is non-open. This contradicts the Invariance of Domain Theorem.

PROBLEM I.3 (†). Recall the definition of a weakly Hausdorff space from Definition 17.1.

1. Prove that any weakly Hausdorff space is a T_1 space. Give an example of a T_1 space which is not weakly Hausdorff.
2. Prove that any Hausdorff space is weakly Hausdorff. Give an example of a weakly Hausdorff space which is not Hausdorff.

3. Prove that if X be a weakly Hausdorff space and $f\colon K \to X$ is a continuous map from a compact Hausdorff space then $f(K)$ is a compact Hausdorff subset of X.

SOLUTION.

1. For every $p \in X$ take the continuous map

$$\begin{aligned} f\colon \{*\} &\to X \\ * &\mapsto p. \end{aligned}$$

As $\{*\}$ is a compact Hausdorff space, $f(\{*\}) = \{p\}$ is closed.

We take the real line with two copies of the origin. This is the quotient of two copies of the real line, where corresponding non-zero points on the two lines are identified. This space is certainly T_1 with respect to the quotient topology. It is however not weakly Hausdorff. To see this consider the interval $[-1,1]$ with only one origin. This is the image of an embedding of a compact Hausdorff space but it is not closed, as the second origin is not contained.

2. Let $p \in f(K)^c$ and define for every $q \in f(K)$ two disjoint open neighbourhoods $q \in V_q$ and $p \in U_q$. The set V_q for all $q \in K$ form an open cover of $f(K)$. As K is compact and f is continuous also $f(K)$ is compact. Hence we find $q_1, \ldots, q_n$ such that $f(K) \subset V_{q_1} \cup \ldots \cup V_{q_n}$. Now $U := U_{q_1} \cap \ldots \cap U_{q_n} \subset f(K)^c$ is an open neighbourhood of p and disjoint form $f(K)$. Since $p \in f(K)^c$ was arbitrary, $f(K)^c$ is open and hence $f(K)$ is closed.

We define the co-countable topology on an uncountable set S by defining the open sets to be the sets who's complement is countable. Let S_1 and S_2 be two non-empty open sets. Then the complement of S_1 is countable and S_2 is uncountable. Hence S_1 and S_2 must intersect, which shows that S is not Hausdorff. Let $f\colon K \to X$ be a continuous map from a compact Hausdorff space. As K is compact also $f(K)$ is compact. A compact subset A of S must be countable. Indeed, for every $p \in S$ take the open set defined by the complement of p. This defines an open cover of A, which has a finite subcover if and only if A is finite. We conclude that $f(K)$ must be finite and thus also closed, which shows that S is weakly Hausdorff.

3. It is clear that $L := f(K)$ is compact with respect to the subspace topology. We need to show that L is Hausdorff. Note that f is a closed map: if $F \subseteq K$ is closed then F is a compact Hausdorff space and hence $f(F)$ is closed. Now let $x \neq y \in L$. Then $f^{-1}(x)$ and $f^{-1}(y)$ are are disjoint closed subspaces of the compact Hausdorff space K, and hence admit disjoint open neighbourhoods U and V respectively. Set

$$U' := \{z \in X \mid f^{-1}(z) \subseteq U\} = L \setminus f(K \setminus U),$$

and

$$V' := \{z \in X \mid f^{-1}(z) \subseteq V\} = L \setminus f(K \setminus V).$$

Then $x \in U'$ and $y \in V'$. Moreover $U' \cap V' = \emptyset$. Since $K \setminus U$ is compact, $f(K \setminus U)$ is closed in L, and hence U' is open in L. Similarly V' is open in L. Thus L is Hausdorff. This completes the proof.

PROBLEM I.4 (†). Let X and Y be topological spaces, and let $X' \subseteq X$ be a closed subspace. Let $f\colon X' \to Y$ be continuous, and let $X \cup_f Y$ denote the adjunction space. The canonical inclusions $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$ induce maps $g\colon X \to X \cup_f Y$ and $j\colon Y \to X \cup_f Y$. Prove that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \imath}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xrightarrow[\ g\ ]{} & X \cup_f Y \end{array}$$

is a pushout in Top. Then prove:

1. The map j is a closed embedding,
2. The restriction of g to $X \setminus X'$ is an open embedding.
3. If X and Y are T_1 spaces then so is $X \cup_f Y$.
4. The quotient map $X \sqcup Y \to X \cup_f Y$ is closed if and only if f is closed.
5. If X and Y are Hausdorff and $X' \subseteq X$ is compact then $X \cup_f Y$ is Hausdorff.
6. If X is compact and $X \cup_f Y$ is Hausdorff then $X \mapsto g(X)$ is a quotient map.

SOLUTION. We already know that the set-theoretic pushout exists. Let $i_X\colon X \hookrightarrow X \sqcup Y$ and $i_Y\colon Y \hookrightarrow X \sqcup Y$ denote the two inclusions and $p\colon X \sqcup Y \to X \cup_f Y$ the quotient map. Notice that $g = p \circ i_X$ and $j = p \circ i_Y$. The topology on $X \cup_f Y$ is the quotient topology of $X \sqcup Y/\sim$, hence $p\colon X \sqcup Y \to X \sqcup Y/\sim$ is continuous and hence also g and j are continuous. Given another pushout C, then there exists a unique morphism $\phi\colon X \cup_f Y \to C$ in the set-theoretic sense. But since the topology of $X \cup_f Y$ is the smallest topology such that the maps g and j are continuous, the map ϕ will automatically be continuous. Let $[x]$ denote the equivalence class of $x \in X \sqcup Y/\sim$.

1. Since j is injective, to show that j is a closed embedding it suffices to show that $j(C)$ is closed in $X \cup_f Y$ whenever C is closed in Y. Since $X \cup_f Y$ has (by definition) the quotient topology induced by p, a set $B \subseteq X \cup_f Y$ is closed if and only if both $g^{-1}(B)$ and $j^{-1}(B)$ is closed. Since j is injective, $j^{-1}(j(C)) = C$, and thus it suffices to check that $g^{-1}(j(C))$ is closed. But $g^{-1}(j(C)) = \imath(f^{-1}(C))$, which is closed as $\imath$ is a closed embedding.
2. The image $[X \setminus X']$ of g is the quotient of the set $X \setminus X'$. The map

$$\begin{aligned} [X \setminus X'] &\to X \setminus X' \\ [x] &\mapsto x \end{aligned}$$

is a well-defined, continuous inverse of $g|_{X \setminus X'}$. Hence $g|_{X \setminus X'}$ is a homeomorphism onto its image. Moreover, if U is an open neighbourhood of $x \in X \setminus X'$ then U is open in $X \sqcup Y$ as $X \setminus X'$ open in X. Then we have $p^{-1}([U]) = U$, since $U \in x \setminus X'$. This is open in $X \sqcup Y$.

3. A point $q \in X \cup_f Y$ is closed if and only if $p^{-1}([q])$ is closed in $X \sqcup Y$. If $q \in Y$ then $p^{-1}([q]) = f^{-1}(q) \sqcup q$, which is closed. If $q \in X \setminus X'$ then $p^{-1}([q]) = q$ is also closed.
4. Let $Z := Z_1 \sqcup Z_2 \subset X \sqcup Y$. Then one checks that
$$p^{-1}([Z]) = Z \cup f(Z_1 \cap X') \cup f^{-1}(f(Z_1 \cap X')) \cup f^{-1}(Z_2).$$
Thus $[Z]$ is closed in $X \cup_f Y$ if and only if $f(Z_1 \cap X') \cup Z_2$ is closed in Y. It follows that p is closed if and only if f is closed.
5. Notice that if X' is closed and Y is Hausdorff then the quotient map $p\colon X \sqcup Y \to X \cup_f Y$ is closed. Indeed, let $Z \subset X'$ be closed, then Z is compact since X' is compact. Hence $f(Z) \subset Y$ is a compact subset of a Hausdorff space and thus f is closed. Then by the previous part the quotient map p is also closed.

   Now let $a, b \in X \cup_f Y$ be distinct points. Then $p^{-1}(a)$ and $p^{-1}(b)$ are disjoint non-empty compact subsets of $X \sqcup Y$. Since $X \sqcup Y$ is Hausdorff (as X and Y are), there exist disjoint open sets U, V with $p^{-1}(a) \subset U$ and $p^{-1}(b) \subset V$. Since p is closed, there exist[1] open subsets U' and V' of $X \cup_f Y$ with $a \in U'$, $b \in V'$ and $p^{-1}(U') \subset U$, $p^{-1}(V') \subset V$. Moreover the sets U' and V' are disjoint as U and V were.
6. In order to show that $g\colon X \to g(X)$ is a quotient map we need to prove it is surjective and a set $[Z] \subset g(X)$ is closed if and only if $g^{-1}([Z])$ is closed. As X is compact an g continuous also $g(X)$ is compact and hence closed, since $X \cup_f Y$ is Hausdorff. Conversely, if $[Z] \subset g(X)$ is closed then $p^{-1}([Z])$ is closed in X by the definition of the topology of $X \cup_f Y$. But as $Z \subset g(X)$ it follows that $g^{-1}([Z]) = i_x^{-1} \circ p^{-1}([Z])$ is closed in X.

Problem I.5 (†). Compute the homology of $\mathbb{C}P^n$. Why doesn't this work for $\mathbb{R}P^n$?[2]

Solution. By induction we want to show
$$H_i(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z} & i = \text{ even}, i \leq 2n \\ 0 & \text{else.} \end{cases}$$
Notice that $\mathbb{C}P^0 = S^1/\sim$ is a point and thus
$$H_i(\mathbb{C}P^0) \cong \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = \text{else} \end{cases}$$

[1] This is a general fact about closed maps: if $h\colon X \to Y$ is a closed map between two topological spaces, $W \subset Y$, and $U \subset X$ is an open subset of X with $h^{-1}(W) \subset U$ then there exists an open set $V \subset Y$ with $W \subset V$ and $h^{-1}(V) \subset U$. Indeed, take
$$V := Y \setminus h(X \setminus U).$$
Since U is open, $X \setminus U$ is closed, and hence as h is closed so is $h(X \setminus U)$. Thus V is open. Since $h^{-1}(W) \subset U$ we have $W \cap h(X \setminus U) = \emptyset$, and thus $W \subset V$. Moreover $h^{-1}(V) = X \setminus h^{-1}(h(X \setminus U)) \subset X \setminus (X \setminus U) = U$ as required.

[2] We will eventually be able to compute the homology of $\mathbb{R}P^n$, but not until Lecture 21.

By Example 18.9 we know that $\mathbb{C}P^n \cong B^{2n} \cup_p \mathbb{C}P^{n-1}$, where $p\colon S^{2n-1} \to \mathbb{C}P^{n-1}$ is the quotient map. By Proposition 18.13 we have an exact sequence

$$\begin{aligned} \cdots \to H_{2n}(\mathbb{C}P^{n-1}) \to H_{2n}\big(B^{2n} \cup_f \mathbb{C}P^{n-1}\big) \to H_{2n-1}(S^{2n-1}) \\ \to H_{2n-1}(\mathbb{C}P^{n-1}) \to H_{2n-1}\big(B^{2n} \cup_f \mathbb{C}P^{n-1}\big) \to H_{2n-2}(S^{2n-1}) \to \dots \end{aligned}$$

But as $H_{2n}(\mathbb{C}P^{n-1}) = 0$, $H_{2n-1}(\mathbb{C}P^{n-1}) = 0$ and $H_{2n-2}(S^{2n-1}) = 0$, we have an isomorphism

$$H_{2n}(B^{2n} \cup_p \mathbb{C}P^{n-1}) \cong H_{2n-1}(S^{2n-1}) \cong \mathbb{Z}$$

and

$$H_{2n-1}(B^{2n} \cup_p \mathbb{C}P^{n-1}) = 0.$$

Moreover, by Corollary 18.14 $H_i(\mathbb{C}P^n) \cong H_i(\mathbb{C}P^{n-1})$ for $i \leq 2n-2$.

This argument breaks down for $\mathbb{R}P^n$ if $n = 2$.

Problem I.6 (†). Prove that for any $n, m \geq 0$, the space $S^m \times S^n$ can be obtained from $S^m \vee S^n$ by attaching a $(m+n)$-cell. Use this to compute the homology $H_\bullet(S^n \times S^m)$ for all $n, m \geq 0$.

Solution. Notice that $B^{m+n} \cong B^m \times B^n$ and $\partial(B^m \times B^n) = \partial(B^m) \times B^n \cup B^m \times \partial(B^n) \cong S^{m-1} \times B^n \cup B^m \times S^{n-1}$. Let $\pi_n\colon B^n \to B^n/S^{n-1} \cong S^n$ denote the quotient map. Assume that $S^m \vee S^n = (S^m \times \{p\}) \cup (\{q\} \times S^n)$. Let $f\colon B^{m+n} = B^m \times B^n \to S^m \times S^n$ be the map defined by $f(x,y) = (\pi_m(x), \pi_n(y))$. As $\pi_n\colon E^n \to S^n \setminus \{*\}$ is a homeomorphism also $f|_{E^{m+n}}\colon E^{m+n} \to S^m \times S^n \setminus S^m \vee S^n$ is a homeomorphism and we conclude by Proposition 18.6 that $S^n \times S^m$ can be obtained from $S^n \vee S^m$ by attaching a $m+n$-cell.

Let $g = f|_{\partial B^{m+n}}\colon S^{m+n-1} \to S^m \vee S^n$ be the attaching map. If both m and n are zero then $S^0 \times S^0$ is the topological space consisting of four points and we have

$$H_i(S^0 \times S^0) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $m = 1$ and $n = 0$ then $S^1 \times S^0 \cong S^1 \sqcup S^1$ and

$$H_i(S^1 \times S^0) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & i = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

and likewise for $m = 0$, $n = 1$. In the case $m = n = 1$, consider the long exact sequence from Proposition 18.13

$$\begin{aligned} \cdots \to H_2(S^1 \vee S^1) \to H_2\big(B^2 \cup_g S^1 \vee S^1\big) \to H_1(S^1) \xrightarrow{H_1(g)} H_1(S^1 \vee S^1) \\ \to H_1\big(B^2 \cup_g S^1 \vee S^1\big) \to H_0(S^1) \xrightarrow{H_0(g)} H_0(S^1 \vee S^1) \to \dots \end{aligned}$$

The induced map $H_1(g)$ sends a generator of $H_1(S^1)$ to the class $a + b - a - b = 0$, where $a, b \in H_1(S^1 \vee S^1)$ are the two generators. (Notice that $S^1 \times S^1$ is the torus, which we can view as the square with opposite edges identified. (See Problem C.3). The attaching map send the generator of $H_1(S^1)$ to the boundary of the square, which has class $a + b - a - b$).) Hence $H_1(g)$ is zero and as $H_2(S^1 \vee S^1) = 0$ we conclude $H_2(S1 \times S^1) = H_2\big(B^2 \cup_f S^1 \vee S^1\big) \cong H_1(S^1) \cong \mathbb{Z}$. $H_0(g)\colon H_0(S^1) \to H_0(S^1 \vee S^1)$ sends the class of a point to the class of a point and is therefor an isomorphism. Hence $H_1\big(B^2 \cup_f S^1 \vee S^1\big) \cong H_1(S^1 \vee S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus

$$H_i(S^1 \times S^1) \cong \begin{cases} \mathbb{Z} & i = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The remaining cases can be calculated by using Proposition 18.13 and Corollary 18.14. If $m \neq n$ we get

$$H_i(S^m \times S^n) \cong \begin{cases} \mathbb{Z} & i = 0, m, n, m + n \\ 0 & \text{otherwise} \end{cases}$$

and if $m = n$ we have

$$H_i(S^m \times S^n) \cong \begin{cases} \mathbb{Z} & i = 0, m + n \\ \mathbb{Z} \oplus \mathbb{Z} & i = m = n \\ 0 & \text{otherwise.} \end{cases}$$

Problem Sheet J

This Problem Sheet is based on Lecture 19 and Lecture 20. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

Problem J.1. Let X be the subset of the closed strip in $\mathbb{R}^2$ between the line $x = 0$ and $x = 1$ consisting of the unit interval I and all the line segment through the origin having slope $1/n$ for $n \in \mathbb{N}$. See Figure J.1. Prove that I is a deformation retract of X but not a strong deformation retract.

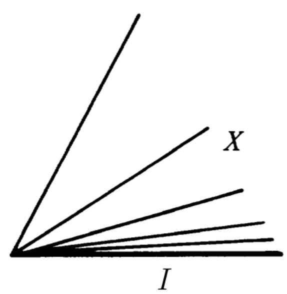


Figure J.1: The space X.

Problem J.2 (†). Let $X' \subset X$ be a closed subspace with the property that there exists a neighbourhood U of X' in X such that X' is a strong deformation retract of U. Let $*$ the point in X/X' corresponding to X'/X'. Prove that $\{*\}$ is a strong deformation retract of U/X' in X/X'.

Problem J.3. Let X be a finite cell complex, and define the **Euler characteristic** of X to be the number

$$\chi(X) = \sum_{i \geq 0} (-1)^k \operatorname{rank} H_k(X).$$

Let N_k denote the number of k-cells of X. Prove that

$$\chi(X) := \sum_{k \geq 0} (-1)^k N_k.$$

Problem J.4. Let X and Y be finite cell complexes. Show that $X \times Y$ also carries the structure of a finite cell complex. Prove that $\chi(X \times Y) = \chi(X)\chi(Y)$.

Problem J.5. Let X be a topological space with cell like filtration $\mathcal{F}$. Suppose $\langle\zeta\rangle \in H_n(X,\mathcal{F})$ comes from a cycle in $\zeta \in H_n(F^n, F^{n-1})$ (i.e. $\partial^{\mathcal{F}}\zeta = 0$.) Let

$$\imath\colon F^n \hookrightarrow X, \qquad \jmath\colon (F^n, \emptyset) \hookrightarrow (F^n, F^{n-1}),$$

denote inclusions. Show that there exists a homology class $\langle c\rangle \in H_n(F^n)$ such that $H_n(\jmath)\langle c\rangle = \zeta$. Denote by $\langle c\rangle' := H_n(\imath)\langle c\rangle$. Show that the isomorphism from Theorem 20.5 can be given explicitly by

$$\Theta\colon H_n(X) \to H_n(X,\mathcal{F}) \qquad \langle c\rangle' \mapsto \langle\zeta\rangle.$$

Problem J.6. Suppose X and Y are topological spaces with cell-like filtrations $\mathcal{F} = (F^n)$ and $\mathcal{G} = (G^n)$ respectively. Let us say a continuous map **respects the filtrations** if $f(F^n) \subseteq G^n$ for all $n \geq 0$.

1. Show that such a map f gives rise to a chain map $f_{\#}\colon C_\bullet(X,\mathcal{F}) \to C_\bullet(Y,\mathcal{G})$, and hence also a map $H_n(f)\colon H_n(X,\mathcal{F}) \to H_n(Y,\mathcal{G})$ for all $n \geq 0$.
2. Show that there is a well-defined category Filt whose objects are pairs $(X,\mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a cell-like filtration and whose morphisms are those continuous maps which respect the filtration. Prove that the operation $(X,\mathcal{F}) \mapsto C_\bullet(X,\mathcal{F})$ defines a functor $\mathsf{Filt} \to \mathsf{Comp}$.
3. Prove that the isomorphism between singular homology and the homology of the cell-like filtration is natural in the sense that if $f\colon (X,\mathcal{F}) \to (Y,\mathcal{G})$ is a continuous map respecting the filtrations then the following diagram commutes:

$$\begin{array}{ccc} H_n(X) & \xrightarrow{\Theta} & H_n(X,\mathcal{F}) \\ {\scriptstyle H_n(f)}\downarrow & & \downarrow{\scriptstyle H_n(f)} \\ H_n(Y) & \xrightarrow[\Theta]{} & H_n(Y,\mathcal{G}) \end{array}$$

where horizontal maps are the isomorphisms from the previous problem, and the two vertical maps are the induced maps on homology. The right-hand one comes from part (1) above, and the left-hand one is just the normal map induced by singuar homology.

Solutions to Problem Sheet J

This Problem Sheet is based on Lecture 19 and Lecture 20. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

PROBLEM J.1. Let X be the subset of the closed strip in $\mathbb{R}^2$ between the line $x = 0$ and $x = 1$ consisting of the unit interval I and all the line segment through the origin having slope $1/n$ for $n \in \mathbb{N}$. See Figure J.1. Prove that I is a deformation retract of X but not a strong deformation retract.

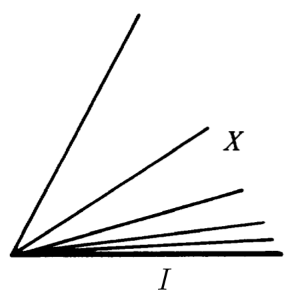


Figure J.1: The space X.

SOLUTION. The space X is given by $\{(x, \frac{1}{n}x) \mid x \in [0,1]\} \cup \{(x,0) \mid x \in [0,1]\}$, where we define $I := \{(x,0) \mid x \in [0,1]\}$. The homotopy

$$
\begin{aligned}
H\colon X \times [0,1] &\to I \\
((x, \frac{1}{n}x), t) &\to \begin{cases} (x(1-2t), \frac{1}{n}x(1-2t) & 0 \leq t \leq 1/2 \\ (x(2t-1), 0) & 1/2 \leq t \leq 1 \end{cases} \\
((x,0),t) &\to \begin{cases} (x(1-2t), 0) & 0 \leq t \leq 1/2 \\ (x(2t-1), 0) & 1/2 \leq t \leq 1 \end{cases}
\end{aligned}
$$

is such that $H(\cdot, 0) = \mathrm{id}_X$, $H(x,1) \in I$ for all $x \in X$ and $H(x',1) = x'$ for all $x' \in I$ and hence defines a deformation retract of X onto I. Suppose we could find a strong deformation retract, i.e. such that $H(x',t) = x'$ for all t and all $x' \in I$. Let G denote the corresponding homotopy. Let $(x,0) \in I$ with $x > 0$. Every δ-neighbourhood of $(x,0)$ contains a point $(x, \frac{1}{n}x)$ for some n big enough. G must map $((x, 1/nx), t)$ to the point $(0,0)$ for some t but leaves $(x,0)$ constant for every t. This contradicts the continuity of G.

Will J. Merry and Berit Singer. Last modified: Sept 01, 2018.

Problem J.2 (†). Let $X' \subset X$ be a closed subspace with the property that there exists a neighbourhood U of X' in X such that U is a strong deformation retract of X. Let $*$ the point in X/X' corresponding to X'/X'. Prove that $\{*\}$ is a strong deformation retract of U/X' in X/X'.

Solution. Let $H\colon U \times I \to X'$ define a strong deformation retract. In particular $H(x', t) = x'$ for all $x' \in X'$. Hence H descends to a map $G\colon U/X' \times I \to X'/X'$ such that $G(\{*\}, t) = \{*\}$ for all t. This defines a strong deformation retract of U/X' to $\{*\}$.

Problem J.3. Let X be a finite cell complex, and define the **Euler characteristic** of X to be the number

$$\chi(X) = \sum_{i \geq 0} (-1)^k \text{rank}\, H_k(X).$$

Let N_k denote the number of k-cells of X. Prove that

$$\chi(X) := \sum_{k \geq 0} (-1)^k N_k.$$

Solution. Let n be the dimension of X. By Theorem 20.5 we have $H^{\text{cell}}_\bullet(X) \cong H_\bullet(X)$ and hence also their ranks coincide in all degrees. Moreover, by definition we have $H^{\text{cell}}_k(X) = \frac{\ker \partial_k}{\operatorname{im} \partial_{k+1}}$, where $\partial = \partial^{\text{cell}}$ denotes the differential of the cellular chain complex. The exact sequence

$$0 \to \ker(\partial_k) \to C_k \to \operatorname{im}(\partial_k) \to 0$$

shows that $N_k = \text{rank} \ker(\partial_k) + \text{rank}(\operatorname{im} \partial_k)$. Similarly the exact sequence

$$0 \to \operatorname{im}(\partial_k) \to \ker(\partial_k) \to H_k \to 0$$

shows that $\text{rank} H_k(X) = \text{rank} \ker(\partial_k) - \text{rank} \operatorname{im}(\partial_{k+1})$. Now

$$\begin{aligned}
\chi(X) &= \sum_{i \geq 0} (-1)^k \text{rank}\, H_k(X) \\
&= \sum_{k \geq 0} (-1)^k \text{rank} H^{\text{cell}}_k(X) \\
&= \sum_{k \geq 0} (-1)^k \big(\text{rank} \ker(\partial_k) - \text{rank} \operatorname{im}(\partial_{k+1})\big) \\
&= \sum_{k \geq 0} (-1)^k \text{rank} \ker(\partial_k) + (-1)^{k+1} \text{rank} \operatorname{im}(\partial_{k+1}) \\
&= (-1)^{n+1} \text{rank} \operatorname{im}(\partial_{n+1}) + \sum_{n \geq k \geq 1} (-1)^k \big(\text{rank} \ker(\partial_k) + \text{rank} \operatorname{im}(\partial_k)\big) + \text{rank} \ker(\partial_0) \\
&= \sum_{k \geq 0} (-1)^k N_k,
\end{aligned}$$

where the last equality follows from the fact that $\partial_0 = 0$ and hence $\text{rank} \ker(\partial_0) = N_0$ and $\operatorname{im} \partial_{n+1} = 0$.

Problem J.4. Let X and Y be finite cell complexes. Show that $X \times Y$ also carries the structure of a finite cell complex. Prove that $\chi(X \times Y) = \chi(X)\chi(Y)$.

Solution. Given a structure of a cell complex on X with cells $E_\alpha^{n_\alpha}$ of dimension n_α and attaching maps f_α. Likewise a cell structure on Y with cells $E_\beta^{n_\beta}$ of dimension n_β and attaching maps f_β. A structure of a cell complex on $X \times Y$ is given by the cells $(E_\alpha \times E_\beta)^{n_\alpha + n_\beta}$, which have attaching maps $f_\alpha \times f_\beta$ and dimension $n_\alpha + n_\beta$. Define recursively the subspace $(X \times Y)^n$ as the space obtained from $(X \times Y)^{n-1}$ by attaching cells of the form $(E_\alpha \times E_\beta)^{n_\alpha + n_\beta}$, where $n = n_\alpha + n_\beta$. As $X \times Y$ carries the product topology and X and Y carry the colimit topology of their cell decompositions, also $X \times Y$ carries the colimit topology for the cell decomposition defined above.

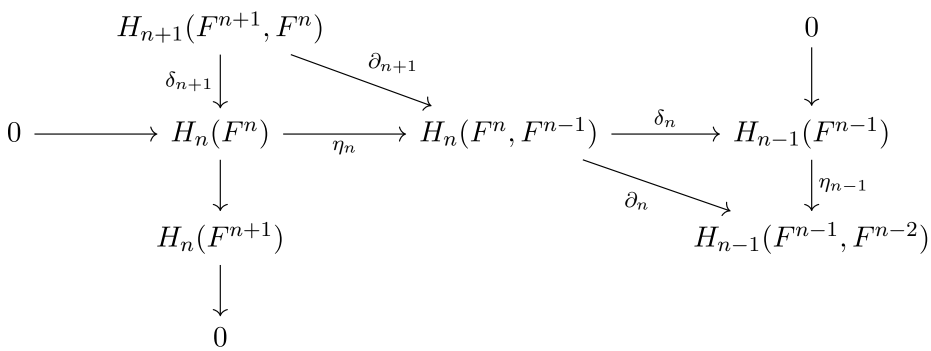
Problem J.5. Let X be a topological space with cell like filtration $\mathcal{F}$. Suppose $\langle\zeta\rangle \in H_n(X, \mathcal{F})$ comes from a cycle in $\zeta \in H_n(F^n, F^{n-1})$ (i.e. $\partial^{\mathcal{F}}\zeta = 0$.) Let

$$\imath\colon F^n \hookrightarrow X, \qquad \jmath\colon (F^n, \emptyset) \hookrightarrow (F^n, F^{n-1}),$$

denote inclusions. Show that there exists a homology class $\langle c\rangle \in H_n(F^n)$ such that $H_n(\jmath)\langle c\rangle = \zeta$. Denote by $\langle c\rangle' := H_n(\imath)\langle c\rangle$. Show that the isomorphism from Theorem 20.5 can be given explicitly by

$$\Theta\colon H_n(X) \to H_n(X, \mathcal{F}) \qquad \langle c\rangle' \mapsto \langle\zeta\rangle.$$

Solution. As $\zeta \in H_n(F^n, F^{n-1})$ is a cycle $\partial^{\mathcal{F}}\zeta = \eta_{n-1} \circ \delta_n(\zeta) = 0$. Recall that δ_n is the connecting homomorphism of the exact sequence of the pair (F^n, F^{n-1}) and $\eta_{n-1} = H_{n-1}(j_{n-1})$, where $j_{n-1}\colon (F^{n-1}, \emptyset) \hookrightarrow (F^{n-1}, F^{n-2})$ is the inclusion. The map η_{n-1} is injective, hence $\delta_{n-1}(\zeta) = 0$ and thus it follows from the long exact sequence of the pair (F^n, F^{n-1}) that there exists a $\langle c\rangle \in H_n(F^n)$ with $H_n(j_n)\langle c\rangle) = \zeta$. Consider the diagram as in the proof of Theorem 20.5:



We have isomorphisms (see the proof of Theorem 20.5)

$$\begin{aligned} H_n(X) &\cong H_n(F^{n+1}) \cong H_n(F^n)/\operatorname{im}(\delta_{n+1}) \\ &\cong \frac{\operatorname{im}(\eta_n)}{\operatorname{im}(\partial_{n+1})} \cong \frac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}. \end{aligned}$$

Clearly, $\langle c'\rangle \in H_n(X)$ is mapped to the equivalence class of $[\langle c\rangle] \in H_n(F^n)/\operatorname{im}(\delta_{n+1})$. The isomorphism $H_n(F^n)/\operatorname{im}(\delta_{n+1}) \cong \frac{\operatorname{im}\eta_n}{\operatorname{im}\partial_{n+1}}$ is induced by the map η_n. Hence $[\langle c\rangle]$ is mapped to $\langle\zeta\rangle \in \frac{\ker\partial_n}{\operatorname{im}\partial_{n+1}} = H_n(X, \mathcal{F})$.

Problem J.6. Suppose X and Y are topological spaces with cell-like filtrations $\mathcal{F} = (F^n)$ and $\mathcal{G} = (G^n)$ respectively. Let us say a continuous map **respects the filtrations** if $f(F^n) \subseteq G^n$ for all $n \geq 0$.

1. Show that such a map f gives rise to a chain map $f_\#\colon C_\bullet(X, \mathcal{F}) \to C_\bullet(Y, \mathcal{G})$, and hence also a map $H_n(f)\colon H_n(X, \mathcal{F}) \to H_n(Y, \mathcal{G})$ for all $n \geq 0$.
2. Show that there is a well-defined category Filt whose objects are pairs $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a cell-like filtration and whose morphisms are those continuous maps which respect the filtration. Prove that the operation $(X, \mathcal{F}) \mapsto C_\bullet(X, \mathcal{F})$ defines a functor Filt $\to$ Comp.
3. Prove that the isomorphism between singular homology and the homology of the cell-like filtration is natural in the sense that if $f\colon (X, \mathcal{F}) \to (Y, \mathcal{G})$ is a continuous map respecting the filtrations then the following diagram commutes:

$$\begin{array}{ccc} H_n(X) & \xrightarrow{\Theta} & H_n(X, \mathcal{F}) \\ {\scriptstyle H_n(f)}\downarrow & & \downarrow{\scriptstyle H_n(f)} \\ H_n(Y) & \xrightarrow[\Theta]{} & H_n(Y, \mathcal{G}) \end{array}$$

   where horizontal maps are the isomorphisms from the previous problem, and the two vertical maps are the induced maps on homology. The right-hand one comes from part (1) above, and the left-hand one is just the normal map induced by singular homology.

Solution.

1. The chain map $f_\#\colon C_\bullet(X, \mathcal{F}) \to C_\bullet(Y, \mathcal{G})$ is defined as

$$H_n(f_n)\colon H_n(F^n, F^{n-1}) \to H_n(G^n, G^{n-1}).$$

   Let $\eta_{n-1} = H_{n-1}(j_{n-1})$, where $j_{n-1}\colon (F^{n-1}, \emptyset) \hookrightarrow (F^{n-1}, F^{n-2})$ is the inclusion. And let δ_n be the connecting homomorphism of the exact sequence of the pair (F^n, F^{n-1}). Recall that the boundary operator ∂_n of the chain complex $C_\bullet(X, \mathcal{F})$ is given by the composition of those two maps, i.e. $\partial_n = \eta_{n-1} \circ \delta_n$. It therefore suffices to show that $H_\bullet(f_\bullet)$ commutes with both η and δ. It follows from the exact sequence of the pair (F^n, F^{n-1}) (see Proposition 12.3) that $\delta_n \circ H_n(f_n) = H_{n-1}(f_{n-1}) \circ \delta_{n-1}$. As f_n respects the filtration and j_n is just the inclusion we must have that $f_{n-1} j_{n-1} = j_{n-1} f_{n-1}$. Thus $\eta_{n-1} \circ H_{n-1}(f_{n-1}) = H_{n-1}(j_{n-1}) \circ H_{n-1}(f_{n-1}) = H(j_{n-1} f_{n-1}) = H(f_{n-1} j_{n-1}) = H_{n-1}(f_{n-1}) \circ \eta_{n-1}$, where we used the fact that $H_\bullet$ is a functor.
2. One easily sees that the first two axioms of a category are fulfilled. The third axiom is also fulfilled, as the identity map of the topological space X is continuous and respects the filtration. It is left to show that the operation $(X, \mathcal{F}) \mapsto C_\bullet(X, \mathcal{F})$ defines a functor Filt $\to$ Comp. The functor send $(X, \mathcal{F})$ to $C_\bullet(X, \mathcal{F})$ and a continuous map $f\colon X \to Y$ which respects the filtration to the chain map $H_n(f_n)\colon H_n(F^n, F^{n-1}) \to H_n(G^n, G^{n-1})$. As $H_\bullet$ is a well-defined functor the claim follows.

3. The isomorphism Θ is the compositon of the maps $\eta_n := H_n(j_n)$ and $H_n(i_n)^{-1}|_{\operatorname{im} H_n(i_n)} = H_n(i_n^{-1}|_{\operatorname{im} i_n})$. As f respects the filtration it commutes with i and j. Since $H_\bullet$ is a functor also $H_n(f)$ commutes with $H_n(i_n^{-1}|_{\operatorname{im} i_n})$ and $H_n(j)$.

Problem Sheet K

This Problem Sheet is based on Lectures 21 and 22. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem K.1. Let p and q be relatively prime integers. Think of S^3 as the pairs $(z,w) \in \mathbb{C}^2$ with $|z|^2 + |w|^2 = 1$. Let $\zeta = e^{2\pi i/p}$ be a primitive pth root of unity, and define $f\colon S^3 \to S^3$ by

$$f(z,w) := (\zeta z, \zeta^q w).$$

Define an equivalence relation on S^3 by saying that $(z,w) \sim (z',w')$ if $(z',w') = f^k(z,w)$ for some integer k. The quotient space $S^3/\sim$ is called a **lens space** and denoted by $L(p,q)$.

1. Show that $L(p,q)$ is a compact Hausdorff space.
2. Identify $L(1,1)$ and $L(2,1)$.
3. Show that if $q \equiv q' \bmod p$ then $L(p,q) = L(p,q')$.

Problem K.2. Show that the following[1] defines a cellular decomposition of S^3 having p cells in dimensions $0, 1, 2$ and 3:

$$E_k^0 := \left\{ (z,0) \in S^3 \mid \arg z = \frac{2\pi k}{p} \right\}_{k=0,1,2,\dots p-1}$$
$$E_k^1 := \left\{ (z,0) \in S^3 \mid \frac{2\pi k}{p} < \arg z < \frac{2\pi(k+1)}{p} \right\}_{k=0,1,2,\dots p-1}$$
$$E_k^2 := \left\{ (z,w) \in S^3 \mid \arg w = \frac{2\pi k}{p} \right\}_{k=0,1,2,\dots p-1}$$
$$E_k^3 := \left\{ (z,w) \in S^3 \mid \frac{2\pi k}{p} < \arg w < \frac{2\pi(k+1)}{p} \right\}_{k=0,1,2,\dots p-1}.$$

Use the cellular boundary formula and this cell decomposition to give another computation of the homology of S^3.

Problem K.3. Show that the quotient map $S^3 \to L(p,q)$ gives rise to a cellular decomposition of $L(p,q)$ with exactly one cell in dimension $0, 1, 2$ and 3. Use this to compute the homology of $L(p,q)$. *Hint:* Use the decomposition of S^3 from the previous question!

Problem K.4 (†). Let C and D be two categories and $S, T\colon \mathsf{C} \to \mathsf{D}$ two functors. Suppose $\Phi\colon S \to T$ is a natural transformation. Prove that Φ is a natural isomorphism if and only if there is a natural transformation $\Psi\colon T \to S$ such that $\Phi \circ \Psi = \mathrm{id}_S$ and $\Psi \circ \Phi = \mathrm{id}_T$.

[1] If $z = re^{i\theta}$ with $r > 0$ and $0 \le \theta < 2\pi$ then $\arg z := \theta$.

Problem K.5. Let C be a category and $C \in \mathrm{obj}(C)$. Consider the **hom-functor**

$$T\colon \mathsf{C} \to \mathsf{Sets}$$

defined as follows:

- **Objects**: For $D \in \mathrm{obj}(\mathsf{C})$, set $T(D) := \mathrm{Hom}(C, D)$.
- **Morphisms**: If $f\colon A \to B$ in C then

  $$T(f)\colon \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B),$$

  is defined by

  $$T(f)(h) := f \circ h.$$

One normally writes this functor as $T = \mathrm{Hom}(C, \square)$.

1. Let $\{*\}$ be a set with one element. Prove that $\mathrm{Hom}(\{*\}, \square)\colon \mathsf{Sets} \to \mathsf{Sets}$ is naturally isomorphic to the identity functor on Sets.
2. Let C be a category, $C \in \mathrm{obj}(C)$ and $S\colon \mathsf{C} \to \mathsf{Sets}$ a functor. Prove the **Yoneda Lemma**[2]: there is a bijection from $\mathrm{Nat}(\mathrm{Hom}(C, \square), S)$ to $S(C)$ given by

   $$\Phi \mapsto \Phi(C)(\mathrm{id}_C).$$

   In particular, this shows that $\mathrm{Nat}(\mathrm{Hom}(C, \square), S)$ is always a set.

Problem K.6 (†). Let $(\mathcal{H}_\bullet, \delta)$ be a homology theory satisfying the first four axioms (homotopy, exact sequence, excision and dimension). Let (X_n, X'_n), $1 \leq n \leq N$ be a *finite* family of pairs of spaces. Denote by

$$\imath_n\colon (X_n, X'_n) \hookrightarrow \left(\bigsqcup_{n=1}^{N} X_n, \bigsqcup_{n=1}^{N} X'_n \right)$$

the inclusion. Prove that for all $k \geq 0$, the map

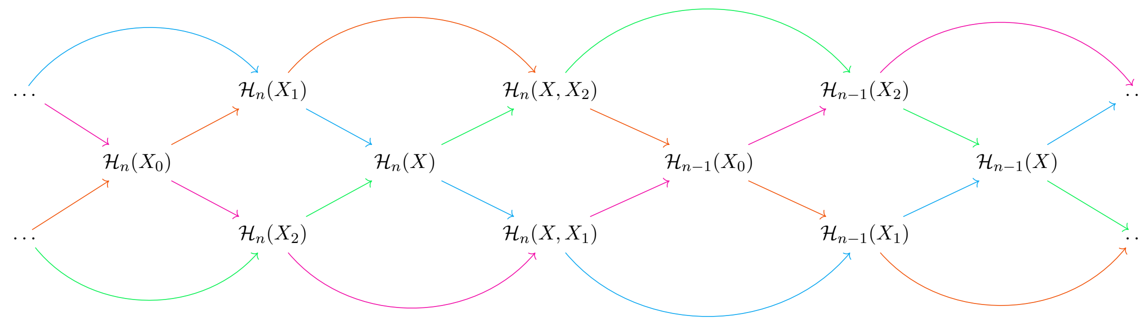
$$\sum_{n=1}^{N} \mathcal{H}_k(\imath_n)\colon \bigoplus_{n=1}^{N} \mathcal{H}_k(X_n, X'_n) \to \mathcal{H}_k \left(\bigsqcup_{n=1}^{N} X_n, \bigsqcup_{n=1}^{N} X'_n \right).$$

is an isomorphism.

Problem K.7 (†). Let X_1 and X_2 be subspaces of X such that $X = X_1^\circ \cup X_2^\circ$. Set $X_0 := X_1 \cap X_2$. Let $(\mathcal{H}_\bullet, \delta)$ be a homology theory satisfying the first four axioms.

[2] This result is a generalisation of Cayley's Theorem from group theory: that every group G is isomorphic to a subgroup of the symmetric group acting on G. Meta-exercise: Make this statement precise.

Prove there is a commutative braid of the form:



where all four braids are exact. Deduce that the Mayer-Vietoris exact sequence holds for $(\mathcal{H}_\bullet, \delta)$.

Solutions to Problem Sheet K

This Problem Sheet is based on Lectures 21 and 22. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

Problem K.1. Let p and q be relatively prime integers. Think of S^3 as the pairs $(z, w) \in \mathbb{C}^2$ with $|z|^2 + |w|^2 = 1$. Let $\zeta = e^{2\pi i/p}$ be a primitive pth root of unity, and define $f\colon S^3 \to S^3$ by

$$f(z,) := (\zeta z, \zeta^q w).$$

Define an equivalence relation on S^3 by saying that $(z, w) \sim (z', w')$ if $(z', w') = f^k(z, w)$ for some integer k. The quotient space $S^3/\sim$ is called a **lens space** and denoted by $L(p, q)$.

1. Show that $L(p, q)$ is a compact Hausdorff space.
2. Identify $L(1, 1)$ and $L(2, 1)$.
3. Show that if $q \sim q'$ mod p then $L(p, q) = L(p, q')$.

Solution.

1. $S^3/\sim$ is compact as $p : S^3 \to S^3/\sim$ is continuous and S^3 is compact. Let $p : S^3 \to S^3/\sim$ be the quotient map. It is open since $S^3/\sim$ is equipped with the quotient topology. Therefor the map

$$f : S^3 \times S^3 \to S^3/\sim \times S^3/\sim$$
$$((z, w), (z', w')) \mapsto (p(z, w), p(z', w'))$$

is also open. Let $R := \{((z, w), (z', w')) | (z, w) \sim (z', w')\}$. The set $S^3 \times S^3 \setminus R$ is open (if $(z, w) \nsim (z', w')$ then this identity also holds in an open neighbourhood of $((z, w), (z', w'))$). Now $S^3/\sim \times S^3/\sim \setminus \Delta = f(S^3 \times S^3 \setminus R)$ and this is open as f is open, which shows that $S^3/\sim \times S^3/\sim$ is Hausdorff.

2. If $p = q = 1$ then $\zeta = 1$ and thus $L(1, 1) \cong S^3$. If $p = 2$ and $q = 1$ then $(z, w) \sim (z', w') \iff (z, w) = \pm(z', w')$ and thus $L(2, 1) \cong \mathbb{R}P^3$.

3. If $q \equiv q' \mod p$ then $\{\zeta^0, \zeta^1, \dots, \zeta^{qp}\} = \{\zeta^0, \zeta^1, \dots, \zeta^{q'p}\}$ and hence $R = R'$ which shows that $L(p, q) \cong L(p, q')$.

Problem K.2. Show that the following[1] defines a cellular decomposition of S^3 having

[1]If $z = re^{i\theta}$ with $r > 0$ and $0 \le \theta < 2\pi$ then $\arg z := \theta$.

p cells in dimensions $0, 1, 2$ and 3:

$$E^0_k := \left\{(z,0) \in S^3 \mathrm{id} \arg z = \frac{2\pi k}{p}\right\}_{k=0,1,2,\dots p-1}$$
$$E^1_k := \left\{(z,0) \in S^3 \mathrm{id} \frac{2\pi k}{p} < \arg z < \frac{2\pi(k+1)}{p}\right\}_{k=0,1,2,\dots p-1}$$
$$E^2_k := \left\{(z,w) \in S^3 \mathrm{id} \arg w = \frac{2\pi k}{p}\right\}_{k=0,1,2,\dots p-1}$$
$$E^3_k := \left\{(z,w) \in S^3 \mathrm{id} \frac{2\pi k}{p} < \arg w < \frac{2\pi(k+1)}{p}\right\}_{k=0,1,2,\dots p-1}.$$

Use the cellular boundary formula and this cell decomposition to give another computation of the homology of S^3.

SOLUTION. The space S^3 has a cell decomposition, which is given by $X^0 := \cup_{k=1}^{p-1} E^0_k$, $X^1 := \{(z,0) \in S^3\}$, $X^2 := \{(z,w) \in S^3 | \arg(w) = \frac{2\pi k}{p}, k \in \{0,\dots,p-1\}\}$ and $X^3 = S^3$. X^1 is obtained from X^0 by attaching the $p-1$ 1-cells E^0_k with the attaching maps

$$\begin{aligned} f^1|_{S^0_k} : S^0_k &\to X^0 \\ 0 &\mapsto E^0_k \\ 1 &\mapsto E^0_{k+1}. \end{aligned}$$

Similarly X^2 is obtained from X^1 by attaching the $p-1$ 2-cells with attaching maps

$$\begin{aligned} f^2|_{S^1_k} : S^1_k &\to X^1 \\ e^{i\phi} &\mapsto (e^{i\phi}, 0). \end{aligned}$$

Finally, X^3 is obtained from X^2 by attaching $p-1$ 3-cells with attaching maps

$$\begin{aligned} f^3|_{S^2_k} : S^2_k &\to X^2 \\ ((1-z^2)e^{i\phi}, z) &\mapsto \begin{cases} (\sqrt{(1-z^2)}e^{i\phi}, ze^{2\pi k/p}) & z \geq 0 \\ (\sqrt{(1-z^2)}e^{i\phi}, ze^{2\pi(k+1)/p}) & z \leq 0 \end{cases}, \end{aligned}$$

where $z \in [-1, 1]$, $\phi \in [0, 2\pi)$ and $(\sqrt{(1-z^2)}e^{i\phi}, z)$ gives a parametrisation of the closed 3-ball. Hence the cell decomposition has p cells in dimensions $0, 1, 2, 3$ and no other cells. The cellular chain complex thus looks (up to isomorphism) like

$$\dots \longrightarrow 0 \longrightarrow \mathbb{Z}^p \xrightarrow{\partial_3} \mathbb{Z}^p \xrightarrow{\partial_2} \mathbb{Z}^p \xrightarrow{\partial_1} \mathbb{Z}^p \longrightarrow 0 \longrightarrow \dots$$

The boundary operator ∂_i is the composition of δ_i, which is the connecting homomorphism of the pair (X^i, X^{i-1}), and $\eta_{i-1} = H_{i-1}(j_{i-1})$, where $j_{i-1} : (X_{i-1}, \emptyset) \hookrightarrow (X_{i-1}, X_{i-2})$ is the inclusion.

Now we are ready to compute the boundary operators. By repeatedly using the Cellular Boundary Formula (Theorem 20.11) we see:

1. In dimension 3 we have that $\partial_4 \equiv 0$ and $\partial_3(E_k^3) = E_{k+1}^2 - E_k^2$. Thus $H_3^{\text{cell}}(S^3) \cong H_3(S^3) \cong \mathbb{Z}$.
2. Moreover, $\partial_2(E_k^2) = \sum_{k=0}^{p-1} E_k^1$ for every k. Thus $E_{k+1}^2 - E_k^2$ for $k = 0, \dots, p-2$ form a basis of $\ker(\partial_2) \cong \mathbb{Z}^{p-1} \subset \mathbb{Z}^p$ and we see that $\ker \partial_2 = \operatorname{im} \partial_3$ which shows that $H_2^{\text{cell}}(S^3) \cong H_2(S^3) \cong 0$.
3. Now, $\partial_1(E_k^1) = E_{k+1}^0 - E_k^0$ and $\partial_1(E_{p-1}^1) = E_0^0 - E_{p-1}^0$. Hence $\ker(\partial_1) = \mathbb{Z} \cdot \sum_{k=1}^{p-1} E_k^0$ and $\operatorname{im}(\partial_2) = \ker(\partial_1)$. Thus $H_1^{\text{cell}}(S^3) \cong H_1(S^3) \cong 0$.
4. As $\delta_0 \equiv 0$ a basis of $\ker(\partial_0)$ is given by $\{E^0, E_1^0 - E_0^0, \dots, E_{p-1}^0 - E_{p-2}^0\}$. But $E_{k+1}^0 - E_k^0 \in \operatorname{im}(\partial_1)$. Thus $\ker(\partial_0)/\operatorname{im}(\partial_1) \cong \mathbb{Z} \cdot E_0^0$ and hence $H_0^{\text{cell}}(S^3) \cong H_0(S^3) \cong \mathbb{Z}$.

Problem K.3. Show that the quotient map $S^3 \to L(p,q)$ gives rise to a cellular decomposition of $L(p,q)$ with exactly one cell in dimension $0, 1, 2$ and 3. Use this to compute the homology of $L(p,q)$. *Hint:* Use the decomposition of S^3 from the previous question!

Solution. Let $p : S^3 \to S^3/\sim \equiv L(p,q)$ denote the quotient map. Notice that $p(E_k^d) = p(E_l^d)$ for every $k = 0, 1, 2, 3$ and every $k, l \in \{0, \dots, p-1\}$. Thus the cell decomposition from the previous exercise gives rise to a cell decompositon of $L(p,q)$ with one cell in dimensions $0, 1, 2, 3$ and no other cells. Let $E^d := p(E_k^d)$ denote the d-cell of $L(p,q)$. From the previous exercise it follows immediately that $\partial_3 \equiv 0$, $\partial_2(E^2) = pE^1$, and $\partial_1 \equiv 0$. Thus we see that

$$H_i(L(p,q)) \cong H_i^{\text{cell}}(L(p,q)) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/p\mathbb{Z} & i = 1 \\ 0 & i = 2 \\ \mathbb{Z} & i = 3. \end{cases}$$

Problem K.4 (†). Let C and D be two categories and $S, T \colon \mathsf{C} \to \mathsf{D}$ two functors. Suppose $\Phi \colon S \to T$ is a natural transformation. Prove that Φ is a natural isomorphism if and only if there is a natural transformation $\Psi \colon T \to S$ such that $\Phi \circ \Psi = \mathrm{id}_S$ and $\Psi \circ \Phi = \mathrm{id}_T$.

Solution. If Φ is a natural isomorphism then for every $A, B \in \operatorname{obj}(C)$ there exists $\Psi(A)$ and $\Psi(B)$ such that

$$\begin{array}{ccccc} S(A) & \xrightarrow{\Phi(A)} & T(A) & \xrightarrow{\Psi(A)} & S(A) \\ \downarrow{\scriptstyle S(f)} & & \downarrow{\scriptstyle T(f)} & & \downarrow{\scriptstyle S(f)} \\ S(B) & \xrightarrow[\Phi(B)]{} & T(B) & \xrightarrow[\Psi(B)]{} & S(B), \end{array}$$

where the left rectangle is commutative and $\Psi(A) \circ \Phi(A) = \mathrm{id}_{S(A)}$ and $\Psi(B) \circ \Phi(B) = \mathrm{id}_{S(B)}$. As $\mathrm{id}_{S(A)} \circ S(f) = S(f) \circ \mathrm{id}_{S(A)}$ also the right square is commutative. Similarly one shows that $\Phi(A) \circ \Psi(A) = \mathrm{id}_{T(A)}$. Hence Ψ is a natural transformation. The other direction is obvious.

Problem K.5. Let C be a category and $C \in \mathrm{obj}(C)$. Consider the **hom-functor**

$$T \colon \mathsf{C} \to \mathsf{Sets}$$

defined as follows:

- **Objects**: For $D \in \mathrm{obj}(\mathsf{C})$, set $T(D) := \mathrm{Hom}(C, D)$.
- **Morphisms**: If $f \colon A \to B$ in C then

  $$T(f) \colon \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B),$$

  is defined by

  $$T(f)(h) := f \circ h.$$

One normally writes this functor as $T = \mathrm{Hom}(C, \Box)$.

1. Let $\{*\}$ be a set with one element. Prove that $\mathrm{Hom}(\{*\}, \Box) \colon \mathsf{Sets} \to \mathsf{Sets}$ is naturally isomorphic to the identity functor on Sets.
2. Let C be a category, $C \in \mathrm{obj}(C)$ and $S \colon \mathsf{C} \to \mathsf{Sets}$ a functor. Prove the **Yoneda Lemma**[2]: there is a bijection from $\mathrm{Nat}(\mathrm{Hom}(C, \Box), S)$ to $S(C)$ given by

   $$\Phi \mapsto \Phi(C)(\mathrm{id}_C).$$

   In particular, this shows that $\mathrm{Nat}(\mathrm{Hom}(C, \Box), S)$ is always a set.

Solution. 1. Define the natural transformation $\mathrm{ev}_{\{*\}} : \mathrm{Hom}(\{*\}, \cdot) \to \mathrm{id}$ by

$$\begin{aligned} \mathrm{ev}_*(X) : \mathrm{Hom}(\{*\}, X) &\to \mathrm{id}(X) \\ \lambda &\mapsto \lambda(*). \end{aligned}$$

Moreover, let $\Phi : \mathrm{id} \to \mathrm{Hom}(\{*\}, \cdot)$ be given by

$$\begin{aligned} \Phi : \mathrm{id}(X) &\to Hom(\{*\}, X) \\ x &\mapsto (\lambda_x : * \mapsto x). \end{aligned}$$

Then for every $X \in \mathrm{obj}(Sets)$ we have $\mathrm{ev}_*(X) \circ \Phi(X) = \mathrm{id}_X$ and $\Phi(X) \circ \mathrm{ev}_*(X) = \mathrm{id}_{\mathrm{Hom}(\{*\}, X)}$, so $\mathrm{ev}_* : \mathrm{Hom}(\{*\}, \cdot) \to \mathrm{id}$ is a natural isomorphism.

[2] This result is a generalisation of Cayley's Theorem from group theory: that every group G is isomorphic to a subgroup of the symmetric group acting on G. Meta-exercise: Make this statement precise.

2. Let $J : \mathrm{Nat}(\mathrm{Hom}(C,\square), S) \to S(C) : \phi \mapsto \phi(C)(\mathrm{id}_C)$ denote the Yoneda map. We start by proving that J is surjective. Let $x \in S(C)$. For $D \in \mathrm{obj}(C)$ let

$$\begin{aligned} \phi_x(D) : \mathrm{Hom}(C,D) &\to S(D) \\ \lambda &\mapsto S(\lambda)(x). \end{aligned}$$

Then $\phi_x : \mathrm{Hom}(C,\square) \to S$ is a natural transformation, as for every $D, E \in \mathrm{obj}(C)$ and any $f \in \mathrm{Hom}(D,E)$ the diagram

$$\begin{array}{ccc} \mathrm{Hom}(C,D) & \xrightarrow{\phi_x(D)} & S(D) \\ {\scriptstyle T(f)}\downarrow & & \downarrow{\scriptstyle S(f)} \\ \mathrm{Hom}(C,E) & \xrightarrow[\phi_x(E)]{} & S(E) \end{array}$$

commutes. Indeed $\phi_x(E)\circ T(f)(\lambda) = \phi_x(E)(f\circ\lambda) = S(f\circ\lambda)(x) = S(f)\circ S(\lambda)(x)$ and $S(f)\circ\phi_x(D)(\lambda) = S(f)S(\lambda)(x)$.

To prove injectivity, suppose that $J(\phi) = \phi(C)(\mathrm{id}_C) = \psi(C)(\mathrm{id}_C) = J(\psi)$. For every $D \in \mathrm{obj}(C)$ and for every $f \in \mathrm{Hom}(C,D)$ we have $\phi(D)(f) = \phi(D)\circ T(f)(\mathrm{id}_C) = S(f)\circ\phi(C)(\mathrm{id}_C) = S(f)\psi(C)(\mathrm{id}_X) = \psi(D)\circ T(f)(\mathrm{id}_C) = \psi(D)(f)$. Hence $\phi(D) = \psi(D)$ for every $D \in \mathrm{obj}(C)$, which implies that $\phi = \psi$.

Problem K.6 (†). Let $(\mathcal{H}_\bullet, \delta)$ be a homology theory satisfying the first four axioms (homotopy, exact sequence, excision and dimension). Let (X_n, X_n'), $1 \le n \le N$ be a *finite* family of pairs of spaces. Denote by

$$\imath_n \colon (X_n, X_n') \hookrightarrow \left(\bigsqcup_{n=1}^N X_n . \bigsqcup_{n=1}^N X_n' \right)$$

the inclusion. Prove that for all $k \ge 0$, the map

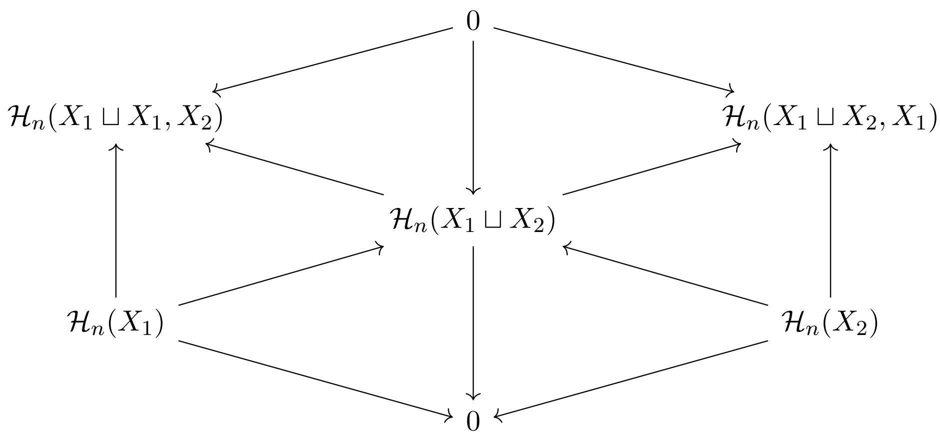
$$\sum_{n=1}^N \mathcal{H}_k(\imath_n) \colon \bigoplus_{n=1}^N \mathcal{H}_k(X_n, X_n') \to \mathcal{H}_k \left(\bigsqcup_{n=1}^N X_n . \bigsqcup_{n=1}^N X_n' \right).$$

is an isomorphism.

Solution. The Excision Axiom tells us that the inclusion induces an isomorphism

$$\mathcal{H}_n(X, X') \cong \mathcal{H}_n(X \sqcup Y, X' \sqcup Y)$$

for any space Y. By induction, we need only consider the case $N = 2$. Moreover, by the usual argument involving the Five Lemma, it is sufficient to consider the absolute groups (see for instance the end of the proof of Theorem 21.12 in Lecture 22.) By functoriality of $\mathcal{H}_n$ and the Exact Sequence axion, the following diagram satisfies the hypotheses of the Hexagon Lemma H.2:

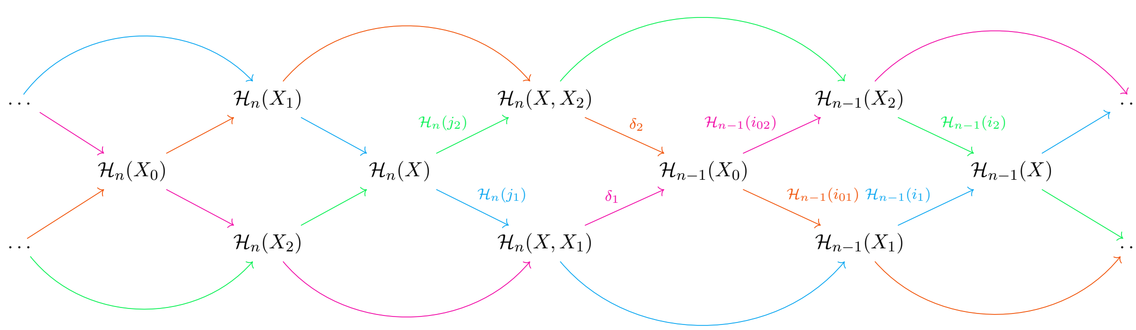


Thus part (1) of the Hexagon Lemma tells us that

$$\mathcal{H}_n(X_1) \oplus \mathcal{H}_n(X_2) \cong \mathcal{H}_n(X_1 \sqcup X_2),$$

which is what we wanted to show.

Problem K.7 (†). Let X_1 and X_2 be subspaces of X such that $X = X_1^\circ \cup X_2^\circ$. Set $X_0 := X_1 \cap X_2$. Let $(\mathcal{H}_\bullet, \delta)$ be a homology theory satisfying the first four axioms. Prove there is a commutative braid of the form:



where all four braids are exact. Deduce that the Mayer-Vietoris exact sequence holds for $(\mathcal{H}_\bullet, \delta)$.

Solution. Consider the long exact sequence of the pair (X_1, X_0). It follows with excision that $H_n(X_1, X_0) \cong H_n(X, X_2)$ and $H_i(X_2, X_0) \cong H_i(X, X_1)$. Hence the long exact sequences of the pair (X_1, X_0) becomes

$$\ldots \to \mathcal{H}_n(X_0) \to \mathcal{H}_n(X_1) \to \mathcal{H}_n(X, X_2) \to \mathcal{H}_{n-1}(X_0) \to \ldots.$$

Similarly there is a long exact sequence

$$\ldots \to \mathcal{H}_n(X_0) \to \mathcal{H}_n(X_2) \to \mathcal{H}_n(X, X_1) \to \mathcal{H}_{n-1}(X_0) \to \ldots.$$

The pairs (X, X_1) and (X, X_2) give rise to long exact sequences

$$\ldots \to \mathcal{H}_n(X_1) \to \mathcal{H}_n(X) \to \mathcal{H}_n(X, X_1) \to \mathcal{H}_{n-1}(X_1) \to \ldots$$

and

$$\ldots \to \mathcal{H}_n(X_2) \to \mathcal{H}_n(X) \to \mathcal{H}_n(X, X_2) \to \mathcal{H}_{n-1}(X_2) \to \ldots.$$

Let $i_k : X_k \hookrightarrow X$ and $i_{0k} : X_0 \hookrightarrow X_k$ be the inclusion for $k = 1, 2$. Similarly let $j_k : (X, \emptyset) \hookrightarrow (X, X_k)$ denote the inclusion and δ_k the boundary homomorphisms for $k = 1, 2$. (See picture.) The maps in the long exact sequences above are induced by these maps. Hence there exists a commutative braid with exact sequences as in the picture. Now we have a sequence

$$\ldots \mathcal{H}_n(X) \xrightarrow{\delta_1 \circ \mathcal{H}_n(j_1)} \mathcal{H}_{n-1}(X_0) \xrightarrow{(\mathcal{H}_{n-1}(i_{01}), \mathcal{H}_{n-1}(i_{02}))} \mathcal{H}_{n-1}(X_1) \oplus \mathcal{H}_{n-1}(X_2) \xrightarrow{D} \mathcal{H}_{n-1}(X) \to \ldots$$

where D is given by $(a, b) \mapsto (\mathcal{H}_{n-1}(i_1)(a) - \mathcal{H}_{n-1}(i_2)(b))$. It follows from the definition that this sequence is exact.

Problem Sheet L

This Problem Sheet is based on Lectures 24 and 25. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem L.1 (†). Suppose A is an abelian group and $\{B_\lambda \mid \lambda \in \Lambda\}$ is a (possibly uncountable) family of abelian groups. Prove there is an isomorphism

$$A \otimes \bigoplus_{\lambda \in \Lambda} B_\lambda \cong \bigoplus_{\lambda \in \Lambda} (A \otimes B_\lambda).$$

Problem L.2 (†). Suppose F and F' are free abelian groups. Prove that $F \otimes F'$ is also free.

Problem L.3 (†). Suppose B is a torsion-free abelian group. Prove that $\square \otimes B$ and $B \otimes \square$ are exact functors.

Problem L.4 (†). Suppose $T\colon \mathsf{Ab} \to \mathsf{Ab}$ is an additive functor and $0 \to A \to B \to C \to 0$ is a split exact sequence. Prove that $0 \to T(A) \to T(B) \to T(C) \to 0$ is also a split exact sequence.

Problem L.5. Let $A = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_5$ and let $B = \mathbb{Z}_3 \oplus \mathbb{Z}_5$. Compute $A \otimes B$ and $\mathrm{Tor}(A, B)$.

Problem L.6 (⋆). Fix $n \geq 2$. Consider the projection map $p\colon S^n \to \mathbb{R}P^n$. Let $a\colon S^n \to S^n$ be the antipodal map. For this problem you may assume the following fact:

> **Fact:** *If $\sigma\colon \Delta^m \to \mathbb{R}P^n$ is a singular m-simplex, then are precisely two singular m-simplices $\tilde{\sigma}_1, \tilde{\sigma}_2\colon \Delta^m \to S^n$ that satisfy $p \circ \tilde{\sigma}_i = \sigma$ for $i = 1, 2$. Moreover $\tilde{\sigma}_2 = a \circ \tilde{\sigma}_1$*[1].

Define a chain map

$$q_m\colon C_m(\mathbb{R}P^n; \mathbb{Z}_2) \to C_m(S^n; \mathbb{Z}_2)$$

by $\sigma \mapsto \tilde{\sigma}_1 + \tilde{\sigma}_2$.

1. Show that the following is a short exact sequence of chain complexes:

$$0 \to C_\bullet(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{q} C_\bullet(S^n; \mathbb{Z}_2) \xrightarrow{p_\#} C_\bullet(\mathbb{R}P^n; \mathbb{Z}_2) \to 0.$$

[1] This follows from standard covering space theory that we will cover later on in the semester.

2. Suppose $f\colon S^n \to S^n$ is an odd map (that is, $f \circ a = a \circ f$). Let $h\colon \mathbb{R}P^n \to \mathbb{R}P^n$ denote the induced map (so that $p \circ f = h \circ p$). Show that the following diagram commutes for every m:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_m(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{q_m} & C_m(S^n;\mathbb{Z}_2) & \xrightarrow{p_\#} & C_m(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle h_\#} & & \\
0 & \longrightarrow & C_m(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow[q_m]{} & C_m(S^n;\mathbb{Z}_2) & \xrightarrow[p_\#]{} & C_m(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0
\end{array}$$

3. Deduce that $f\colon H_n(S^n;\mathbb{Z}_2) \to H_n(S^n;\mathbb{Z}_2)$ is an isomorphism. *Hint:* Argue by induction on m, using the associated long exact sequence in homology from the short exact sequence of chain complexes in part (1).

4. Deduce that f has odd degree.

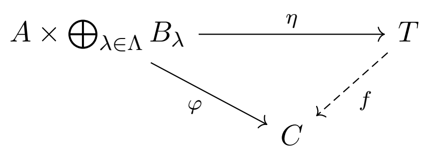
Solutions to Problem Sheet L

This Problem Sheet is based on Lectures 24 and 25. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

PROBLEM L.1 (†). Suppose A is an abelian group and $\{B_\lambda \mid \lambda \in \Lambda\}$ is a (possibly uncountable) family of abelian groups. Prove there is an isomorphism

$$A \otimes \bigoplus_{\lambda \in \Lambda} B_\lambda \cong \bigoplus_{\lambda \in \Lambda} (A \otimes B_\lambda).$$

SOLUTION. Let T be an abelian group and $\eta \colon A \times \bigoplus_{\lambda\in\Lambda} B_\lambda \to T$ a bilinear map. Consider the following universal property. For an abelian group C and $\varphi \colon A \times \bigoplus_{\lambda\in\Lambda} B_\lambda \to C$ a bilinear map, there exists a *unique* homomorphism $f \colon T \to C$ such that the following diagram commutes:



(L.1)

It suffices to show that $\bigoplus_{\lambda\in\Lambda}(A \otimes B_\lambda)$ together with the bilinear map $\eta \colon A \times \bigoplus_{\lambda\in\Lambda} B_\lambda \to \bigoplus_{\lambda\in\Lambda}(A \otimes B_\lambda) \colon (a, (b_\lambda)_{\lambda\in\Lambda}) \mapsto (a \otimes b_\lambda)_{\lambda\in\Lambda}$ satisfies this universal property. Then Lemma 24.4 ensures that $A \otimes \bigoplus_{\lambda\in\Lambda} B_\lambda \cong \bigoplus_{\lambda\in\Lambda}(A \otimes B_\lambda)$.

Let C be an abelian group and $\varphi \colon A \times \bigoplus_{\lambda\in\Lambda} B_\lambda \to C$ a bilinear map. Let $\bar{b}_\lambda \in \bigoplus_{\lambda\in\Lambda} B_\lambda$ be the element that has b_λ as the entry for index λ and entries zero otherwise. Define $\varphi_\lambda \colon A \times B_\lambda \to C$ by $(a, b_\lambda) \mapsto \varphi(a, \bar{b}_\lambda)$. This map is bilinear and thus by the universal property of $A \otimes B_\lambda$ there exists a unique homomorphism $f_\lambda \colon A \otimes B_\lambda \to C$ such that $f_\lambda(a \otimes b_\lambda) = \varphi(a, \bar{b}_\lambda)$. By the properties of the direct sum there exists a unique homomorphism

$$\begin{array}{rcc} f \colon \bigoplus_{\lambda\in\Lambda}(A \otimes B_\lambda) & \to & C \\ (a \otimes b_\lambda)_{\lambda\in\Lambda} & \mapsto & \sum_{\lambda\in\Lambda} f_\lambda(a \otimes b_\lambda). \end{array}$$

But $\sum_{\lambda\in\Lambda} f_\lambda(a \otimes b_\lambda) = \sum_{\lambda\in\Lambda} \varphi(a \otimes \bar{b}_\lambda) = \varphi(a \otimes \sum_{\lambda\in\Lambda} \bar{b}_\lambda) = \varphi(a, (b_\lambda)_{\lambda\in\Lambda})$ and thus $f \circ \eta = \varphi$. In other words $\bigoplus_{\lambda\in\Lambda}(A \otimes B_\lambda)$ together with the bilinear map $\eta \colon A \times \bigoplus_{\lambda\in\Lambda} B_\lambda \to \bigoplus_{\lambda\in\Lambda}(A \otimes B_\lambda) \colon (a, (b_\lambda)) \mapsto (a \otimes b_\lambda)$ satisfies the universal property (L.1). Using the universal property one can also show that an explicit isomorphism is given by

$$\begin{array}{ccc} A \otimes \bigoplus_{\lambda\in\Lambda} B_\lambda & \to & \bigoplus_{\lambda\in\Lambda}(A \otimes B_\lambda) \\ a \otimes (b_\lambda)_{\lambda\in\Lambda} & \mapsto & (a \otimes b_\lambda)_{\lambda\in\Lambda}. \end{array}$$

PROBLEM L.2 (†). Suppose F and F' are free abelian groups. Prove that $F \otimes F'$ is also free.

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Solution. Let B and B' be basis of F and F' respectively. Then $F \cong \bigoplus_{b\in B}\langle b\rangle$ and $F' \cong \bigoplus_{b'\in B'}\langle b'\rangle$. By the previous exercise we have

$$F \otimes F' \cong \bigoplus_{b\in B}\langle b\rangle \otimes \bigoplus_{b'\in B'}\langle b'\rangle \cong \bigoplus_{b\in B, b'\in B'}\langle b\rangle \otimes \langle b'\rangle \cong \bigoplus_{b\otimes b'}\langle b\otimes b'\rangle.$$

This proves that $F\otimes F'$ is free with basis $\{b\otimes b' \mid b\in B, b'\in B'\}$.

Problem L.3 (†). Suppose B is a torsion-free abelian group. Prove that $\Box\otimes B$ and $B\otimes\Box$ are exact functors.

Solution. Notice that a functor T is exact if and only if for every exact sequence $0\to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ the sequence $0 \to T(C) \xrightarrow{T(f)} T(D) \xrightarrow{T(g)} T(E) \to 0$ is also exact. If B is finitely generated and torsion-free then it is free and thus isomorphic to $\mathbb{Z}^n$ for some n. Then $\mathbb{Z}^n\otimes G\cong G^n$ by Proposition 24.8 and $f\otimes \mathrm{id}_B$ becomes the map $(f,\ldots,f)$. Moreover, applying $\Box\otimes B$ to a short exact sequence $0\to C\xrightarrow{f} D\xrightarrow{g} E\to 0$ yields the sequence $0\to C^n \xrightarrow{(f,\ldots,f)} D^n \xrightarrow{(g,\ldots,g)} E^n\to 0$, which is also exact.

For the general case, we start with the following claim:

Claim. Let $0\to C\xrightarrow{f} D$ be exact, and let A be any abelian group. If $x\in\ker(f\otimes\mathrm{id}_A)$ then there exists a finitely generated subgroup $A'\subseteq A$ and an element $x'\in C\otimes A'$ such that $x'\in\ker(f\otimes\mathrm{id}_{A'}\colon C\otimes A'\to D\otimes A')$ and such that $x=(\mathrm{id}_C\otimes i)(x')$, where $i\colon A'\to A$ is the inclusion.

Proof. Suppose $x=\sum_k c_k\otimes a_k$. Then by assumption

$$\sum_{k=1}^n f(c_k)\otimes a_k = 0\in D\otimes A.$$

Let F be the free abelian group with basis $D\times A$ and let N be the subgroup of F generated by all the relations that define the tensor product $D\otimes A$ (i.e. so that $D\otimes A = F/N$.) Then we have an exact sequence $0\to N\to F\xrightarrow{u} D\otimes A\to 0$, where $u(d,a)=d\otimes a$. Since $\sum_k f(c_k)\otimes a_k = 0\in D\otimes A$, there $m\geq 1$ and elements $a_j'\in A$ and $d_j\in D$ for $j=1,\ldots,m$ such that

$$\sum_{k=1}^n f(c_k)\otimes a_k = \sum_{j=1}^m u(d_j, a_j').$$

Now let A' denote the subgroup of A generated by the a_k and the a_j'. Then A' is finitely generated. Set

$$x' := \sum_{k=1}^n c_k\otimes a_k\in C\otimes A'.$$

Then certainly $(\mathrm{id}_C\otimes i)(x')=x$. Moreover $(f\otimes\mathrm{id}_{A'})(x')=0\in D\otimes A'$, since we have ensured that all the relations that make $(f\otimes\mathrm{id}_A)(x)=0$ are present in $D\otimes A'$. ■

Now we prove the result in the case where B is torsion-free. Let $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$ be a short exact sequence. Consider the sequence $0 \to C \otimes B \xrightarrow{f \otimes \mathrm{id}_B} D \otimes B \xrightarrow{g \otimes \mathrm{id}_B} E \otimes B \to 0$. By Proposition 24.10 it is left to show that $\ker(f \otimes \mathrm{id}_B) = 0$. Suppose $x := \sum_{i=1}^{k} c_i \otimes b_i) = \in \ker(f \otimes \mathrm{id}_B)$. Let B' be the subgroup of B given to us by the claim, and let $x' \in C \otimes B'$ be such that $(\mathrm{id}_C \otimes i)(x') = x$, where $i \colon B' \to B$ is the inclusion. Then B' is torsion-free and finitely generated and thus it is free. But this forces $x' = 0$, since we already know that $\square \otimes B'$ is exact. Thus also $x = 0$, and this completes the proof.

Problem L.4 (†). Suppose $T \colon \mathsf{Ab} \to \mathsf{Ab}$ is an additive functor and $0 \to A \to B \to C \to 0$ is a split exact sequence. Prove that $0 \to T(A) \to T(B) \to T(C) \to 0$ is also a split exact sequence.

Solution. Notice that

Claim. A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is split exact if and only if there exist homomorphisms $k \colon B \to A$ and $h \colon C \to B$ such that

$$\begin{aligned} gf &= 0 \\ kf &= \mathrm{id}_A \\ gh &= \mathrm{id}_C \\ kh &= 0 \\ \mathrm{id}_B &= fk + hg \\ &. \end{aligned}$$

Proof. If the sequence splits then k and h exist and by Proposition 12.15 the first three equalities hold. By Proposition 12.16 we also have that $B = \operatorname{im} f \oplus \operatorname{im} h$. As g and k are surjective this implies $B = \operatorname{im} fk \oplus \operatorname{im} hg$ and thus $\mathrm{id}_B = fk + hg$. Moreover, as $kf = \mathrm{id}_A$ this also proves that $kh = 0$. For the other direction it suffices to show that $\ker g \subset \operatorname{im} f$. Let $x \in B$ such that $g(x) = 0$. Then $x = \mathrm{id}_B(x) = fk(x) + hg(x) = fk(x)$ and hence $x \in \operatorname{im} f$. ∎

As T is a functor it preserves the first four equalities. If T is also additive the last equality is also preserved. This proves that T preserves split exact sequences.

Problem L.5. Let $A = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_5$ and let $B = \mathbb{Z}_3 \oplus \mathbb{Z}_5$. Compute $A \otimes B$ and $\operatorname{Tor}(A, B)$.

Solution. We know from Proposition 24.8 and 25.6 and Problem L.1 that for abelian groups A and B_λ the following identities hold:

$$\begin{aligned} A \otimes \mathbb{Z} &\cong A \\ A \otimes \bigoplus_{\lambda \in \Lambda} B_\lambda &\cong \bigoplus_{\lambda \in \Lambda} (A \otimes B_\lambda) \\ \operatorname{Tor}(\mathbb{Z}, A) &= 0 \\ \operatorname{Tor}(A, \bigoplus_{\lambda \in \Lambda} B_\lambda) &\cong \bigoplus_{\lambda \in \Lambda} \mathrm{T}(A, B_{\lambda \in \Lambda}) \end{aligned}$$

We will prove the following general fact:

Claim. $\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_{\gcd(n,m)}$.

Proof. The homomorphism

$$\begin{array}{rcl} \phi\colon \mathbb{Z} & \to & \mathbb{Z}_n \otimes \mathbb{Z}_m \\ 1 & \mapsto & 1 \otimes 1 \end{array}$$

is surjective. The greates common divisor $\gcd(n, m)$ can be defined as the smallest positive integer d which can be written in the form $d = ax + by$, where x and y are integers. If follows that kernel of ϕ is given by $\gcd(n, m)\mathbb{Z}$, which proves the claim. ∎

Another general fact is

Claim. $\mathrm{Tor}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$

Proof. A short free resolution of $\mathbb{Z}_n$ is

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto n} \mathbb{Z} \xrightarrow{1 \mapsto 1} \mathbb{Z}_n \to 0.$$

Applying the fuctor $\square \otimes \mathbb{Z}_m$ gives

$$0 \to \mathbb{Z}_m \xrightarrow{1 \mapsto n} \mathbb{Z}_m \xrightarrow{1 \mapsto 1 \otimes 1} \mathbb{Z}_n \otimes \mathbb{Z}_m \to 0.$$

Then $\mathrm{Tor}(\mathbb{Z}_n, \mathbb{Z}_m) = \ker(\mathbb{Z}_m \xrightarrow{1 \mapsto n} \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$. ∎

Using these properties we compute

$$\begin{aligned} A \otimes B \cong & (\mathbb{Z} \otimes \mathbb{Z}_3) \oplus (\mathbb{Z} \otimes \mathbb{Z}_5) \oplus (\mathbb{Z} \otimes \mathbb{Z}_3) \oplus (\mathbb{Z} \otimes \mathbb{Z}_5) \\ & \oplus (\mathbb{Z}_6 \otimes \mathbb{Z}_3) \oplus (\mathbb{Z}_6 \otimes \mathbb{Z}_5) \oplus (\mathbb{Z}_5 \otimes \mathbb{Z}_3) \oplus (\mathbb{Z}_5 \otimes \mathbb{Z}_5) \\ \cong & \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus 0 \oplus 0 \oplus \mathbb{Z}_5 \\ \cong & \mathbb{Z}_3^3 \oplus \mathbb{Z}_3^3 \end{aligned}$$

and

$$\begin{aligned} \mathrm{Tor}(A, B) \cong & \big(\mathrm{Tor}(\mathbb{Z}, \mathbb{Z}_3) \oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}_5)\big)^2 \oplus \mathrm{Tor}(\mathbb{Z}_6, \mathbb{Z}_3) \oplus \mathrm{Tor}(\mathbb{Z}_6, \mathbb{Z}_5) \\ & \oplus \mathrm{Tor}(\mathbb{Z}_5, \mathbb{Z}_3) \oplus \mathrm{Tor}(\mathbb{Z}_5, \mathbb{Z}_5) \\ \cong & \mathrm{Tor}(\mathbb{Z}_6, \mathbb{Z}_3) \oplus \mathrm{Tor}(\mathbb{Z}_5, \mathbb{Z}_5) \\ \cong & \mathbb{Z}_3 \oplus \mathbb{Z}_5. \end{aligned}$$

Problem L.6 ($\star$). Fix $n \geq 2$. Consider the projection map $p\colon S^n \to \mathbb{R}P^n$. Let $a\colon S^n \to S^n$ be the antipodal map. For this problem you may assume the following fact:

Fact: *If $\sigma\colon \Delta^m \to \mathbb{R}P^n$ is a singular m-simplex, then are precisely two singular m-simplices $\tilde{\sigma}_1, \tilde{\sigma}_2\colon \Delta^m \to S^n$ that satisfy $p \circ \tilde{\sigma}_i = \sigma$ for $i = 1, 2$. Moreover*

$\tilde{\sigma}_2 = a \circ \tilde{\sigma}_1$[1].

Define a chain map

$$q_m \colon C_m(\mathbb{R}P^n; \mathbb{Z}_2) \to C_m(S^n; \mathbb{Z}_2)$$

by $\sigma \mapsto \tilde{\sigma}_1 + \tilde{\sigma}_2$.

1. Show that the following is a short exact sequence of chain complexes:

$$0 \to C_\bullet(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{q_\bullet} C_\bullet(S^n; \mathbb{Z}_2) \xrightarrow{p_\#} C_\bullet(\mathbb{R}P^n; \mathbb{Z}_2) \to 0$$

2. Suppose $f \colon S^n \to S^n$ is an odd map (that is, $f \circ a = a \circ f$). Let $h \colon \mathbb{R}P^n \to \mathbb{R}P^n$ denote the induced map (so that $p \circ f = h \circ p$). Show that the following diagram commutes for every m:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_m(\mathbb{R}P^n; \mathbb{Z}_2) & \xrightarrow{q_m} & C_m(S^n; \mathbb{Z}_2) & \xrightarrow{p_\#} & C_m(\mathbb{R}P^n; \mathbb{Z}_2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle h_\#} & & \\ 0 & \longrightarrow & C_m(\mathbb{R}P^n; \mathbb{Z}_2) & \xrightarrow[q_m]{} & C_m(S^n; \mathbb{Z}_2) & \xrightarrow[p_\#]{} & C_m(\mathbb{R}P^n; \mathbb{Z}_2) & \longrightarrow & 0 \end{array}$$

3. Deduce that $H_n(f) \colon H_n(S^n; \mathbb{Z}_2) \to H_n(S^n; \mathbb{Z}_2)$ is an isomorphism. *Hint:* Argue by induction on m, using the associated long exact sequence in homology from the short exact sequence of chain complexes in part (1).

4. Deduce that f has odd degree.

Solution. We call the simplices $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ the lifts of σ.

1. The map $p_\#$ is surjective as every $\sigma \colon \Delta^k \to \mathbb{R}P^n$ has two lifts $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$. Since we work with coefficients in $\mathbb{Z}_2$ the kernel of $p_\#$ is generated by $\tilde{\sigma}_1 + \tilde{\sigma}_2$. Thus $\ker p_\# = \operatorname{im} q_k$. The map q_k is injective, as the lift of a cycle will be a cycle again. (If $\delta(\tau) = \sigma$, then $\delta(\tilde{\tau}_i) = \tilde{\sigma}_i$ for $i = 1, 2$.) The boundaries $\partial\tilde{\sigma}_1$ and $\partial\tilde{\sigma}_1$ lift the boundary $\partial\sigma$, i.e. $p_\#(\partial\tilde{\sigma}_i) = \partial\sigma$ for $i = 1, 2$. If follows that $p_\#$ and q commute with the boundary operators.

2. Notice that $pf\tilde{\sigma}_i = hp\tilde{\sigma}_i = h\sigma$, since $pf = hp$. Thus for every k-simplex $\sigma \colon \Delta^k \to \mathbb{R}P^n$ with lifts $\tilde{\sigma}_i$ $i = 1, 2$, the two lifts of $h\sigma$ are $f\tilde{\sigma}_i$ $i = 1, 2$. This immediately implies that the left square in the diagram commutes. The right square commutes since $pf = hp$.

3. Since $\operatorname{Tor}(\mathbb{Z}, \mathbb{Z}_2) = 0$ it follows from the Universal Coefficient Theorem that

$$H_i(S^n; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & i = 0, n \\ 0, & i \neq 0, n. \end{cases}$$

Furthermore, $\operatorname{Tor}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$ and thus

$$H_i(\mathbb{R}P^n; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2, & 0 \leq i \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

[1] This follows from standard covering space theory that we will cover later on in the semester.

Using these facts and part two of the exercise we get a long exact sequence in homology

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & H_n(\mathbb{R}P^n,\mathbb{Z}_2) & \xrightarrow{H_n(q)} & H_n(S^n,\mathbb{Z}_2) & \xrightarrow{H_n(p)} & H_n(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\delta} & H_{n-1}(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 & \longrightarrow & ... \\
 & & {\scriptstyle H_n(\overline{f})}\downarrow & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_n(\overline{f})} & & {\scriptstyle H_{n-1}(\overline{f})}\downarrow & & & & \\
0 & \longrightarrow & H_n(\mathbb{R}P^n,\mathbb{Z}_2) & \xrightarrow{H_n(q)} & H_n(S^n,\mathbb{Z}_2) & \xrightarrow{H_n(p)} & H_n(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\delta} & H_{n-1}(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 & \longrightarrow & ...
\end{array}$$

$$\begin{array}{ccccccccccc}
... & \longrightarrow & 0 & \longrightarrow & H_i(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\delta} & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 & \longrightarrow & ... \\
 & & & & \downarrow{\scriptstyle H_i(\overline{f})} & & {\scriptstyle H_{i-1}(\overline{f})}\downarrow & & & & \\
... & \longrightarrow & 0 & \longrightarrow & H_i(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\delta} & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 & \longrightarrow & ...
\end{array}$$

$$\begin{array}{ccccccccccccc}
... & \longrightarrow & 0 & \longrightarrow & H_1(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\delta} & H_0(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{H_0(q)} & H_0(S^n;\mathbb{Z}_2) & \xrightarrow{H_0(p)} & H_0(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle H_1(\overline{f})} & & {\scriptstyle H_0(\overline{f})}\downarrow & & \downarrow{\scriptstyle H_0(f)} & & \downarrow{\scriptstyle H_n(\overline{f})} & & \\
... & \longrightarrow & 0 & \longrightarrow & H_1(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\delta} & H_0(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{H_0(q)} & H_0(S^n;\mathbb{Z}_2) & \xrightarrow{H_0(p)} & H_0(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0
\end{array}$$

In this diagram all non-trivial homology groups are isomorphic to $\mathbb{Z}_2$ (apart from $H_0(S^n;\mathbb{Z}_2)$ which is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.) Notice that a homomorphism $\mathbb{Z}_2 \to \mathbb{Z}_2$ that is either injective or surjective must be an isomorphism. Therefore by exactness of the rows the above diagram simplifies to

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & H_n(\mathbb{R}P^n,\mathbb{Z}_2) & \xrightarrow{\cong} & H_n(S^n,\mathbb{Z}_2) & \xrightarrow{0} & H_n(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\cong} & H_{n-1}(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 & \longrightarrow & ... \\
 & & {\scriptstyle H_n(\overline{f})}\downarrow & & \downarrow{\scriptstyle H_n(f)} & & \downarrow{\scriptstyle H_n(\overline{f})} & & {\scriptstyle H_{n-1}(\overline{f})}\downarrow & & & & \\
0 & \longrightarrow & H_n(\mathbb{R}P^n,\mathbb{Z}_2) & \xrightarrow{\cong} & H_n(S^n,\mathbb{Z}_2) & \xrightarrow{0} & H_n(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\cong} & H_{n-1}(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 & \longrightarrow & ...
\end{array}$$

$$\begin{array}{ccccccccccc}
... & \longrightarrow & 0 & \longrightarrow & H_i(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\cong} & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 & \longrightarrow & ... \\
 & & & & \downarrow{\scriptstyle H_i(\overline{f})} & & {\scriptstyle H_{i-1}(\overline{f})}\downarrow & & & & \\
... & \longrightarrow & 0 & \longrightarrow & H_i(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\cong} & H_{i-1}(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 & \longrightarrow & ...
\end{array}$$

$$\begin{array}{ccccccccccccc}
... & \longrightarrow & 0 & \longrightarrow & H_1(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\cong} & H_0(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{0} & H_0(S^n;\mathbb{Z}_2) & \xrightarrow{\cong} & H_0(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle H_1(\overline{f})} & & {\scriptstyle H_0(\overline{f})}\downarrow & & \downarrow{\scriptstyle H_0(f)} & & \downarrow{\scriptstyle H_0(\overline{f})} & & \\
... & \longrightarrow & 0 & \longrightarrow & H_1(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{\cong} & H_0(\mathbb{R}P^n;\mathbb{Z}_2) & \xrightarrow{0} & H_0(S^n;\mathbb{Z}_2) & \xrightarrow{\cong} & H_0(\mathbb{R}P^n;\mathbb{Z}_2) & \longrightarrow & 0
\end{array}$$

The maps $H_0(f)\colon H_0(S^n;\mathbb{Z}_2) \to H_0(S^n;\mathbb{Z}_2)$ and $H_0(\overline{f})\colon H_0(\mathbb{R}P^n;\mathbb{Z}_2) \to H_0(\mathbb{R}P^n;\mathbb{Z}_2)$ are obviously isomorphisms (as a singular 0-simplex always generates $H_0(X)$ for X path connected, cf. Proposition 8.3.) Using the isomorphisms in the diagram above we conclude by induction on the dimension i that $H_i(f)$ and $H_i(\overline{f})$ are isomorphisms for all i.

4. Notice that for any pair $X' \subset X$ and a group homomorphism $\phi\colon A \to A'$ induces a chain map $C_n(X, X'; A) \to C_n(X, X'; A')$ by $\phi_\#(\sum_i a_i\sigma_i) = \sum_i \phi(a_i)\sigma_i$. Indeed,

$$\partial\phi_\#(\sum_i a_i\sigma_i) = \sum_i \phi(a_i)\partial(\sigma_i) = \phi_\#\partial(\sum_i a_i\sigma_i).$$

Thus it also induces a homomorphism $H_n(\phi)\colon H_n(X, X'; A) \to H_n(X, X'; A')$. If $f\colon (X, X') \to (Y, Y')$, one can see that $H_n(\phi)$ commutes with $H_n(f)$ as

$$H_n(f)H_n(\phi)\left\langle \sum_i a_i\sigma_i \right\rangle = \left\langle \sum_i \phi(a_i) f\circ\sigma_i \right\rangle = H_n(\phi)H_n(f)\left\langle \sum_i a_i\sigma_i \right\rangle.$$

Let $f\colon S^n \to S^n$ be a map of degree d and let $\phi\colon \mathbb{Z} \xrightarrow{1\mapsto 1} \mathbb{Z}_2$ be a group homomorphism. Then we have the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z} & \xrightarrow{\cong} & H_n(S^n;\mathbb{Z}) & \xrightarrow{H_n(f)} & H_n(S^n;\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle H_n(\phi)} & & \downarrow{\scriptstyle H_n(\phi)} & & \downarrow{\scriptstyle \phi} \\
\mathbb{Z}_2 & \xrightarrow{\cong} & H_n(S^n;\mathbb{Z}_2) & \xrightarrow{H_n(f)} & H_n(S^n;\mathbb{Z}_2) & \xrightarrow{\cong} & \mathbb{Z}_2.
\end{array}$$

Commutativity and the fact that the map $H_n(f)$ across the top is multiplication by d implies that also the map $H_n(f)$ across the bottom is multiplication by d. Hence the map $H_n(f)\colon H_n(S^n;\mathbb{Z}_2) \to H_n(S^n;\mathbb{Z}_2)$ is an isomorphism if only only if $d = 1 \mod 2$, or in other words if d is odd. By part (4) an odd map $f\colon S^n \to S^n$ has odd degree.

Problem Sheet M

This Problem Sheet is based on Lectures 26 and 27. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

PROBLEM M.1 (†). Let $C_\bullet, C'_\bullet, D_\bullet, D'_\bullet$ be four non-negative chain complexes, and let

$$f, f' \colon C_\bullet \to D_\bullet, \qquad g, g' \colon C'_\bullet \to D'_\bullet,$$

be chain maps. Assume that f and f' are chain homotopic and g and g' are chain homotopic. Prove that $f \otimes g$ is chain homotopic to $f' \otimes g'$.

PROBLEM M.2 (†). Let $0 \to C_\bullet \xrightarrow{f} C'_\bullet \xrightarrow{g} C''_\bullet \to 0$ be a short exact sequence of non-negative chain complexes. Let $D_\bullet$ be a non-negative free chain complex. Prove that

$$0 \to C_\bullet \otimes D_\bullet \xrightarrow{f \otimes \mathrm{id}} C'_\bullet \otimes D_\bullet \xrightarrow{g \otimes \mathrm{id}} C''_\bullet \otimes D_\bullet \to 0$$

is another short exact sequence of chain complexes.

PROBLEM M.3. Compute the homology of $\mathbb{R}P^n \times \mathbb{R}P^m$ for all even m and n.

PROBLEM M.4 (⋆). In each of the following three examples, show that X and Y have the same singular homology groups but they are *not* homotopy equivalent.

1. $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$.
2. $X = \mathbb{R}P^3$ and $Y = \mathbb{R}P^2 \vee S^3$.
3. $X = S^1 \vee S^2 \vee S^3$ and $Y = S^1 \times S^2$.

Solutions to Problem Sheet M

This Problem Sheet is based on Lectures 26 and 27. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem M.1 (†). Let $C_\bullet, C'_\bullet, D_\bullet, D'_\bullet$ be four non-negative chain complexes, and let

$$f, f' \colon C_\bullet \to D_\bullet, \qquad g, g' \colon C'_\bullet \to D'_\bullet,$$

be chain maps. Assume that f and f' are chain homotopic and g and g' are chain homotopic. Prove that $f \otimes g$ is chain homotopic to $f' \otimes g'$.

Solution. Suppose that $Q \colon C_\bullet \to D_\bullet$ is a chain homotopy from f to f', that is $f - f' = Q\partial + \partial Q$. We claim that $Q \otimes g$ is chain homotopy from $f \otimes g$ to $f' \otimes g$. Indeed, if $c \in C_n$ and $d \in D_\bullet$ then

$$\begin{aligned}((Q\otimes g)\circ\Delta + \Delta\circ(Q\otimes g))(c\otimes d) &= (Q\otimes g)\,(\partial c\otimes d + (-1)^n(c\otimes\partial d)) + \Delta(Qc\otimes gd)\\ &= Q\partial c\otimes gd + (-1)^n Qc\otimes g\partial d\\ &\quad + (\partial Qc\otimes gd) + (-1)^{n+1}(Qc\otimes\partial gd)\\ &= ((Q\partial c + \partial Qc)\otimes gd) + (-1)^n(Qc, \underbrace{g\partial d - \partial gd}_{=0})\\ &= ((f-f')(c)\otimes gd)\\ &= (f\otimes g)(c\otimes d) - (f'\otimes g)(c\otimes d).\end{aligned}$$

Similarly if P is a chain homotopy from g to g' then $f' \otimes P$ is a chain homotopy from $f' \otimes g$ to $f' \otimes g'$. Putting these together we have chain homotopies:

$$f\otimes g \underset{Q\otimes g}{\simeq} f'\otimes g \underset{f'\otimes P}{\simeq} f'\otimes g',$$

which completes the proof.

Problem M.2 (†). Let $0 \to C_\bullet \xrightarrow{f} C'_\bullet \xrightarrow{g} C''_\bullet \to 0$ be a short exact sequence of non-negative chain complexes. Let $D_\bullet$ be a non-negative free chain complex. Prove that

$$0 \to C_\bullet\otimes D_\bullet \xrightarrow{f\otimes \mathrm{id}} C'_\bullet\otimes D_\bullet \xrightarrow{g\otimes\mathrm{id}} C''_\bullet\otimes D_\bullet \to 0$$

is another short exact sequence of chain complexes.

Solution. As D_n is free for every n we have with Problem L.3 that

$$0 \to C_i\otimes D_n \xrightarrow{f_i\otimes \mathrm{id}} C'_i\otimes D_n \xrightarrow{g_i\otimes\mathrm{id}} C''_i\otimes D_n \to 0$$

is exact for every i. Thus also

$$0 \to \bigoplus_{i+j=n} C_i \otimes D_j \xrightarrow{\sum f_i \otimes \mathrm{id}} \bigoplus_{i+j=n} C'_i \otimes D_j \xrightarrow{\sum g_i \otimes \mathrm{id}} \bigoplus_{i+j=n} C''_i \otimes D_j \to 0$$

is exact for every n. By definition $(C_\bullet \otimes C'_\bullet)_n = \sum_{i+j=n} C_i \otimes C'_j$ and thus

$$0 \to C_\bullet \otimes D_\bullet \xrightarrow{f \otimes \mathrm{id}} C'_\bullet \otimes D_\bullet \xrightarrow{g \otimes \mathrm{id}} C''_\bullet \otimes D_\bullet \to 0$$

is exact.

Problem M.3. Compute the homology of $\mathbb{R}P^n \times \mathbb{R}P^m$ for all even m and n.

Solution. Assume that $m < n$ and that i is odd. Then for any $0 \leq k \leq i$, at least one of k and $i-k$ are even. Thus

$$\bigoplus_{0 \leq k \leq i} (H_k(\mathbb{R}P^n) \otimes H_{i-k}(\mathbb{R}P^m)) = H_0(\mathbb{R}P^n) \otimes H_i(\mathbb{R}P^m) \oplus H_i(\mathbb{R}P^n) \otimes H_0(\mathbb{R}P^m).$$

Thus:

$$\bigoplus_{0 \leq k \leq i} (H_k(\mathbb{R}P^n) \otimes H_{i-k}(\mathbb{R}P^m)) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2, & i < m \\ \mathbb{Z}_2 & m < i < n, \\ 0, & i > n. \end{cases}$$

Likewise the torsion groups

$$\bigoplus_{0 \leq k \leq i-1} \mathrm{Tor}\,((H_k(\mathbb{R}P^n), H_{i-k-1}(\mathbb{R}P^m))$$

contribute $r(n,m,i)$ many copies of $\mathbb{Z}_2$, where $r(n,m,i)$ is the number of $0 \leq k \leq i$ such that k is odd and both $k < n$ and $i-k-1 < m$. Explicitly,

$$r(n,m,i) = \begin{cases} \frac{i-1}{2}, & i < m, \\ \frac{m}{2}, & m < i < n, \\ \max\{\frac{n+m+1-i}{2}, 0\}, & i > n. \end{cases}$$

Putting this altogether, we obtain for i odd that:

$$H_i(\mathbb{R}P^n \times \mathbb{R}P^m) = \bigoplus_{s(n,m,i)} \mathbb{Z}_2,$$

where

$$s(n,m,i) = \begin{cases} 2 + \frac{i-1}{2}, & i < m, \\ 1 + \frac{m}{2}, & m < i < n, \\ \max\{\frac{n+m+1-i}{2}, 0\}, & i > n, \end{cases}$$

A similar analysis works if i is even. The case $m = n$ is easier.

Here is an alternative proof, which uses the idea of a *double complex*. The chain complex of $\mathbb{R}P^n$ and $\mathbb{R}P^m$ are both of the form

$$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Taking their tensor product yields

$$
\begin{array}{ccccccccccc}
\mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \cdots & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle -2} & & \downarrow{\scriptstyle 2} & & & & \downarrow{\scriptstyle -2} & & \downarrow{\scriptstyle 2} \\
\mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \cdots & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
\mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \cdots & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle -2} & & \downarrow{\scriptstyle 2} & & & & \downarrow{\scriptstyle -2} & & \downarrow{\scriptstyle 2} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \cdots & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\
\downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle -2} & & \downarrow{\scriptstyle 2} & & & & \downarrow{\scriptstyle -2} & & \downarrow{\scriptstyle 2} \\
\mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \cdots & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
\mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \cdots & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z}.
\end{array}
$$

Some of the vertical maps become -2, which follows from the fact that $\Delta|_{C_i \otimes C_j} = \partial \otimes \mathrm{id} + (-1)^i \mathrm{id} \otimes \partial$. $(C_\bullet(\mathbb{R}P^n) \otimes C_\bullet(\mathbb{R}P^m))_k$ is the direct sum of the groups at postion (i, j) such that $i + j = k$. Consider the square

$$
\begin{array}{ccccc}
 & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & \\
\xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} \\
 & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle -2} & \\
\xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} \\
 & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} &
\end{array}
\tag{M.1}
$$

The homology groups at the top right, top left and bottom left of (M.1) are all zero. At the bottom right the homology is isomorphic to $\mathbb{Z}_2$. Suppose (i, j) is the index of the group at the bottom right of this square. The boundary operator (for notational simplicity, we will write C_j for all the groups, etc)

$$\Delta|_{(C_{i+1} \otimes C_j) \oplus (C_i \otimes C_{j+1})} : (C_{i+1} \otimes C_j) \oplus (C_i \otimes C_{j+1}) \to C_i \otimes C_j$$

maps the element $1 \otimes 1 + 1 \otimes 1$ to $2 \otimes 1 + 1 \otimes (-2) = 0$. But

$$\Delta|_{(C_{i+1} \otimes C_j + 1)} : (C_{i+1} \otimes C_{j+1}) \to (C_{i+1} \otimes C_j) \otimes (C_i \otimes C_{j+1})$$

sends $1 \otimes 1$ to $2 \otimes 1 + 1 \otimes 2$. Thus $1 \otimes 1 + 1 \otimes 1$ is a boundary and $2(1 \otimes 1 + 1 \otimes 1)$ is a cycle. Hence this contributes another $\mathbb{Z}_2$ factor to the homology group $H_{i+j+1}(C_\bullet(\mathbb{R}P^n) \otimes$

$C_\bullet(\mathbb{R}P^m))$. We will write this $\mathbb{Z}_2$ factor at position $(i+1, j)$. (Although strictly speaking this is not the homology at the index $(i+1, j)$. However, we only care to represented it somewhere in the diagonal corresponding to the indexes (i', j') such that $i' + j' = i + j + 1$.) The homology groups can therefor pictorially be represented as follows:

$$\begin{array}{ccccccc}
0 & \mathbb{Z}_2 & 0 & \dots & 0 & \mathbb{Z}_2 & 0 \\
0 & \mathbb{Z}_2 & 0 & \dots & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
0 & \mathbb{Z}_2 & 0 & \dots & 0 & \mathbb{Z}_2 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & \mathbb{Z}_2 & 0 & \dots & 0 & \mathbb{Z}_2 & 0 \\
0 & \mathbb{Z}_2 & 0 & \dots & 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\
0 & \mathbb{Z}_2 & 0 & \dots & 0 & \mathbb{Z}_2 & \mathbb{Z}
\end{array}$$

By Theorem 27.6 and 26.5, $H_k(\mathbb{R}P^n \times \mathbb{R}P^m) \cong H_k(C_\bullet(\mathbb{R}P^n) \otimes C_\bullet(\mathbb{R}P^m))$. Thus $H_k(\mathbb{R}P^n \times \mathbb{R}P^m)$ is the sum of the groups in the diagram at position $(n-i, j)$ with $i + j = k$.

PROBLEM M.4 ($\star$). In each of the following three examples, show that X and Y have the same homology groups but they are *not* homotopy equivalent.

1. $X = S^1 \times S^1$ and $Y = S^1 \vee S^1 \vee S^2$.
2. $X = \mathbb{R}P^3$ and $Y = \mathbb{R}P^2 \vee S^3$.
3. $X = S^1 \vee S^2 \vee S^3$ and $Y = S^1 \times S^2$.

SOLUTION.

1. With the Künneth Formula we compute

$$\begin{aligned}
H_2(S^1 \times S^1) &\cong H_1(S^1) \otimes H_1(S^1) \cong \mathbb{Z} \\
H_1(S^1 \times S^1) &\cong H_1(S^1) \otimes H_0(S^1) \oplus H_0(S^1) \otimes H_1(S^1) \cong \mathbb{Z}^2 \\
H_0(S^1, S^1) &\cong H_0(S^1) \otimes H_0(S_1) \cong \mathbb{Z},
\end{aligned}$$

where we used the fact that $\mathrm{Tor}(H_0(S^1), \square) = 0$ since $H_0(S^1)$ is free. In order to compute the homology groups of Y, let X_1 be a small open neighbourhood of $S^1 \vee S^1$ that deformation retracts onto $S^1 \vee S^1$. Likewise, let X_2 be a small open neighbourhood of S^2 that deformation retracts onto S^2. Mayer Vietoris gives

$$\begin{aligned} H_2(S^1 \vee S^1 \vee S^2) &\cong H_2(S^2) \cong \mathbb{Z} \\ H_1(S^1 \vee S^1 \vee S^2) &\cong H_1(S^1) \oplus H_1(S^1) \cong \mathbb{Z}^2 \\ H_0(S^1 \vee S^1 \vee S^2) &\cong \mathbb{Z}. \end{aligned}$$

X and Y are *not* homotopy equivalent since they have different fundamental groups. The fundamental group of X has been computed in Problem C.3:

$$\pi_1(S^1 \times S^1, p) \cong \mathbb{Z}^2.$$

The fundamental group of Y can be computed with Seifert-van-Kampen. As before, let X_1 be a small open neighbourhood of $S^1 \vee S^1$ that deformation retracts onto $S^1 \vee S^1$. Let X_2 be a small open neighbourhood of S^2 that deformation retracts onto S^2. Then

$$\pi_1(S^1 \vee S^1 \vee S^2, p) \cong \pi_1(S^1 \vee S^1) * \pi_1(S^2) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}.$$

2. We already know the homology groups of $\mathbb{R}P^3$:

$$\begin{aligned} H_3(\mathbb{R}P^3) &\cong \mathbb{Z} \\ H_2(\mathbb{R}P^3) &\cong 0 \\ H_1(\mathbb{R}P^3) &\cong \mathbb{Z}_2 \\ H_0(\mathbb{R}P^3) &\cong \mathbb{Z} \end{aligned}$$

Using Mayer Vietoris we can also show that

$$\begin{aligned} H_3(\mathbb{R}P^2 \vee S^3) &\cong H_3(S^3) \cong \mathbb{Z} \\ H_2(\mathbb{R}P^2 \vee S^3) &\cong H_2(\mathbb{R}P^2) \cong 0 \\ H_1(\mathbb{R}P^2 \vee S^3) &\cong H_1(\mathbb{R}P^2) \cong \mathbb{Z}_2 \\ H_0(\mathbb{R}P^2 \vee S^3) &\cong \mathbb{Z}, \end{aligned}$$

which proves that X and Y have the same homology groups.

One way of seeing that X and Y are not homotopy equivalent is to show that they have non-isomorphic cohomology rings. The cohomology ring of $\mathbb{R}P^3$ with coefficients in $\mathbb{Z}_2$ is isomorphic to $\mathbb{Z}_2[t]/t^3$. The cohomology ring of Y is isomorphic to $\mathbb{Z}_2[t]/t^2 \oplus \mathbb{Z}_2$. These facts will be proved shortly in lectures.

Another way of seeing that they are not homotopy equivalent is by showing that their universal covers are different. The universal cover of X is S^3 and the universal cover of Y is $S^2 \vee S^3$. This will also be covered in the lecture in a few weeks.

3. Using Mayer Vietoris we compute

$$\begin{aligned}
H_3(S^1 \vee S^2 \vee S^3) &\cong H_3(S^2) \cong \mathbb{Z} \\
H_2(S^1 \vee S^2 \vee S^3) &\cong H_2(S^2) \cong \mathbb{Z} \\
H_1(S^1 \vee S^2 \vee S^3) &\cong H_1(S^1) \cong \mathbb{Z} \\
H_0(S^1 \vee S^2 \vee S^3) &\cong \mathbb{Z}
\end{aligned}$$

and the Künneth formula yields

$$\begin{aligned}
H_3(S^1 \times S^2) &\cong H_1(S^1) \otimes H_2(S^2) \cong \mathbb{Z} \\
H_2(S^1 \times S^2) &\cong H_1(S^1) \otimes H_1(S^2) \oplus H_0(S^1) \otimes H_2(S^2) \cong \mathbb{Z} \\
H_1(S^1 \times S^2) &\cong H_1(S^1) \otimes H_0(S^2) \oplus H_0(S^1) \otimes H_1(S^2) \cong \mathbb{Z} \\
H_0(S^1 \times S^2) &\cong \mathbb{Z}
\end{aligned}$$

There are again several methods to show these spaces are not homotopy equivalent. Here is one using the universal cover.

Suppose $f\colon S^1 \vee S^2 \vee S^3 \to S^1 \times S^2$ is a homotopy equivalence. Then f induces an isomorphism of the homology groups in every degree. In particular $H_3(f)\colon H_3(S^1 \vee S^2 \vee S^3) \to H_3(S^1 \times S^2)$ is an isomorphism. Consider the inclusion $i\colon S^3 \hookrightarrow S^1 \vee S^2 \vee S^3$. Set $g := f \circ i$. Using for example cellular homology we see that $H_3(i)\colon H_3(S^3) \to H_3(S^1 \vee S^2 \vee S^3)$ is an isomorphism, and thus also $H_3(g)\colon H_3(S^3) \to H_3(S^1 \times S^2)$ is an isomorphisms.

The universal cover of Y is $p\colon \mathbb{R} \times S^2 \to S^1 \times S^2$. By the properties of the universal cover a map $S^3 \to S^1 \times S^2$ must factor through p:

$$\begin{array}{ccc}
S^3 & \xrightarrow{\tilde{g}} & \mathbb{R} \times S^2 \\
 & {\scriptstyle g}\searrow & \downarrow {\scriptstyle p} \\
 & & S^1 \times S^2
\end{array}$$

and we get the following commutative diagram:

$$\begin{array}{ccc}
H_3(S^3) & \xrightarrow{H_3(\tilde{g})} & H_3(\mathbb{R} \times S^1) \\
 & {\scriptstyle H_3(g)}\searrow & \downarrow {\scriptstyle H_3(p)} \\
 & & H_3(S^1 \times S^2).
\end{array}$$

As $H_3(\mathbb{R} \times S^2) \cong H_3(S^2) \cong 0$ this contradicts the fact that $H_3(g)$ is an isomorphism. Thus the homotopy equivalence f cannot exist.

Problem Sheet N

This Problem Sheet is based on Lectures 28-31. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

PROBLEM N.1 (†). Let A be any abelian group. Prove that $\mathrm{Hom}(\square, A)$ and $\mathrm{Hom}(A, \square)$ are left exact. That is, prove that if $B \xrightarrow{f} B' \xrightarrow{g} B'' \to 0$ is exact then so is

$$0 \to \mathrm{Hom}(B'', A) \xrightarrow{\mathrm{Hom}(g,A)} \mathrm{Hom}(B', A) \xrightarrow{\mathrm{Hom}(f,A)} \mathrm{Hom}(B, A),$$

and if $0 \to B \xrightarrow{f} B' \xrightarrow{g} B''$ is exact then so is

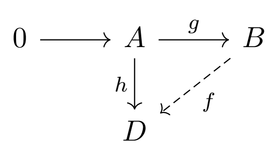
$$0 \to \mathrm{Hom}(A, B) \xrightarrow{\mathrm{Hom}(A,f)} \mathrm{Hom}(A, B') \xrightarrow{\mathrm{Hom}(A,g)} \mathrm{Hom}(A, B'').$$

Now let F be a free abelian group. Prove that $\mathrm{Hom}(F, \square)$ is an exact functor.

PROBLEM N.2 (†). Let A be a finitely generated abelian group, with torsion subgroup $T(A)$. Prove that $\mathrm{Hom}(A, \mathbb{Z}) \cong A/T(A)$ and that $\mathrm{Ext}(A, \mathbb{Z}) \cong T(A)$.

PROBLEM N.3 (†). This problem explores divisible groups.

1. Prove that an abelian group D is divisible if and only if the following property holds: Suppose $g\colon A \to B$ is an injective homomorphism of abelian groups and $h\colon A \to D$ is a homomorphism. Then there exists a homomorphism $f\colon B \to D$ such that $fg = h$:



   (Compare this to Lemma 22.3.)

2. Deduce that if D is divisible then $\mathrm{Hom}(\square, D)$ is an exact functor, and hence $\mathrm{Ext}(A, D) = 0$ for any abelian group D.

3. Prove that any abelian group B is a subgroup of a divisible group D. Deduce that for any abelian group B, there exists a short exact sequence

$$0 \to B \to D \xrightarrow{f} E \to 0$$

   where both D and E are divisible. In analogy with short free resolutions (Definition 24.11), let us[1] call such a sequence a *short divisible resolution* of B.

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[1] As with the terminology "short free resolution", this is *not* standard.

4. Let A and B be abelian groups. Let $0 \to B \to D \xrightarrow{f} E \to 0$ denote a short divisible resolution of B. Prove there is an exact sequence

$$0 \to \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, D) \to \operatorname{Hom}(A, E) \to \operatorname{Ext}(A, B) \to 0,$$

and hence that

$$\operatorname{Ext}(A, B) \cong \operatorname{coker} \operatorname{Hom}(A, f).$$

Hint: Use (29.2) from Theorem 29.3. As remarked there, the proof of (29.2) is essentially identical to the corresponding exact sequences in Theorem 25.6, and hence we may take this as already proved. This is *not* true of the first exact sequence (29.1) from Theorem 29.3. (In fact, you will prove (29.1) in part (8) of this problem below.)

5. Let $0 \to B \to D \xrightarrow{f} E \to 0$ denote a short divisible resolution of B. Consider a cochain complex $(C^\bullet, d)$ with

$$C^n := \begin{cases} D, & n = 0, \\ E, & n = 1, \\ 0, & n \neq 0, 1. \end{cases}$$

and differential $d\colon C^0 \to C^1$ given by $f\colon D \to E$. Prove that $H^0(C^\bullet) = B$. Now let A denote any other abelian group, and consider the new[2] cochain complex $\operatorname{Hom}(A, C^\bullet)$. Prove that $H^1(\operatorname{Hom}(A, C^\bullet)) = \operatorname{Ext}(A, B)$.

6. Prove the following analogue of Proposition 22.4: if $h\colon B \to B'$ is a homomorphism of abelian groups, and $0 \to B \to D \to E \to 0$ and $0 \to B' \to D' \to E' \to 0$ are short divisible resolutions of B and B' respectively, with corresponding cochain complexes (as in the previous part) $C^\bullet$ and $(C')^\bullet$, then there exists a chain map $m\colon C^\bullet \to (C')^\bullet$ such that $H^0(m) = h$. Prove moreover that any two such chain maps m and m' are chain homotopic.

7. Deduce that $\operatorname{Ext}(\square, A)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a well-defined contravariant functor.

8. If $0 \to A \to A' \to A'' \to 0$ is an exact sequence of abelian groups then for any abelian group B there is an exact sequence

$$0 \to \operatorname{Hom}(A'', B) \to \operatorname{Hom}(A', B) \to \operatorname{Hom}(A, B) \to \operatorname{Ext}(A'', B) \to \operatorname{Ext}(A', B) \to \operatorname{Ext}(A, B) \to 0.$$

Problem N.4. Let X be a topological space and R a commutative ring. Assume that the additive group $H^\star(X; R)$ has no elements of order 2. Prove that if $\langle \alpha \rangle \in H^\star(X; R)$ has odd degree then $\langle \alpha \rangle \smile \langle \alpha \rangle = 0$.

Problem N.5 ($\star$). Compute the cohomology ring structure of $H^\star(S^n)$ and $H^\star(T^n)$, where T^n is the n-torus $\underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$.

[2]This is again a cochain complex, since $\operatorname{Hom}(A, \square)$ is a covariant functor.

Solutions to Problem Sheet N

Problem N.1 (†). Let A be any abelian group. Prove that $\operatorname{Hom}(\square, A)$ and $\operatorname{Hom}(A, \square)$ are left exact. That is, prove that if $B \xrightarrow{f} B' \xrightarrow{g} B'' \to 0$ is exact then so is

$$0 \to \operatorname{Hom}(B'', A) \xrightarrow{\operatorname{Hom}(g,A)} \operatorname{Hom}(B', A) \xrightarrow{\operatorname{Hom}(f,A)} \operatorname{Hom}(B, A), \tag{N.1}$$

and if $0 \to B \xrightarrow{f} B' \xrightarrow{g} B''$ is exact then so is

$$0 \to \operatorname{Hom}(A, B) \xrightarrow{\operatorname{Hom}(A,f)} \operatorname{Hom}(A, B') \xrightarrow{\operatorname{Hom}(A,g)} \operatorname{Hom}(A, B''). \tag{N.2}$$

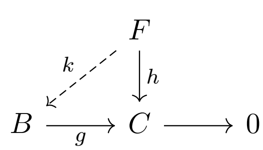
Now let F be a free abelian group. Prove that $\operatorname{Hom}(F, \square)$ is an exact functor.

Solution. Let us prove that (N.1) is exact. There are three things to show:

1. $\operatorname{Hom}(g, A)$ is injective: If $h \in \operatorname{Hom}(B'', A)$ satisfies $\operatorname{Hom}(g, A)(h) = 0$ then h vanishes on the image of g. But g is surjective and thus h is identically zero.
2. $\operatorname{im} \operatorname{Hom}(g, A) \subseteq \ker \operatorname{Hom}(f, A)$: If If $h \in \operatorname{Hom}(B'', A)$ then $\operatorname{Hom}(f, A) \circ \operatorname{Hom}(g, A)(h) = h \circ g \circ f = 0$ as $g \circ f = 0$.
3. $\ker \operatorname{Hom}(f, A) \subseteq \operatorname{im} \operatorname{Hom}(g, A)$: This one is harder. Suppose $k \in \operatorname{Hom}(B', A)$ satisfes $k \circ f = 0$. Define $h\colon B'' \to A$ by $h(b'') = k(b')$ if $g(b') = b''$. This is well defined, since if $g(b') = g(b'_1) = b''$ then $b' - b'_1 \in \ker g = \operatorname{im} f$. Thus $b' - b'_1 = f(b)$ for some $b \in B$, and thus $g(b') - g(b'_1) = gf(b) = 0$. Now observe that $\operatorname{Hom}(g, A)(h) = h \circ g = k$, since $k \circ g = h$ for every $b'' \in B$. Thus $k \in \operatorname{im} \operatorname{Hom}(g, A)$.

The proof that (N.2) is similar, and we omit it.

Finally, let us show that $\operatorname{Hom}(F, \square)$ is exact. By (N.2), we need only show that if if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact then $\operatorname{Hom}(F, g)\colon \operatorname{Hom}(F, B) \to \operatorname{Hom}(F, C)$ is surjective. So suppose $h\colon F \to C$. By Lemma 22.3, there exists $k\colon F \to B$ such that $g \circ k = h$:



Thus $\operatorname{Hom}(F, g)(k) = h$. This completes the proof.

Problem N.2 (†). Let A be a finitely generated abelian group, with torsion subgroup $T(A)$. Prove that $\operatorname{Hom}(A, \mathbb{Z}) \cong A/T(A)$ and that $\operatorname{Ext}(A, \mathbb{Z}) \cong T(A)$.

Solution.

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1. Let $T(A) \subset A$ be the torsion subgroup of A. Then $A/T(A)$ is torsion free and finitely generated and thus it is free. Consider the short exact sequence

$$0 \to T(A) \to A \to A/T(A) \to 0. \tag{N.3}$$

Since $A/T(A)$ is free this sequence splits, and hence by Lemma 28.14 the sequence remains exact after applying $\mathrm{Hom}(\square, \mathbb{Z})$.

Now observe that $\mathrm{Hom}(T(A), \mathbb{Z}) = 0$. Indeed, if $a \in T(A)$ then there exists $n \in \mathbb{N}$ such that $na = 0$. If $\phi\colon T(A) \to \mathbb{Z}$ is a homomorphism, then $n\phi(a) = \phi(na) = 0$ which implies $\phi(a) = 0$ as $\mathbb{Z}$ is torsion free. Since a was arbitrary, ϕ is identically zero.

Thus $\mathrm{Hom}(A, \mathbb{Z}) \cong \mathrm{Hom}(A/T(A), \mathbb{Z})$.

But now if F is any finitely generated free abelian group then $\mathrm{Hom}(F, \mathbb{Z}) \cong F$. Indeed, let $\{b_1, \dots, b_r\}$ be a basis of F. Then the homomorphisms $\phi_i \in \mathrm{Hom}(F, \mathbb{Z})$ defined by

$$b_j \mapsto \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

form a basis of $\mathrm{Hom}(F, \mathbb{Z})$ and the map $F \to \mathrm{Hom}(F, \mathbb{Z})\colon a_i \to \phi_{a_i}$ extends to an isomorphism of groups.

We conclude

$$\mathrm{Hom}(A, \mathbb{Z}) \cong \mathrm{Hom}(A/T(A), \mathbb{Z}) \cong A/T(A).$$

2. Since $A/T(A)$ is free, $\mathrm{Ext}(A/T(A), \mathbb{Z}) = 0$. Thus from the long exact sequence from (29.1) (this sequence will be proved in the next problem!) applied to the short exact sequence (N.3) we obtain

$$\mathrm{Ext}(A, \mathbb{Z}) \cong \mathrm{Ext}(T(A), \mathbb{Z}).$$

It thus suffices to show that if T is a finitely generated torsion subgroup then $\mathrm{Ext}(T, \mathbb{Z}) \cong T$. For this, let $a_1, \dots, a_s$ be a minimal generating set of T. Note that

$$n_a := \min\{k \in \mathbb{N} \mid ka = 0\}$$

is finite for every $a \in T$. Let F be the free group generated by $a_1, \dots, a_s$ and let $f\colon F \to F$ denote the homomorphism defined by $a \mapsto n_a a$ on the basis and extended linearly. A short free resolution of T is given by

$$0 \to F \xrightarrow{f} F \to T \to 0.$$

Then by definition,

$$\mathrm{Ext}(T, \mathbb{Z}) = \mathrm{Hom}(F, \mathbb{Z}) / \operatorname{im} \mathrm{Hom}(f, \mathbb{Z}) \cong F / \operatorname{im} f \cong T,$$

where the second last isomorphism follows from the fact that as we have just seen above, $\mathrm{Hom}(F, \mathbb{Z}) \cong F$ for a finitely generated free group F. Thus $\mathrm{Ext}(A, \mathbb{Z}) \cong \mathrm{Ext}(T(A), \mathbb{Z}) \cong T(A)$.

Problem N.3 (†). This problem explores divisible groups.

1. Prove that an abelian group D is divisible if and only if the following property holds: Suppose $g\colon A \to B$ is an injective homomorphism of abelian groups and $h\colon A \to D$ is a homomorphism. Then there exists a homomorphism $f\colon B \to D$ such that $fg = h$:

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{g} & B \\ & & {\scriptstyle h}\big\downarrow & \swarrow{\scriptstyle f} & \\ & & D & & \end{array}$$

(Compare this to Lemma 22.3.)

2. Deduce that if D is divisible then $\operatorname{Hom}(\square, D)$ is an exact functor, and hence $\operatorname{Ext}(A, D) = 0$ for any abelian group D.

3. Prove that any abelian group B is a subgroup of a divisible group D. Deduce that for any abelian group B, there exists a short exact sequence

$$0 \to B \to D \xrightarrow{f} E \to 0$$

where both D and E are divisible. In analogy with short free resolutions (Definition 24.11), let us[1] call such a sequence a *short divisible resolution* of B.

4. Let A and B be abelian groups. Let $0 \to B \to D \xrightarrow{f} E \to 0$ denote a short divisible resolution of B. Prove there is an exact sequence

$$0 \to \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, D) \to \operatorname{Hom}(A, E) \to \operatorname{Ext}(A, B) \to 0,$$

and hence that

$$\operatorname{Ext}(A, B) \cong \operatorname{coker} \operatorname{Hom}(A, f).$$

Hint: Use (29.2) from Theorem 29.3. As remarked there, the proof of (29.2) is essentially identical to the corresponding exact sequences in Theorem 25.6, and hence we may take this as already proved. This is *not* true of the first exact sequence (29.1) from Theorem 29.3. (In fact, you will prove (29.1) in part (8) of this problem below.)

5. Consider a cochain complex $(C^\bullet, d)$ with

$$C^n := \begin{cases} D, & n = 0, \\ E, & n = 1, \\ 0, & n \neq 0, 1. \end{cases}$$

and differential $d\colon C^0 \to C^1$ given by $f\colon D \to E$. Prove that $H^0(C^\bullet) = B$ and $H^1(\operatorname{Hom}(A, C^\bullet)) = \operatorname{Ext}(A, B)$.

[1] As with the terminology "short free resolution", this is *not* standard.

6. Prove the following analogue of Proposition 22.4: if $h\colon B \to B'$ is a homomorphism of abelian groups, and $0 \to B \to D \to E \to 0$ and $0 \to B' \to D' \to E' \to 0$ are short divisible resolutions of B and B' respectively, with corresponding cochain complexes (as in the previous part) $C^\bullet$ and $(C')^\bullet$, then there exists a chain map $m\colon C^\bullet \to (C')^\bullet$ such that $H^0(m) = h$. Prove moreover that any two such chain maps m and m' are chain homotopic.
7. Deduce that $\mathrm{Ext}(\square, A)\colon \mathsf{Ab} \to \mathsf{Ab}$ is a well-defined contravariant functor.
8. If $0 \to A \to A' \to A'' \to 0$ is an exact sequence of abelian groups then for any abelian group B there is an exact sequence

$$0 \to \mathrm{Hom}(A'', B) \to \mathrm{Hom}(A', B) \to \mathrm{Hom}(A, B) \to \mathrm{Ext}(A'', B) \to \mathrm{Ext}(A', B) \to \mathrm{Ext}(A, B) \to 0.$$

Solution.

1. Let us begin with a lemma. Let B be an abelian group and let A and A' be subgroups. Suppose C is another abelian group and $f\colon A \to C$ and $f'\colon A' \to C$ are homomorphisms such that

$$f|_{A\cap A'} = f'|_{A \cap A'}.$$

Then there is a unique extension $f + f'$ of f and f' to the subgroup $A + A'$ of B given by

$$(f + f')(a + a') := f(a) + f'(a').$$

For this consider the short exact sequence

$$0 \to A \cap A' \xrightarrow{h} A \oplus A' \xrightarrow{k} A + A' \to 0$$

where $h(a) := (a, -a)$ and $k(a, a') := a + a'$. The assumption implies that the $\operatorname{im} h$ lies in the kernel of $f \oplus f'\colon A \oplus A' \to B$. Thus $f \oplus f'$ factors to give a well-defined map on the quotient $A + A'$.

Now going to the question, if $a + E$ has infinite order in D/E then the group generated by E and a is just $E \oplus \mathbb{Z} \cdot a$ and we extend by $f \oplus 0$.

First suppose D is divisible. Let $g\colon A \to B$ is an injective homomorphism of abelian groups and $h\colon A \to D$ is a homomorphism. Set $A' := g(A)$. We can view h as a homomorphism $A' \to D$, and our goal is to extend h to homomorphism defined on all of B. We will use Zorn's Lemma. Consider all pairs (f, E) where $A' \subseteq E \subseteq B$ is a subgroup and $f\colon E \to D$ is a homomorphism such that $f|_{A'} = h$. We introduce a partial order on these pairs by inclusions. The union of any totally ordered chain is itself, and hence by Zorn's Lemma there exists a maximal pair (f, E). We claim that $E = B$. Suppose $a \in B \setminus E$.

If $a + E$ has infinite order in B/E then we immediately arrive a contradiction, since then we can extend f to $E \oplus \mathbb{Z} \cdot a$ by setting $f(e + na) := f(e)$ for any $e \in E$ and $n \in \mathbb{Z}$. This contradicts maximality of E.

If however $a + E$ has finite order m then we use divisibility to D to find $d \in D$ such that $md = f(ma)$. We then apply the above claim with $f'\colon \mathbb{Z} \cdot a \to D$

given by $f'(a) = d$. Then the claim allows us to extend f to $E + \mathbb{Z} \cdot a$ (which is bigger than E) via $f + f'$, which again contradicts maximality of E. Thus we conclude that $E = B$.

For the reverse direction, fix $d \in D$ and $n \neq 0$. We consider the inclusion $n\mathbb{Z} \hookrightarrow \mathbb{Z}$ and the homomorphism $n\mathbb{Z} \to D$ given by $nm \mapsto md$:

$$\begin{array}{ccccc} 0 & \longrightarrow & n\mathbb{Z} & \hookrightarrow & \mathbb{Z} \\ & & {\scriptstyle nm\mapsto md}\downarrow & \swarrow {\scriptstyle f} & \\ & & D & & \end{array}$$

Then if $a := f(1)$ one has $na = d$. Thus D is divisible.

2. The first statement follows exactly as in the solution of Problem N.1. The second statement is then immediate from the definition of Ext.

3. First let us remark that any quotient of a divisible group is divisible, as follows readily from the definition. Let F denote the free abelian group with basis the elements of B. Then there is a surjection $F \to B$, and $B \cong F/R$ for some $R \subset F$. Let Q be the rational vector space on with the elements of B. Then Q is divisible, and hence so is $D := Q/R$. Since $B \cong F/R \subset Q/R$, we see that B is a subgroup of D. The quotient group D/B is again divisible, and thus we have a short divisible resolution $0 \to B \to D \to D/B \to 0$.

4. Let A and B be abelian groups. Let $0 \to B \to D \to E \to 0$ be a short divisible resolution of B. Then from (29.2) from Theorem 29.3, there is an exact sequence

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, D) \to \mathrm{Hom}(A, E) \to \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, D) \to \mathrm{Ext}(A, E) \to 0.$$

Since D and E are divisible, this sequence actually reads

$$0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, D) \to \mathrm{Hom}(A, E) \to \mathrm{Ext}(A, B) \to 0.$$

5. This is obvious.

6. Our goal is find homomorphisms g, g' such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{i} & D & \xrightarrow{j} & E & \longrightarrow & 0 \\ & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} & & \\ 0 & \longrightarrow & B' & \xrightarrow[i']{} & D' & \xrightarrow[j']{} & E & \longrightarrow & 0 \end{array} \tag{N.4}$$

The composition $i' \circ h \colon B \to D'$ extends to a map $g \colon D \to D'$ by divisibility of D' (cf. part (1)). The first square then commutes by construction. Next, regarding i, i' as inclusion, there is an induced diagram with exact rows:

$$\begin{array}{ccccc} 0 & \longrightarrow & D/B & \xrightarrow{\bar{j}} & E \\ & & {\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle \bar{g}'} \\ 0 & \longrightarrow & D'/B' & \xrightarrow[\bar{j}']{} & E' \end{array}$$

Then by divisibility of E', the map $\bar{j}' \circ \bar{g}$ extends to a map $g' \colon E \to E'$, and by construction the second square of (N.4) commutes.

Now suppose k, k' are two other choices, i.e. so that the following commutes as well:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{i} & D & \xrightarrow{j} & E & \longrightarrow & 0 \\ & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k'} & & \\ 0 & \longrightarrow & B' & \xrightarrow[i']{} & D' & \xrightarrow[j']{} & E & \longrightarrow & 0 \end{array}$$

The desired chain homotopy is equivalent to the existence of a map $p \colon E' \to D$ such that

$$p \circ j = g - k, \qquad j' \circ p = g' - k'.$$

But this is easy: the map $g - k$ vanishes on B, and hence induces a map $D/B \to D'$. By divisibility this extends to a map $p \colon E \to D'$, giving us the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{i} & D & \xrightarrow{j} & E & \longrightarrow & 0 \\ & & {\scriptstyle 0}\downarrow & & {\scriptstyle g-k}\downarrow & \swarrow{\scriptstyle p} & \downarrow{\scriptstyle g'-k'} & & \\ 0 & \longrightarrow & B' & \xrightarrow[i']{} & D' & \xrightarrow[j']{} & E & \longrightarrow & 0 \end{array}$$

The lower right-hand triangle commutes because j is surjective.

7. This follows from the previous part in exactly the same way that we showed in Proposition 25.3 that $\mathrm{Tor}(\square, A)$ is a well-defined covariant functor.
8. Let $0 \to B \to D \to E \to 0$ denote a short divisible resolution of B, and let $C^\bullet$ denote the corresponding cochain complex. If $0 \to A \to A' \to A'' \to 0$ is exact then by part (1) we have a short exact sequence of cochain complexes:

$$0 \to \mathrm{Hom}(A'', C^\bullet) \to \mathrm{Hom}(A', C^\bullet) \to \mathrm{Hom}(A, C^\bullet) \to 0.$$

The desired long exact sequence is then the long exact sequence in cohomology associated to this short exact sequence of cochain complexes.

PROBLEM N.4. Let X be a topological space and R a commutative ring. Assume that the additive group $H^\star(X; R)$ has no elements of order 2. Prove that if $\langle \alpha \rangle \in H^\star(X; R)$ has odd degree then $\langle \alpha \rangle \smile \langle \alpha \rangle = 0$.

SOLUTION. Let d denote the degree of $\langle \alpha \rangle$ and suppose that d is odd. Then also d^2 is odd and

$$\langle \alpha \rangle \smile \langle \alpha \rangle = (-1)^{d^2} \langle \alpha \rangle \smile \langle \alpha \rangle = -\langle \alpha \rangle \smile \langle \alpha \rangle,$$

and thus $2 \cdot \langle \alpha \rangle \smile \langle \alpha \rangle = 0$. As $H^\star(X; R)$ has no elements of order 2 this implies that $\langle \alpha \rangle \smile \langle \alpha \rangle = 0$.

PROBLEM N.5 ($\star$). Compute the cohomology ring structure of $H^\star(S^n)$ and $H^\star(T^n)$, where T^n is the n-torus $\underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$.

Solution.

1. Theorem 29.5 implies that

$$H^k(S^n) \cong \mathrm{Hom}(H_k(S^n), \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & k \neq 0, n \end{cases}.$$

   Moreover, the isomorphism is

$$\zeta\colon H^n(\mathrm{Hom}(C_\bullet, \mathbb{Z})) \to \mathrm{Hom}(H_n(C_\bullet), \mathbb{Z}))$$

   given by

$$\zeta\langle\gamma\rangle\langle c\rangle := \gamma(c).$$

   Let $\sigma_n^+\colon \Delta^n \to S^n$ be the n-simplex such that $\sigma|_{\Delta^n \setminus \partial\Delta^n}$ is a homeomorphism onto the upper hemisphere B_+^{n+1} and such that $\partial\sigma_n$ is mapped to the equator S^{n-1}. Similarly, let $\sigma_n^-\colon \Delta^n \to S^n$ be the n-simplex such that $\sigma|_{\Delta^n \setminus \partial\Delta^n}$ is a homeomorphism onto the lower hemisphere B_-^{n+1} and such that $\partial\sigma_n$ is mapped to the equator S^{n-1}. We may assume that $\partial\sigma_n^+ = -\partial\sigma_n^-$ such that $\sigma^+ + \sigma^-$ is a cycle and $\langle\sigma^+ + \sigma^-\rangle$ is a generator of $H_n(S^n)$. Notice that for any 0-simplex σ_0 the class $\langle\sigma_0\rangle$ generates $H_0(S^n)$. Moreover, if $\tilde{\sigma_0}$ is another 0-simplex then $\langle\sigma_0\rangle = \langle\tilde{\sigma_0}\rangle$. Let $\alpha\colon C_n(S^n) \to \mathbb{Z}$ be a cochain such that $\langle\alpha\rangle$ generates $H^n(S^n)$. We may assume that $\zeta\langle\alpha\rangle\colon H_n(S^n) \to \mathbb{Z}$ is the isomorphism that sends $\langle\sigma_n^+ + \sigma_n^-\rangle$ to 1. Let $\beta\colon C_0(S^n) \to \mathbb{Z}$ be a cochain such that $\langle\beta\rangle$ generates $H^0(S^n)$. We may assume that $\zeta\langle\beta\rangle\colon H_0(S^n) \to \mathbb{Z}$ is the isomorphism that sends $\langle\sigma_0\rangle$ to 1.
   In order to compute $\langle\alpha \smile \beta\rangle$ it suffices to evaluate $\alpha \smile \beta$ on $\sigma_n^+ + \sigma_n^-$.

$$\begin{aligned}\alpha \smile \beta(\sigma_n^+ + \sigma_n^-) &= \alpha(\sigma_n^+ \circ F_n) \cdot \beta(\sigma_n^+ \circ B_0) + \alpha(\sigma_n^- \circ F_n) \cdot \beta(\sigma_n^- \circ B_0) \\ &= \alpha(\sigma_n^+) \cdot 1 + \alpha(\sigma_n^-) \cdot 1 \\ &= \alpha(\sigma_n^+ + \sigma_n^-),\end{aligned}$$

   where we used the fact that $F_n^n = id_{\Delta^n}$ and that B_0^n has image $e_0 = (0, \ldots, 0, 1)$. Thus $\langle\alpha \smile \beta\rangle = \langle\alpha\rangle$.
   The product $\langle\alpha\rangle \smile \langle\alpha\rangle$ must be zero by degree reasons. Indeed $\langle\alpha\rangle \smile \langle\alpha\rangle \in H^{2n}(S^n) = 0$.
   Finally,

$$\beta \smile \beta(\sigma_0) = \beta(\sigma_0 \circ F_0) \cdot \beta(\sigma_0 \circ B_0) = \beta(\sigma_0) \cdot \beta(\sigma_0) = 1 \cdot 1 = 1.$$

   The last equality follows, since for a 0-simplex σ_i we have $\sigma_i \circ F_0 = \sigma_i = \sigma_i \circ B_0$. We conclude that $\langle\beta\rangle \smile \langle\beta\rangle = \langle\beta\rangle$. The cohomology ring $H^\star(S^n)$ is therefor isomorphic to $\mathbb{Z}[\langle\alpha\rangle]/(\langle\alpha\rangle \smile \langle\alpha\rangle = 0)$

2. The Künneth formula gives $H_k(S^1 \times \ldots \times S^1) \cong \bigoplus_{i_1+\ldots+i_n=k} H_{i_1}(S^1) \otimes \ldots \otimes H_{i_n}(S^1)$. Moreover, since all groups $H_k(S^1 \times \ldots \times S^1)$ are free Theorem 29.5 gives

$$\begin{aligned}H^k(S^1 \times \ldots \times S^1) &\cong \mathrm{Hom}(H_k(S^1 \times \ldots \times S^1), \mathbb{Z}) \\ &\cong \bigoplus_{i_1+\ldots+i_n=k} \mathrm{Hom}(H_{i_1}(S^1) \otimes \ldots \otimes H_{i_n}(S^1), \mathbb{Z}).\end{aligned}$$

Let $\sigma_1 \colon \Delta^1 \to S^1$ be the 1-cycle such that $\sigma_1|_{\Delta^1 \setminus \partial\Delta^1}$ is a homeomorphism onto $S^1 \setminus \{p\}$ and such that $\partial\sigma_1$ is mapped to the point p. We use a superscript to indicate the factor of the product $S^1 \times \ldots \times S^1$. A basis of $H_1(S^1 \times \ldots \times S^1)$ is given by $s_1^j := \langle\sigma_0^1\rangle \otimes \ldots \langle\sigma_0^{j-1}\rangle \otimes \langle\sigma_1^j\rangle \otimes \langle\sigma_0^{j+1}\rangle \otimes \ldots \otimes \langle\sigma_0^n\rangle$ for $j \in \{1, \ldots, n\}$. Similarly, a basis of $H_k(S^1 \times \ldots \times S^1)$ is given by $\{s_k^{i_1,\ldots,i_k} := \langle\sigma_{\delta_1}^1\rangle \otimes \ldots \otimes \langle\sigma_{\delta_n}^n\rangle | i_1 < i_2 < \ldots < i_k\}$,where

$$\delta_j = \begin{cases} 1 & j \in \{i_1, \ldots, i_k\} \\ 0 & \text{otherwise} \end{cases}.$$

Let $\langle\alpha^j\rangle$ be the elements in $H^1(S^1 \times \ldots \times S^1)$ corresponding to the homomorphism

$$H_1(S^1 \times \ldots \times S^1) \to \mathbb{Z}, \qquad s_1^j \mapsto \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then the $\langle\alpha^j\rangle$'s form a basis of $H^1(S^1 \times \ldots \times S^1)$. Moreover we see (similar as in the first part of this exercise) that $\langle\alpha^j\rangle \smile \langle\alpha^i\rangle$ vanishes if $i = j$ and if $i \neq j$ it gives a generator of $H^2(S^1 \times \ldots \times S^1)$ that corresponds to the homomorphism

$$H_2(S^1 \times \ldots \times S^1) \to \mathbb{Z}, \qquad s_k^{i_1,i_2} \mapsto \begin{cases} 1 & (i_1, i_2) = (i, j) \\ 0 & \text{otherwise} \end{cases}.$$

The other products can be calculated in a similar way. The cohomology ring $H^\star(T^n)$ is therefore isomorphic to $\mathbb{Z}[\langle\alpha^1\rangle, \ldots, \langle\alpha^n\rangle]/(\langle\alpha^j\rangle \smile \langle\alpha^j\rangle = 0)$.

Remark: An alternative way to compute $H^\star(T^n)$ is to use induction on n, starting from the fact that we know the case $n = 1$ from the previous part, and $T^n = S^1 \times T^{n-1}$, and by Corollary 32.9, $H^\star(T^n) \cong H^\star(S^1) \otimes H^\star(T^{n-1})$.

Problem Sheet O

This Problem Sheet is based on Lectures 32-35. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem O.1 (†). Let $(C_\bullet, \partial)$ be a non-negative chain complex. Prove that the function

$$\mathrm{twist}\colon C_\bullet \otimes C_\bullet \to C_\bullet \otimes C_\bullet$$

given by

$$\mathrm{twist}(c \otimes c') = (-1)^{nm} c' \otimes c, \qquad c \in C_n,\ c' \in C_m,$$

is a natural chain equivalence.

Problem O.2 (†). Denote by $\mathbb{F}$ either the real numbers $\mathbb{R}$, or the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$. For $m = 1, 2$ or 4, we can view the sphere $S^{m(n+1)-1}$ as a subset of $\mathbb{F}^{n+1}$:

$$S^{m(n+1)-1} = \left\{ (x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{F} \text{ and } \sum_{i=0}^{n} |x_i|^2 = 1 \right\}.$$

Set $\mathbb{F}P^n = \big(\mathbb{F}^{n+1} \setminus \{0\}\big) / \sim$, where

$$(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n) \qquad \Leftrightarrow \qquad \exists\, \mu \in \mathbb{F}\setminus\{0\} \text{ such that } x_i = \mu y_i,\ \forall\, i = 0, \dots, n.$$

Let $p\colon S^{m(n+1)-1} \to \mathbb{F}P^n$ denote the function that sends a tuple (x_i) to its equivalence class. Prove that

$$S^{m-1} \to S^{m(n+1)-1} \xrightarrow{p} \mathbb{F}P^n$$

is a fibre bundle.

Problem O.3 (†). Let $n \geq 1$. Let $\sigma_n\colon \Delta^n \to \mathbb{R}^n$ denote the unique affine map such that[1]

$$\sigma_n(e_0) = -\sum_{i=1}^{n} q_i, \qquad \text{and} \qquad \sigma_n(e_i) = q_i, \qquad i = 1, \dots, n.$$

Prove that σ_n determines a generator $\langle \sigma_n \rangle$ of $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{Z}$.

Problem O.4 (†).

1. Let X be a topological space and let X', X'' be open subsets of X. Let R be a ring. Prove that the cup product induces a relative product

$$H^\star(X, X'; R) \otimes H^\star(X, X''; R) \xrightarrow{\smile} H^\star(X, X' \cup X''; R).$$

[1]See Definition 35.9 for an explanation of this notation.

2. Let X be a topological space and let X' be an open subset of X. Let R be a ring. Let $i\colon (X,\emptyset) \hookrightarrow (X,X')$ denote the inclusion. Suppose $\langle\alpha\rangle \in H^n(X;R)$ and $\langle\beta\rangle \in H^m(X,X';R)$. Prove that

$$\langle\alpha\rangle \smile H^m(i)\langle\beta\rangle = H^{n+m}(i)(\langle\alpha\rangle \smile \langle\beta\rangle),$$

where the left-hand side is the normal cup product in X, and the right-hand side is the relative cup product $H^\star(X;R)\otimes H^\star(X,X';R) \xrightarrow{\smile} H^\star(X,X';R)$ from part (1).

Problem O.5 (†). Let X, Y be topological spaces. Let $X' \subseteq X$ and $Y' \subseteq Y$. Assume that Y' is a closed retract of Y and moreover that there exists a neighbourhood W of Y' in Y such that Y' is a strong deformation retract of W. Let R be a commutative ring, and assume that $H^n(Y,Y';R)$ is a finitely generated free R-module for all $n \geq 0$. Prove that the relative cross product from Definition 33.34 is an isomorphism:

$$H^\star(X,X';R)\otimes_R H^\star(Y,Y';R) \xrightarrow{\times} H^\star(X\times Y,(X'\times Y)\cup(X\times Y');R).$$

Hint: The case $Y' = \emptyset$ was proved in Theorem 33.26.

Problem O.6 ($\star$). In this problem you may use the following result without proof[2]:

Theorem. Let $F \to E \xrightarrow{p} X$ be a fibre bundle and assume that the fibre F is contractible. Then for all $n \geq 0$ and any abelian group A, the map $H^n(p)\colon H^n(X;A) \to H^n(E;A)$ is an isomorphism.

We will prove this result in a few lectures time. Now let $(B^n, S^{n-1}) \to (E,E') \xrightarrow{p} X$ be a fibre bundle pair with X path connected. Let $t \in H^n(E,E';\mathbb{Z}_2)$ denote the Thom class. Let $\jmath\colon (E,\emptyset) \hookrightarrow (E,E')$ denote the inclusion.

1. Prove there is a unique class $\varepsilon \in H^n(X;\mathbb{Z}_2)$ such that

$$H^n(p)(\varepsilon) = H^n(\jmath)(t).$$

One calls ε the **Euler class** of the bundle.

2. Prove there is a long exact sequence called the **Gysin Sequence** given by

$$\cdots \to H^i(X;\mathbb{Z}_2) \xrightarrow{\smile\epsilon} H^{i+n}(X;\mathbb{Z}_2) \xrightarrow{H^{i+n}(p|_{E'})} H^{i+n}(E';\mathbb{Z}_2) \to H^{i+1}(X;\mathbb{Z}_2) \to \dots$$

Hint: Consider the long exact sequence in cohomology associated to the pair (E,E') and try to fit this in to a commutative diagram involving the desired Gysin Sequence. You will need to use part (2) of Problem O.4 in order to show the diagram commutes.

3. Use the Gysin sequence to compute the cohomology ring $H^\star(\mathbb{R}P^n;\mathbb{Z}_2)$ for all $n \geq 1$.

[2]We will sketch the proof of this in Lecture 46, cf. Corollary 46.11. However for vector bundles this result is obvious: the map $s\colon X \to E$ defined by $x \mapsto 0_x$ satisfies $p\circ s = \mathrm{id}_X$ and $s \circ p \simeq \mathrm{id}_E$. Similarly if we start with a sphere bundle $S^n \to E \to X$ and then form the disk bundle $B^{n+1} \to Z \to X$ (following the method in Remark 34.16) that has the given sphere bundle as its boundary sphere bundle, then again the result is obvious, as there is a section $s\colon X \to Z$ that sends x to the point in Z_x corresponding to $E_x \times \{0\}$.

Solutions to Problem Sheet O

Problem O.1 (†). Let $(C_\bullet, \partial)$ be a non-negative chain complex. Prove that the function

$$\text{twist}\colon C_\bullet \otimes C_\bullet \to C_\bullet \otimes C_\bullet$$

given by

$$\text{twist}(c \otimes c') = (-1)^{nm} c' \otimes c, \qquad c \in C_n,\ c' \in C_m,$$

is a natural chain equivalence.

Solution. Let Δ denote the boundary operator on $C_\bullet \otimes C_\bullet$. Then if $c \in C_n$ and $c' \in C_m$ then

$$\begin{aligned}\Delta \circ \tau(c \otimes c') = (-1)^{nm} \Delta(c' \otimes c) \\ &= (-1)^{nm} \left(\partial c' \otimes c + (-1)^m c' \otimes \partial c\right) \\ &= (-1)^{nm} \partial c' \otimes c + (-1)^{nm+m} c' \otimes \partial c.\end{aligned}$$

Going the other way round:

$$\begin{aligned}\tau \circ \Delta(c \otimes c') &= \tau\left(\partial c \otimes c' + (-1)^n c \otimes \partial c'\right) \\ &= (-1)^{(n-1)m} c' \otimes \partial c + (-1)^{n+n(m-1)} \partial c' \otimes c \\ &= (-1)^{nm-m} c' \otimes \partial c + (-1)^{nm} \partial c' \otimes c.\end{aligned}$$

Since $(-1)^m = (-1)^{-m}$, it follows that $\Delta \circ \tau = \tau \circ \Delta$. It is then clear that τ is a natural isomorphism.

Problem O.2 (†). Denote by $\mathbb{F}$ either the real numbers $\mathbb{R}$, or the complex numbers $\mathbb{C}$, or the quaternions $\mathbb{H}$. For $m = 1, 2$ or 4, we can view the sphere $S^{m(n+1)-1}$ as a subset of $\mathbb{F}^{n+1}$:

$$S^{m(n+1)-1} = \left\{ (x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{F} \text{ and } \sum_{i=0}^{n} |x_i|^2 = 1 \right\}.$$

Set $\mathbb{F}P^n = \left(\mathbb{F}^{n+1} \setminus \{0\}\right) / \sim$, where

$$(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n) \qquad \Leftrightarrow \qquad \exists\, \mu \in \mathbb{F}\setminus\{0\} \text{ such that } x_i = \mu y_i,\ \forall\, i = 0, \dots, n.$$

Let $p\colon S^{m(n+1)-1} \to \mathbb{F}P^n$ denote the function that sends a tuple (x_i) to its equivalence class. Prove that

$$S^{m-1} \to S^{m(n+1)-1} \xrightarrow{p} \mathbb{F}P^n$$

is a fibre bundle.

Will J. Merry and Berit Singer. Last modified: Sept 01, 2018.

Solution. We define an open cover $\{V_0, V_1, \dots, V\}$ of $\mathbb{F}P^n$ by setting

$$V_i := \{[x_0, x_1, \dots, x_n] \in \mathbb{F}P^n \mid x_i \neq 0\},$$

where $[\cdots]$ denotes the equivalence class under p. Then define maps $h_i\colon V_i \times S^{m-1} \to p^{-1}(V_i)$ by

$$h_i([x_0, x_1, \dots, x_n], \lambda) := \frac{|x_i|\lambda}{x_i\sqrt{\sum_k |x_k|^2}}(x_0, x_1, \dots, x_n).$$

The map $k_i\colon p^{-1}(V_i) \to V_i \times S^{m-1}$ given by

$$k_i(x_0, x_1, \dots, x_n) := \left([x_0, x_1, \dots, x_n], \frac{x_i}{|x_i|}\right)$$

is an inverse for h_i. Thus the h_i are homeomorphisms.

Problem O.3 (†). Let $n \geq 1$. Let $\sigma_n\colon \Delta^n \to \mathbb{R}^n$ denote the unique affine map such that[1]

$$\sigma_n(e_0) = -\sum_{i=1}^{n} q_i, \qquad \text{and} \qquad \sigma_n(e_i) = q_i, \qquad i = 1, \dots, n.$$

Prove that σ_n determines a generator $\langle\sigma_n\rangle$ of $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{Z}$.

Solution. First note that the image $\Sigma := \operatorname{im}(\sigma) \subset \mathbb{R}^n$ of σ is an n-simplex in $\mathbb{R}^n$ satisfying $0 \in \Sigma \setminus \partial\Sigma$ and (e.g. by excision) we have an isomorphism $H_\bullet(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong H_\bullet(\Sigma, \partial\Sigma)$.
Next, note that we also have an isomorphism

$$H_\bullet(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} H_\bullet(\Sigma, \partial\Sigma) \cong H_\bullet(\mathbb{R}^n, \mathbb{R}^n \setminus 0): \quad \langle\tau\rangle \mapsto \langle\sigma_n \circ \tau\rangle$$

and therefore, in order to show that $\langle\sigma_n\rangle$ is a generator of $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ it suffices to check that the identity map $\ell_n : \Delta^n \to \Delta^n$ (viewed a a singular simplex) induces a generator $\langle\ell_n\rangle$ of $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.

Consider the diagram

$$\begin{array}{ccc} H_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\cong} & \tilde{H}_{n-1}(\partial\Delta^n) \\ \downarrow{\cong} & & \\ H_n(\Delta^n/\partial\Delta^n, *) & & \end{array} \tag{O.1}$$

The horizontal map is the boundary map from the (reduced) LES for the pair $(\Delta^n, \partial\Delta^n)$, which is an isomorphism by looking at the neighbouring terms in the LES. The vertical map is induced by the quotient map $(\Delta^n, \partial\Delta^n) \to (\Delta^n/\partial\Delta^n, *)$ and is an isomorphism since $(\Delta^n, \partial\Delta^n)$ satisfies the hypotheses of Theorem 19.2.

Let $\alpha_n : \Delta^n \to \Delta^n/\partial\Delta^n$ be the quotient map. The image of $\langle\ell_n\rangle$ under the vertical map is $\langle\alpha_n\rangle \in H_n(\Delta^n/\partial\Delta^n, *)$, while its image under the horizontal map is the class $\langle\beta_{n-1}\rangle \in \tilde{H}_{n-1}(\partial\Delta^n)$ with

$$\beta_{n-1} = \partial_n \ell_n = \sum_{i=0}^{n} (-1)^i \varepsilon_i^n \in C_{n-1}(\partial\Delta^n),$$

[1] See Definition 35.9 for an explanation of this notation.

where $\varepsilon_i^n : \Delta^{n-1} \to \partial\Delta^n$ is the i-th face map of the simplex Δ^n. So once we know that $\langle\beta_{n-1}\rangle$ generates $\tilde{H}_{n-1}(\partial\Delta^n)$, we can conclude from (O.1) that $\langle\alpha_n\rangle$ generates $H_n(\Delta^n/\partial\Delta^n, *)$.

It is clear that $\langle\beta_0\rangle$ generates $\tilde{H}_0(\partial\Delta^1)$, so we know that $\langle\alpha_1\rangle$ generates $H_1(\Delta^1/\partial\Delta^1, *)$, which is what the problem asks us to prove for $n = 1$. We now proceed by induction; for the inductive step, consider the map $\phi : \partial\Delta^n \to \Delta^{n-1}/\partial\Delta^{n-1}$ which collapses all except the zero-th face to a point, and the induced map $H_{n-1}(\phi)\colon \tilde{H}_{n-1}(\partial\Delta^n) \to H_{n-1}(\Delta^{n-1}/\partial\Delta^{n-1}, *)$. Observe that $H_{n-1}(\phi)\langle\beta_{n-1}\rangle = \langle\alpha_{n-1}\rangle$; since $\langle\alpha_{n-1}\rangle$ generates by inductive assumption, we conclude that $\langle\beta_{n-1}\rangle$ generates.

Problem O.4 (†).

1. Let X be a topological space and let X', X'' be open subsets of X. Let R be a ring. Prove that the cup product induces a relative product
$$H^\star(X, X'; R) \otimes H^\star(X, X''; R) \xrightarrow{\smile} H^\star(X, X' \cup X''; R).$$
2. Let X be a topological space and let X' be an open subset of X. Let R be a ring. Let $i\colon (X, \emptyset) \hookrightarrow (X, X')$ denote the inclusion. Suppose $\langle\alpha\rangle \in H^m(X; R)$ and $\langle\beta\rangle \in H^n(X, X'; R)$. Prove that
$$\langle\alpha\rangle \smile H^n(i)\langle\beta\rangle = H^{n+m}(i)(\langle\alpha\rangle \smile \langle\beta\rangle),$$
where the left-hand side is the normal cup product in X, and the right-hand side is the relative cup product $H^\star(X; R) \otimes H^\star(X, X'; R) \xrightarrow{\smile} H^\star(X, X'; R)$ from part (1).

Solution.

1. We will assume that X is a cell complex (the general case can be dealt with via cellular approximation). We denote by $C^n(X, X' + X''; R)$ the subgroup of $C^n(X; R)$ consisting of cochains vanishing on chains in X' and on chains in X''. Notice that the usual cup product induces a cup product
$$C^m(X, X'; R) \times C^n(X, X''; R) \to C^{m+n}(X, X' + X''; R).$$
Moreover, we denote by $C_n(X' + X'')$ the subgroup of $C_n(X)$ consisting of sums of chains in X' and chains in X'', and by $C^n(X' + X''; R)$ its dual. Recall from Theorem 14.2 that the inclusions $\iota_\# : C_\bullet(X' + X'') \hookrightarrow C_\bullet(X' \cup X'')$ form a chain homotopy equivalence and hence induces isomorphisms on homology (note that X', X'' are open). Therefore, by dualizing we get that the restrictions $\iota_\# : C^\bullet(X' \cup X''; R) \to C^\bullet(X' + X''; R)$ induce isomorphisms on cohomology. Moreover, we have inclusions $j : C^\bullet(X, X' \cup X''; R) \hookrightarrow C^\bullet(X, X' + X''; R)$ and hence we get the following commutative diagram where the rows are short exact sequences:
$$\begin{array}{ccccccccc} 0 \to & C^\bullet(X, X' \cup X''; R) & \to & C^\bullet(X; R) & \to & C^\bullet(X' \cup X''; R) & \to 0 \\ & \downarrow & & \downarrow \mathrm{id} & & \downarrow \iota_\# & \\ 0 \to & C^\bullet(X, X' + X''; R) & \to & C^\bullet(X; R) & \to & C^\bullet(X' + X''; R) & \to 0 \end{array} \quad (O.2)$$

This leads to the following commutative diagram on the cohomology level, where the rows are long exact sequences:

$$\begin{array}{ccccccccc}
\to H^{n-1}(X' \cup X''; R) & \to & H^n(X, X' \cup X''; R) & \to & H^n(X; R) & \to & H^n(X' \cup X''; R) & \to \\
\cong\big\downarrow & & \big\downarrow & & \cong\big\downarrow \mathrm{id} & & \cong\big\downarrow \iota_\# & \\
\to H^{n-1}(X' + X''; R) & \to & H^n(X, X' + X''; R) & \to & H^n(X; R) & \to & H^n(X' + X''; R) & \to
\end{array} \tag{O.3}$$

By the 5-Lemma we conclude that we have isomorphisms

$$H^n(X, X' \cup X''; R) \cong H^n(X, X' + X''; R),$$

and hence the cup product induces a relative cup product

$$H^\star(X, X'; R) \otimes H^\star(X, X''; R) \xrightarrow{\smile} H^\star(X, X' \cup X''; R).$$

2. This follows from the definition of the relative cup product as in the previous part of this problem. We will treat the general case. As we have seen in the previous part, the usual cup product induces a cup product $C^m(X, X'; R) \otimes C^n(X, X''; R) \xrightarrow{\smile} C^{m+n}(X, X' + X''; R)$ and hence we get the following commutative diagram (the vertical maps are induced by the respective inclusions):

$$\begin{array}{ccccc}
C^m(X, X'; R) \otimes C^n(X, X''; R) & \xrightarrow{\smile} & C^{m+n}(X, X' + X''; R) & \xleftarrow[j]{\simeq} & C^{m+n}(X, X' \cup X''; R) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
C^m(X; R) \otimes C^n(X; R) & \xrightarrow{\quad\smile\quad} & C^{m+n}(X; R) & \xleftarrow{\quad=\quad} & C^{m+n}(X; R).
\end{array}$$

Recall from the previous part that j is a quasi-isomorphism, that is, it induces an isomorphism on cohomology. Therefore we obtain the following commutative diagram on the cohomology level:

$$\begin{array}{ccc}
H^m(X, X'; R) \otimes H^n(X, X''; R) & \xrightarrow{\smile} & H^{m+n}(X, X' \cup X''; R) \\
\big\downarrow & & \big\downarrow \\
H^m(X; R) \otimes H^n(X; R) & \xrightarrow{\quad\smile\quad} & H^{m+n}(X; R),
\end{array}$$

from which the result follows.

Problem O.5 (†). Let X, Y be topological spaces. Let $X' \subseteq X$ and $Y' \subseteq Y$. Assume that Y' is a closed retract of Y and moreover that there exists a neighbourhood W of Y' in Y such that Y' is a strong deformation retract of W. Let R be a commutative ring, and assume that $H^n(Y, Y'; R)$ is a finitely generated free R-module for all $n \geq 0$. Prove that the relative cross product from Definition 33.34 is an isomorphism:

$$H^\star(X, X'; R) \otimes_R H^\star(Y, Y'; R) \xrightarrow{\times} H^\star(X \times Y, (X' \times Y) \cup (X \times Y'); R).$$

Hint: The case $Y' = \emptyset$ was proved in Theorem 33.26.

Solution. We already did the case $Y' = \emptyset$ in the proof of Theorem 33.26. Let $*$ denote the point corresponding to Y' in the quotient space Y/Y'. By Theorem 19.2, $H_n(Y,Y') \cong H_n(Y/Y',*)$ for all $n \geq 0$. Using naturality of the Dual Universal Coefficients Theorem 29.5 and the Five Lemma, one sees that the same holds for cohomology with coefficients in R (or indeed, any abelian group): $H^n(Y,Y';R) \cong H^n(Y/Y',*;R)$ for all $n \geq 0$.

Consider the following commutative diagram:

$$\begin{array}{ccc}
H^\star(X,X';R) \otimes_R H^\star(Y/Y',*;R) & \longrightarrow & H^\star(X,X';R) \otimes_R H^\star(Y,Y';R) \\
\times\big\downarrow & & \big\downarrow\times \\
H^\star(X \times (Y/Y'), (X' \times (Y/Y')) \cup (X \times *)) & \longrightarrow & H^\star(X \times Y, (X' \times Y) \cup (X \times Y'); R)
\end{array}$$

The top horizontal map is an isomorphism due to the discussion above. The lower horizontal map is an isomorphism, since the two quotient spaces are homeomorphic. Thus if we prove the left-hand vertical map is an isomorphism, then the right-hand one is too. This reduces the theorem to dealing with the case where Y' is a point.

So assume Y' consists of a single point p. Since p is a retract of Y, the long exact sequence of the pair (Y,p) splits. Thus by Lemma 25.13, we can tensor this sequence with $H^\bullet(X,X';R)$ and remain exact. Thus the top row of the following commutative diagram is exact, where we omit the R from the notation so it fits on the page:

$$\begin{array}{ccccc}
H^\star(X,X') \otimes_R H^\star(Y,p) & \longrightarrow & H^\star(X,X') \otimes_R H^\star(Y) & \longrightarrow & H^\star(X,X') \otimes_R H^\star(p) \\
 & & & & \big\downarrow\times \\
\times\big\downarrow & & \big\downarrow\times & & H^\star(X \times p, X' \times p) \\
 & & & & \big\uparrow\cong \\
H^\star(X \times Y, (X \times p) \cup (X' \times Y)) & \longrightarrow & H^\star(X \times Y, X' \times Y) & \longrightarrow & H^\star((X \times p) \cup (X' \times Y), X' \times Y)
\end{array}$$

The bottom row is also exact, since $(X \times p, X' \times p)$ is a retract of $(X \times Y, X' \times Y)$. The middle and right-hand cross products are isomorphisms by Theorem 33.26, and thus by the Five Lemma the left-hand cross product is also an isomorphism.

Problem O.6 ($\star$). In this problem you may use the following result without proof[2]:

Theorem. Let $F \to E \xrightarrow{p} X$ be a fibre bundle and assume that the fibre F is contractible. Then for all $n \geq 0$ and any abelian group A, the map $H^n(p)\colon H^n(X;A) \to H^n(E;A)$ is an isomorphism.

We will prove this result in a few lectures time. Now let $(B^n, S^{n-1}) \to (E,E') \xrightarrow{p} X$ be a fibre bundle pair with X path connected. Let $t \in H^n(E,E';\mathbb{Z}_2)$ denote the Thom class. Let $\jmath\colon (E,\emptyset) \hookrightarrow (E,E')$ denote the inclusion.

[2] We will sketch the proof of this in Lecture 46, cf. Corollary 46.11. However for vector bundles this result is obvious: the map $s\colon X \to E$ defined by $x \mapsto 0_x$ satisfies $p \circ s = \mathrm{id}_X$ and $s \circ p \simeq \mathrm{id}_E$. Similarly if we start with a sphere bundle $S^n \to E \to X$ and then form the disk bundle $B^{n+1} \to Z \to X$ (following the method in Remark 34.16) that has the given sphere bundle as its boundary sphere bundle, then again the result is obvious, as there is a section $s\colon X \to Z$ that sends x to the point in Z_x corresponding to $E_x \times \{0\}$.

1. Prove there is a unique class $\varepsilon \in H^n(X;\mathbb{Z}_2)$ such that

$$H^n(p)(\varepsilon) = H^n(\jmath)(t).$$

One calls ε the **Euler class** of the bundle.

2. Prove there is a long exact sequence called the **Gysin Sequence** given by

$$\cdots \to H^i(X;\mathbb{Z}_2) \xrightarrow{\smile \epsilon} H^{i+n}(X;\mathbb{Z}_2) \xrightarrow{H^{i+n}(p|_{E'})} H^{i+n}(E';\mathbb{Z}_2) \to H^{i+1}(X;\mathbb{Z}_2) \to \dots$$

Hint: Consider the long exact sequence in cohomology associated to the pair (E, E') and try to fit this in to a commutative diagram involving the desired Gysin Sequence. You will need to use part (2) of Problem O.4 in order to show the diagram commutes.

3. Use the Gysin sequence to compute the cohomology ring $H^\star(\mathbb{R}P^n;\mathbb{Z}_2)$ for all $n \geq 1$.

Solution.

1. This is immediate as $H^n(p)$ is an isomorphism by the Theorem.

2. Consider the following diagram. The top row is the long exact sequence in cohomology for the pair (E, E'), where as above $\jmath\colon (E, \emptyset) \hookrightarrow (E, E')$ is the inclusion. The bottom row is the Gysin sequence.

$$\begin{array}{ccccccc}
H^{i+n}(E,E';\mathbb{Z}_2) & \xrightarrow{H^{i+n}(\jmath)} & H^{i+n}(E;\mathbb{Z}_2) & \longrightarrow & H^{i+n}(E';\mathbb{Z}_2) & \longrightarrow & H^{i+n+1}(E,E';\mathbb{Z}_2) \\
{\scriptstyle L}\uparrow & & \uparrow{\scriptstyle H^{i+n}(p)} & & \uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle L} \\
H^i(X;\mathbb{Z}_2) & \xrightarrow[\smile \epsilon]{} & H^{i+n}(X;\mathbb{Z}_2) & \xrightarrow[H^{i+n}(p|_{E'})]{} & H^{i+n}(E';\mathbb{Z}_2) & \longrightarrow & H^{i+1}(X;\mathbb{Z}_2)
\end{array}$$

The vertical maps are all isomorphisms. We need only check that the diagram commutes, and the only square for which this is not immediate is the first one. But if $\langle \alpha \rangle \in H^i(X : \mathbb{Z}_2)$ then

$$\begin{aligned}
H^{i+n}(\jmath) \circ L\langle \alpha \rangle &= H^{i+n}(\jmath)(H^i(p)\langle \alpha \rangle \smile t) \\
&\overset{(*)}{=} H^i(p)\langle \alpha \rangle \smile H^n(\jmath)(t) \\
&\overset{(\dagger)}{=} H^i(p)\langle \alpha \rangle \smile H^n(p)\langle \varepsilon \rangle \\
&= H^{i+n}(p)(\langle \alpha \rangle \smile \varepsilon).
\end{aligned}$$

Here $(*)$ used part (2) of Problem O.4 and $(\dagger)$ used the definition of the Euler class.

3. Note that $\mathbb{R}P^n$ is path connected. We have a sphere bundle $S^0 \to S^n \xrightarrow{p} \mathbb{R}P^n$ (c.f. Problem O.2), where p is the usual quotient map. Consider the mapping cylinder

$$Z := \Big((S^n \times I) \sqcup \mathbb{R}P^n \Big) / \sim,$$

where $(y,1) \sim p(y)$. There is a natural map $q : Z \to \mathbb{R}P^n$ with fibre $Z_x = (p_x^{-1} \times I)/(p^{-1}(x) \times 1) \cong B^1$ over $x \in \mathbb{R}P^n$. Identifying $p^{-1}(x) \cong p^{-1}(x) \times 1$ we have that

$$(B^1, S^0) \to (Z, S^n) \xrightarrow{q} \mathbb{R}P^n$$

is a fibre bundle pair with $q|_{S^n} = p$ (see Remark 34.16). Let $\varepsilon \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ be the Euler-class. Now by the Gysin sequence

$$\ldots \to H^i(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\smile \varepsilon} H^{i+1}(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{H^{i+1}(p)} H^{i+1}(S^n; \mathbb{Z}_2) \to H^{i+1}(\mathbb{R}P^n; \mathbb{Z}_2) \to \ldots.$$

Thus as $H^i(S^n; \mathbb{Z}_2) = 0$ for $i \neq 0, n$, the map $\smile \varepsilon$ is an isomorphism for $0 < i < n-1$, and in degree 0 it is surjective and in degree $n-1$ it is also injective. In degree 0 is always injective (this can also bee seen by analyzing the starting part of the LES), so to complete the proof that $\square \mapsto \square \smile \varepsilon$ is an isomorphism in all degrees $0 \leq i \leq n-1$, it remains to show that in degree $n-1$ it is also surjective. For this look at the tail end of the sequence:

$$H^{n-1}(\mathbb{R}P^n; \mathbb{Z}_2) \xrightarrow{\smile \varepsilon} H^n(\mathbb{R}P^n; \mathbb{Z}_2) \to H^n(S^n; \mathbb{Z}_2) \xrightarrow{h} H^n(\mathbb{R}P^n; \mathbb{Z}_2) \to H^{n+1}(\mathbb{R}P^n; \mathbb{Z}_2).$$

The last group is zero as $\mathbb{R}P^n$ has a cell structure with no $(n+1)$-dimensional cells. Thus the map h must be surjective. Since both groups are $\mathbb{Z}_2$, h must be an isomorphism. Thus $\smile \varepsilon$ is surjective and hence an isomorphism.
We conclude that $H^\bullet(\mathbb{R}P^n; \mathbb{Z}) \cong \mathbb{Z}_2[\varepsilon]/\varepsilon^{n+1}$ with ε of degree 1.

Problem Sheet P

This Problem Sheet is based on Lectures 36-39. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem P.1 (†). Let M be an n-dimensional topological manifold. Let $K \subseteq M$ be a compact connected subset. Let R be a commutative ring and assume that M is not orientable along K. Prove that

$$H_n(M, M \setminus K; R) = \{r \in R \mid 2r = 0\} .$$

Problem P.2 (†). Let M be an n-dimensional topological manifold. Let $K \subseteq M$ be a closed connected subset. Prove that:

1. If K is non-compact then the torsion subgroup of $H_{n-1}(M, M \setminus K)$ is zero.
2. If K is compact and M is orientable along K then the torsion subgroup of $H_{n-1}(M, M \setminus K)$ is also zero.
3. If K is compact and M is not orientable along K then the torsion subgroup of $H_{n-1}(M, M \setminus K)$ is isomorphic to $\mathbb{Z}_2$.

Problem P.3 (†). Let X be a topological spaces. Let $\alpha \in C^n(X; R)$ and $c \in C_{n+m}(X; R)$. Prove that

$$\partial(\alpha \frown c) = (-1)^n(\alpha \frown \partial c - d\alpha \frown c).$$

Problem P.4 (†). Let $f\colon (L, K) \to (L', K')$ be a continuous map between compact pairs (L, K) and (L', K') contained in some Euclidean neighbourhood retract X. Let A be an abelian group and let $k \geq 0$. Prove that the induced map $\check{H}^k(f)\colon \check{H}^k(L', K'; A) \to \check{H}^k(L, K; A)$ from Definition 38.10 is well defined. Deduce that the Čech cohomology functor from Theorem 38.11 satisfies the homotopy axiom. *Hint:* Use part (2) of Proposition 38.4.

Problem P.5 (⋆). Let M be a closed connected topological manifold of dimension $n - 1$, and let $f\colon M \to S^n$ be a homeomorphism onto its image. Set $K := f(M)$. Prove **Alexander Duality**:

$$\tilde{H}^k(M) \cong \tilde{H}_{n-k-1}(S^n \setminus K).$$

Deduce that $S^n \setminus K$ has two connected components. *Remark:* This is a far-reaching generalisation of the Jordan-Brouwer Separation Theorem 17.11.

Solutions to Problem Sheet P

Problem P.1 (†). Let M be an n-dimensional topological manifold. Let $K \subseteq M$ be a compact connected subset. Let R be a commutative ring and assume that M is not orientable along K. Prove that

$$H_n(M, M \setminus K; R) = \{r \in R \mid 2r = 0\}.$$

Solution. Let $\mathrm{Ori}(M) \to M$ denote the projection and let

$$G := \{\varphi : \mathrm{Ori}(M) \to \mathrm{Ori}(M) \mid \varphi \text{ is a homeomorphism, } p \circ \varphi = p\}$$

be the automorphism-group of $\mathrm{Ori}(M)$. We will use the following two lemmas.

Claim. Let M be a connected n dimensional topological manifold. Then $\mathrm{Ori}(M)$ is a double covering with automorphism group

$$G \cong \mathbb{Z}_2 = \{1, \psi \mid \psi^2 = 1\}.$$

Let ψ act on R and $\mathrm{Ori}(M)$ by multiplication by -1. Then the associated covering

$$\mathrm{Ori}(M) \times_G R := (\mathrm{Ori}(M) \times R)/G$$

is isomorphic to $\mathcal{O}(M; R) = \bigsqcup_{x \in M} H_n(M, M \setminus x; \mathbb{Z}) \otimes R$.

Proof. Let $\varphi \in G$, then φ is fibre-preserving, since $p \circ \varphi = p$. As the fibres of $\mathrm{Ori}(M)$ are discrete and φ is continuous, φ is determined by its value at a single point. (If φ interchanges the two points in the fibre we have $\varphi^2 = \mathrm{id}$, otherwise $\varphi = \mathrm{id}$.) The map

$$f : \mathrm{Ori}(M) \times R \longrightarrow \bigsqcup_{x \in M} H_n(M, M \setminus x; \mathbb{Z}) \otimes R : \quad (u, r) \mapsto u \otimes r$$

descends to

$$\bar{f} : \mathrm{Ori}(M) \times_G R \longrightarrow \bigsqcup_{x \in M} H_n(M, M \setminus x; \mathbb{Z}) \otimes R : \quad [(u, r)] \mapsto u \otimes r$$

where $[(u, g)]$ denotes the equivalence class of (u, g). Any element $u \otimes r \in H_n(M, M \setminus x; \mathbb{Z}) \otimes R$ can be written as $v \otimes s$, where $u \in H_n(M, M \setminus x; \mathbb{Z})$ is a unit. Moreover $f([(u, r)]) = u \otimes r$ vanishes if and only if $r = 0$. Thus $\bar{f}$ is an isomorphism. ∎

Claim. Sections in $\Gamma(K; R)$ are in one-to-one correspondence with continuous maps

$$\lambda : \mathrm{Ori}(M)|_K \longrightarrow R$$

satisfying $\lambda \circ \varphi = -\lambda$.

Proof. By the previous claim, a section

$$s : K \to \bigsqcup_{x \in K} H_n(M, M \setminus x; R)$$

corresponds to a section $s : K \to (\mathrm{Ori}(M) \times_G R)|_K$. Define

$$\lambda : \mathrm{Ori}(M)|_K \to R : \quad u_x \to r_x,$$

where $r_x \in R$ is given by $s(x) = [(u_x, r_x)]$. Then $\lambda(-u_x) = -r_x$ and thus $\lambda \circ \varphi = -\lambda$. Conversely for a given $\lambda : \mathrm{Ori}(M)|_K \to R$ with $\lambda \circ \varphi = -\lambda$, define

$$s : K \to (\mathrm{Ori}(M) \times_G R)|_K : \quad x \mapsto [(u_x, \lambda(u_x)] = [(-u_x, \lambda(-u_x))].$$

■

By the second claim, a section in $\Gamma(K; R)$ corresponds to a continuous map $\lambda : \mathrm{Ori}(M)|_K \to R$ satisfying $\lambda \circ \varphi = -\lambda$. Since M is *non*-orientable along K, $\mathrm{Ori}(M)|_K$ is connected and therefore λ is constant, i.e. $\lambda \equiv r$ for some $r \in R$. But $\lambda \circ \varphi = -\lambda$ implies $r = -r$, i.e. $2r = 0$.

Since K is compact, Theorem 36.19 yields an isomorphism $\Phi(K) : H_n(M, M \setminus K; R) \to \Gamma_c(K; R) = \Gamma(K; R)$ and we obtain the result.

PROBLEM P.2 (†). Let M be an n-dimensional topological manifold. Let $K \subseteq M$ be a closed connected subset. Prove that:

1. If K is non-compact then the torsion subgroup of $H_{n-1}(M, M \setminus K)$ is zero.
2. If K is compact and M is orientable along K then the torsion subgroup of $H_{n-1}(M, M \setminus K)$ is also zero.
3. If K is compact and M is not orientable along K then the torsion subgroup of $H_{n-1}(M, M \setminus K)$ is isomorphic to $\mathbb{Z}_2$.

SOLUTION. Notice that the torsion subgroup $T(G)$ of an abelian group G is

$$T(G) \cong \bigoplus_{p \text{ prime}} T_p(G)$$

where $T_p(G) := \bigoplus_{p \text{ prime}} \{g \in G \mid \exists n \in \mathbb{Z} : \; p^n g = 0\}$. Moreover, recall that we have $\mathrm{Tor}(G, \mathbb{Z}_q) \cong \{g \in G \mid qg = 0\}$.

1. Let $q \in \mathbb{N}_{>1}$. As K is non-compact, Theorem 37.8 yields

$$\begin{aligned} 0 &\cong H_n(M, M \setminus K; \mathbb{Z}_q) \\ &\cong (H_n(M, M \setminus K; \mathbb{Z}) \otimes \mathbb{Z}_q) \oplus \mathrm{Tor}(H_{n-1}(M, M \setminus K; \mathbb{Z}), \mathbb{Z}_q) \\ &\cong \mathrm{Tor}(H_{n-1}(M, M \setminus K; \mathbb{Z}), \mathbb{Z}_q), \end{aligned}$$

where the second identity follows from the universal coefficient theorem and the third identity follows again from Theorem 37.8. Since q was arbitrary, this shows $T(H_{n-1}(M, M \setminus K; \mathbb{Z})) = 0$.

2. Let $q \in \mathbb{N}_{>1}$. As K is compact and M orientable along K, Theorem 37.8 implies

$$\begin{aligned}\mathbb{Z}_q &\cong H_n(M, M \setminus K; \mathbb{Z}_q)\\ &\cong (H_n(M, M \setminus K; \mathbb{Z}) \otimes \mathbb{Z}_q) \oplus \operatorname{Tor}(H_{n-1}(M, M \setminus K; \mathbb{Z}), \mathbb{Z}_q)\\ &\cong \mathbb{Z}_q \oplus \operatorname{Tor}(H_{n-1}(M, M \setminus K; \mathbb{Z}), \mathbb{Z}_q),\end{aligned}$$

where the second identity follows from the universal coefficient theorem and the third identity follows again from Theorem 37.8. Therefore, $\operatorname{Tor}(H_{n-1}(M, M \setminus K; \mathbb{Z}), \mathbb{Z}_q) = 0$ and as q was arbitrary it follows that the torsion subgroup of $H_{n-1}(M, M \setminus K; \mathbb{Z})$ is zero.

3. We know that $H_{n-1}(M, M \setminus K)$ is finitely generated and hence we may write

$$T(H_{n-1}(M, M \setminus K)) \cong \bigoplus_{i=1}^{m} \mathbb{Z}_{\ell_i}, \quad \ell_i \geq 2, m \geq 0.$$

Let $q \in \mathbb{N}$. Using the result from Problem P.1 twice as well as the universal coefficient theorem, we get:

$$\begin{aligned}\{r \in \mathbb{Z}_q \mid 2r = 0\} &\cong H_n(M, M \setminus K; \mathbb{Z}_q)\\ &\cong (H_n(M, M \setminus K) \otimes \mathbb{Z}_q) \oplus \operatorname{Tor}(H_{n-1}(M, M \setminus K), \mathbb{Z}_q)\\ &\cong \operatorname{Tor}(H_{n-1}(M, M \setminus K), \mathbb{Z}_q).\end{aligned}$$

Note that

$$\{r \in \mathbb{Z}_q \mid 2r = 0\} = \begin{cases} 0, & \text{if } q \text{ is odd} \\ \{0, \frac{q}{2}\}, & \text{if } q \text{ is even} \end{cases}$$

and

$$\operatorname{Tor}(H_{n-1}(M, M \setminus K), \mathbb{Z}_q) \cong \operatorname{Tor}(\bigoplus_{i=1}^{m} \mathbb{Z}_{\ell_i}, \mathbb{Z}_q) \cong \bigoplus_{i=1}^{m} \mathbb{Z}_{\gcd(\ell_i, q)}.$$

We conclude that for even q the following holds:

$$\bigoplus_{i=1}^{m} \mathbb{Z}_{\gcd(\ell_i, q)} \cong \mathbb{Z}_2,$$

which implies that $m = 1$ and $\ell_1 = 2$. This shows the result

$$T(H_{n-1}(M, M \setminus K)) \cong \mathbb{Z}_2.$$

PROBLEM P.3 (†). Let X be a topological spaces. Let $\alpha \in C^n(X; R)$ and $c \in C_{n+m}(X; R)$. Prove that

$$\partial(\alpha \frown c) = (-1)^n(\alpha \frown \partial c - d\alpha \frown c).$$

SOLUTION. We will make use of Lemma 30.12 throughout. In particular

$$\varepsilon_i^{n+m} \circ F_n^{n+m-1} = \begin{cases} F_{n+1}^{n+m} \circ \varepsilon_i^{n+1}, & \text{if } i \leq n \\ F_n^{n+m}, & \text{if } i \geq n+1, \end{cases}$$

and

$$\varepsilon_i^{n+m} \circ B_{m-1}^{n+m-1} = \begin{cases} B_{m-1}^{n+m}, & \text{if } i \leq n \\ B_m^{n+m} \circ \varepsilon_{i-n}^m, & \text{if } i \geq n+1. \end{cases}$$

Using these identities we compute the following three expressions.

$$\begin{aligned} \alpha \frown \partial c &= \sum_{i=0}^{n+m} (-1)^i \alpha \frown (c \circ \varepsilon_i^{n+m}) \\ &= \sum_{i=0}^{n+m} (-1)^i \alpha (c \circ \varepsilon_i^{n+m} \circ F_n^{n+m-1}) \cdot (c \circ \varepsilon_i^{n+m} \circ B_{m-1}^{n+m-1}) \\ &= \sum_{i=0}^{n} (-1)^i \alpha (c \circ F_{n+1}^{m+n} \circ \varepsilon_i^{n+1}) \cdot (c \circ B_{m-1}^{n+m}) \\ &\quad + \sum_{i=n+1}^{n+m} (-1)^i \alpha (c \circ F_n^{n+m}) \cdot (c \circ B_m^{n+m} \circ \varepsilon_{i-n}^m) \end{aligned} \tag{P.1}$$

$$\begin{aligned} d\alpha \frown c &= d\alpha (c \circ F_{n+1}^{n+m}) \cdot (c \circ B_{m-1}^{n+m}) \\ &= \alpha (\partial (c \circ F_{n+1}^{n+m})) \cdot (c \circ B_{m-1}^{n+m}) \\ &= \sum_{i=0}^{n+1} (-1)^i \alpha (c \circ F_{n+1}^{n+m} \circ \varepsilon_i^{n+1}) \cdot (c \circ B_{m-1}^{n+m}) \end{aligned} \tag{P.2}$$

$$\begin{aligned} \partial (\alpha \frown c) &= \alpha (c \circ F_n^{n+m}) \cdot \partial (c \circ B_m^{n+m}) \\ &= \alpha (c \circ F_n^{n+m}) \cdot \left(\sum_{i=0}^{m} (-1)^i (c \circ B_m^{n+m} \circ \varepsilon_i^m) \right) \\ &= \sum_{i=n}^{n+m} (-1)^{i-n} \alpha (c \circ F_n^{n+m}) \cdot (c \circ B_m^{n+m} \circ \varepsilon_{i-n}^m) \end{aligned} \tag{P.3}$$

Equations (P.1), (P.2) and (P.3) finally imply the result:

$$\begin{aligned} (-1)^n (\alpha \frown \partial c - d\alpha \frown c) =& (-1)^n \cdot (-1)^{n+2} \alpha (c \circ \underbrace{F_{n+1}^{n+m} \circ \varepsilon_{n+1}^{n+1}}_{F_n^{n+m}}) \cdot (c \circ \underbrace{B_{m-1}^{n+m}}_{B_m^{n+m} \circ \varepsilon_0^m}) \\ &+ (-1)^n \sum_{i=n+1}^{n+m} (-1)^i \alpha (c \circ F_n^{n+m}) \cdot (c \circ B_m^{n+m} \circ \varepsilon_{i-n}^m) \\ =& \sum_{i=n}^{n+m} (-1)^{i-n} \alpha (c \circ F_n^{n+m}) \cdot (c \circ B_m^{n+m} \circ \varepsilon_{i-n}^m) \\ =& \partial (\alpha \frown c). \end{aligned}$$

■

Problem P.4 (†). Let $f\colon (L,K) \to (L',K')$ be a continuous map between pairs (L,K) and (L',K'). Let A be an abelian group and let $k \geq 0$. Prove that the induced map $\check{H}^k(f)\colon \check{H}^k(L',K';A) \to \check{H}^k(L,K;A)$ from Definition 38.10 is well defined. Deduce that the Čech cohomology functor from Theorem 38.11 satisfies the homotopy axiom. *Hint:* Use part (2) of Proposition 38.4.

Solution. We will prove something a little more general. Let $K \subset L \subset X$ and $K \subset L' \subset X'$ be compact pairs in two Euclidean neighbourhood retracts, and suppose $f, g\colon (L,K) \to (L',K')$ are homotopic maps. By part (1) of Proposition 38.4, we may assume that both f and g can be extended to F and G respectively on some neighbourhood of L.

By part (2) or Proposition 38.4, we can find another smaller neighbourhood W of K in X and a homotopy $H_t\colon W \to X'$ such that $H_0|_W = F|_W$ and $H_1|_W = G|_W$.

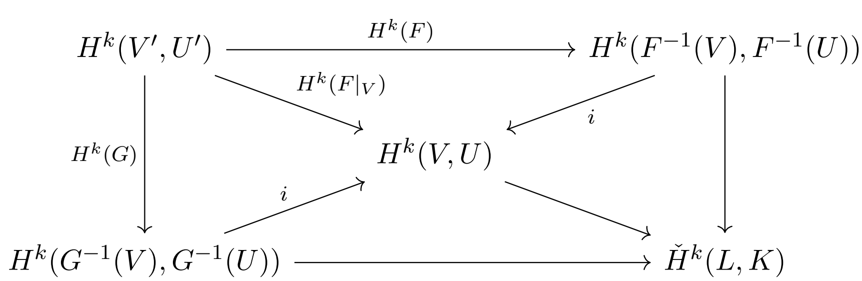
Now fix a pair $U' \subset V' \subset X'$ of open sets in X' such that $K' \subset U'$ and $L' \subset V'$. Set

$$V := \left\{ x \in W \mid H_t(x) \in V',\ \forall\, t \in [0,1] \right\}, \qquad U := \left\{ x \in W \mid H_t(x) \in U',\ \forall\, t \in [0,1] \right\}.$$

Then $U \subset V \subset X$ are open and $K \subset U$, $L \subset V$, and $H_t|_V\colon (V,U) \to (V',U')$ defines a homotopy from $F|_V$ to $G|_V$. In particular, by the homotopy axiom in singular homology,

$$H^k(F|_V) = H^k(G|_V)\colon H^k(V',U') \to H^k(V,U).$$

Now consider the following commutative diagram, where the maps labelled i are all induced from inclusions, and the unlabelled maps are those induced by the universal property of the colimit:



It follows that the map $H^k(V',U') \to \check{H}^k(L,K)$ induced by F (i.e. going clockwise round the outer square) is the same as the map induced by G (i.e. going anticlockwise round the outer square). Since (V',U') was an arbitrary element of $\mathcal{U}_{X'}(L',K')$, it follows that induced maps $\check{H}^k(L,K) \to \check{H}^k(L,K)$ are the same (cf. Definition 38.10.)

Taking $f = g$ shows that the definition of $\check{H}^k(f)$ does not depend on the choice of extension F. Thus $\check{H}^k(f)$ is well-defined, and moreover if $f \simeq g$ as maps $(L,K) \to (L',K')$ then we have just shown that $\check{H}^k(f) = \check{H}^k(g)$. Thus the homotopy axiom holds for Čech cohomology.

Problem P.5 ($\star$). Let M be a closed connected topological manifold of dimension $n-1$, and let $f\colon M \to S^n$ be a homeomorphism onto its image. Set $K := f(M)$.

Prove **Alexander Duality**:

$$\tilde{H}^k(M) \cong \tilde{H}_{n-k-1}(S^n \setminus K).$$

Deduce that $S^n \setminus K$ has two connected components.

Solution. Let y be a point not in K. By the Duality Theorem, using coefficients in $\mathbb{Z}_2$ (remember every manifold is $\mathbb{Z}_2$-orientable) we have

$$\check{H}^k(K, x; \mathbb{Z}_2) \cong H_{n-k}(S^n \setminus x, S^n \setminus K; \mathbb{Z}_2).$$

Since $H_k(S^n \setminus x, y; \mathbb{Z}_2) = 0$ for all $k \geq 0$, from the long exact sequence of the triple we have $H_{n-k}(S^n \setminus x, S^n \setminus K; \mathbb{Z}_2) \cong H_{n-k-1}(S^n \setminus K, y; \mathbb{Z}_2)$. Since both K and x are an Euclidean neighbourhood retracts by Corollary 38.5, we have $\check{H}^k(K, x; \mathbb{Z}_2) \cong H^k(K, x; \mathbb{Z}_2)$, and thus

$$H^k(K, x; \mathbb{Z}_2) \cong H_{n-k-1}(S^n \setminus K, y; \mathbb{Z}_2).$$

Finally using Corollary 12.22, we obtain

$$\tilde{H}^k(M; \mathbb{Z}_2) \cong \tilde{H}_{n-k-1}(S^n \setminus K; \mathbb{Z}_2).$$

Since $H^{n-1}(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$ by Theorem 37.8, we obtain $\tilde{H}_0(S^n \setminus K; \mathbb{Z}_2) = \mathbb{Z}_2$, which implies that $S^n \setminus K$ has two path connected components by Corollary 12.12. Finally, since $S^n \setminus K$ is open (in S^n), it is locally pathwise connected. Thus the path components agree with the connected components, and the result follows.

Problem Sheet Q

This Problem Sheet is based on Lectures 40, 41 and 42. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

Problem Q.1 (†). Let C be a category and $A, B, C \in \mathrm{obj}(\mathsf{C})$. Prove that:

1. If the products $A \sqcap B$ and $B \sqcap A$ exist then they are isomorphic:
$$A \sqcap B \cong B \sqcap A.$$
2. If the coproducts $A \sqcup B$ and $B \sqcup A$ exist then they are isomorphic:
$$A \sqcup B \cong B \sqcup A.$$
3. If the products $A \sqcap B$, $B \sqcap C$, $(A \sqcap B) \sqcap C$ and $A \sqcap (B \sqcap C)$ all exist then:
$$(A \sqcap B) \sqcap C \cong A \sqcap (B \sqcap C).$$
4. If the coproducts $A \sqcup B$, $B \sqcup C$, $(A \sqcup B) \sqcup C$ and $A \sqcup (B \sqcup C)$ all exist then:
$$(A \sqcup B) \sqcup C \cong A \sqcup (B \sqcup C).$$

Problem Q.2 (†). Let R and R' be rings (not necessarily commutative), and let M be an (R, R')-bimodule. Prove that $(\square \otimes_R M, \mathrm{Hom}_{R'}(M, \square))$ forms an adjoint pair. Prove also that $(M \otimes_{R'} \square, \mathrm{Hom}_R(M, \square))$ forms an adjoint pair.

Problem Q.3 (†). Let A be a set, and consider the functor $\mathrm{Hom}(A, \square) \colon \mathsf{Sets} \to \mathsf{Sets}$. Construct a functor $T \colon \mathsf{Sets} \to \mathsf{Sets}$ such that $(T, \mathrm{Hom}(A, \square))$ forms an adjoint pair. Does there exist a functor $S \colon \mathsf{Sets} \to \mathsf{Sets}$ such that $(\mathrm{Hom}(A, \square), S)$ forms an adjoint pair?

Problem Q.4 (†). Let C be a category. Suppose $A_1, A_2, B_1, B_2 \in \mathrm{obj}(\mathsf{C})$ are four objects and $f_i \colon A_i \to B_i$ are morphisms for $i = 1, 2$.

1. Prove that if the products $A_1 \sqcap A_2$ and $B_1 \sqcap B_2$ exist then there is a unique morphism $f_1 \sqcap f_2 \colon A_1 \sqcap A_2 \to B_1 \sqcap B_2$ such that the following diagram commutes for $i = 1, 2$, where the vertical maps are those induced from the limit:
$$\begin{array}{ccc} A_1 \sqcap A_2 & \xrightarrow{f_1 \sqcap f_2} & B_1 \sqcap B_2 \\ \downarrow & & \downarrow \\ A_i & \xrightarrow[f_i]{} & B_i \end{array}$$

2. Prove that if the coproducts $A_1 \sqcup A_2$ and $B_1 \sqcup B_2$ exist then there is a unique morphism $f_1 \sqcup f_2 \colon A_1 \sqcup A_2 \to B_1 \sqcup B_2$ such that the following diagram commutes for $i = 1, 2$, where the vertical maps are those induced from the colimit:

$$\begin{array}{ccc} A_i & \xrightarrow{\quad f_i \quad} & B_i \\ \downarrow & & \downarrow \\ A_1 \sqcup A_2 & \xrightarrow[f_1 \sqcup f_2]{} & B_1 \sqcup B_2 \end{array}$$

Problem Q.5 (†). Suppose C is a category and $B, C_1, C_2, D \in \mathrm{obj}(\mathsf{C})$.

1. Assume the products $B \sqcap B$ and $C_1 \sqcap C_2$ exist. Suppose $f_i \in \mathrm{Hom}(B, C_i)$ for $i = 1, 2$. Prove that
$$(f_1, f_2) = (f_1 \sqcap f_2) \circ \Delta_B,$$
where $\Delta_B \in \mathrm{Hom}(B, B \sqcap B)$ was defined part (1) of Definition 41.5.

2. Assume the coproducts $D \sqcup D$ and $C_1 \sqcup C_2$ exist. Suppose $g_i \in \mathrm{Hom}(C_i, D)$ for $i = 1, 2$. Prove that
$$(g_1, g_2) = \nabla_D \circ (g_1 \sqcup g_2),$$
where $\nabla_D \in \mathrm{Hom}(D \sqcup D, D)$ was defined in part (2) of Definition 41.5.

Problem Q.6 (†). Let (X, x_0) denote a pointed space. Let $\xi \colon X \vee X \hookrightarrow X \times X$ denote the inclusion, where as usual we think of $X \vee X$ as the subspace $(X \times \{x_0\}) \cup (\{x_0\} \times X)$ of $X \times X$.

1. Assume X is an H-group with multiplication m. Prove that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} X \vee X & & \\ {\scriptstyle \xi}\downarrow & \searrow^{\nabla_X} & \\ X \times X & \xrightarrow[m]{} & X \end{array}$$

2. Assume X is an H-cogroup with comultiplication μ. Prove that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & X \vee X \\ & \searrow_{\Delta_X} & \downarrow{\scriptstyle \xi} \\ & & X \times X \end{array}$$

Problem Q.7 (⋆). The aim of this exercise is to deduce that spheres are cogroup objects in hTop_*. Given pointed spaces X and Y, define the **smash product** $X \wedge Y$ to be the pointed space[1] $X \times Y/(X \vee Y)$ (where as usual we think of $X \vee Y$ as a subspace of $X \times Y$.)

[1]**Warning:** The smash product really does depend on the basepoint. For instance, $I \wedge I$ is homeomorphic to $I \times I$ if we choose $0 \in I$ as the basepoint in both factors, but if we choose $1/2$ as the basepoint then it is homeomorphic to the wedge of four copies of $I \times I$.

1. Prove that if X is a locally compact Hausdorff space then ΣX is homeomorphic to $X \wedge S^1$ (as pointed spaces).
2. Given a locally compact and Hausdorff space X, let $X^\infty = X \cup \infty$ denote the one-point compactification of X, which we think of as a pointed space where ∞ is chosen as the basepoint[2]. Prove that if X and Y are locally compact Hausdorff spaces then $X^\infty \wedge Y^\infty$ is homeomorphic to $(X \times Y)^\infty$.
3. Deduce that ΣS^n is homeomorphic to S^{n+1} for all $n \geq 0$, and thus that S^n is a cogroup object in hTop_* for all $n \geq 1$.

[2]Since X is locally compact and Hausdorff, X^∞ is compact and Hausdorff.

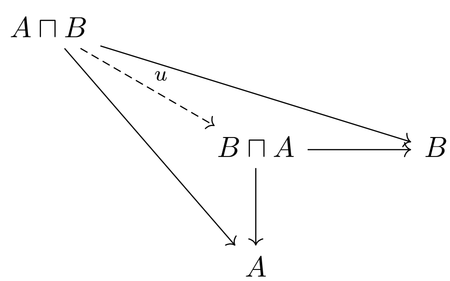
Solutions to Problem Sheet Q

This Problem Sheet is based on Lectures 40, 41 and 42. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

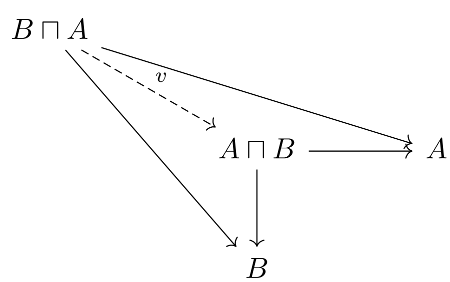
Problem Q.1 (†). Let C be a category and $A, B, C \in \operatorname{obj}(\mathsf{C})$. Prove that:

1. If the products $A \sqcap B$ and $B \sqcap A$ exist then they are isomorphic:
$$A \sqcap B \cong B \sqcap A.$$
2. If the coproducts $A \sqcup B$ and $B \sqcup A$ exist then they are isomorphic:
$$A \sqcap B \cong B \sqcap A.$$
3. If the products $A \sqcap B$, $B \sqcap C$, $(A \sqcap B) \sqcap C$ and $A \sqcap (B \sqcap C)$ all exist then:
$$(A \sqcap B) \sqcap C \cong A \sqcap (B \sqcap C).$$
4. If the coproducts $A \sqcup B$, $B \sqcup C$, $(A \sqcup B) \sqcup C$ and $A \sqcup (B \sqcup C)$ all exist then:
$$(A \sqcup B) \sqcup C \cong A \sqcup (B \sqcup C).$$

Solution. We prove (1) only; the other parts are analogous. The products are limits and hence satisfy the universal property of Definition 40.1. Thus there exist unique morphisms $u : A \sqcap B \to B \sqcap A$ and $v : B \sqcap A \to A \sqcap B$ such that

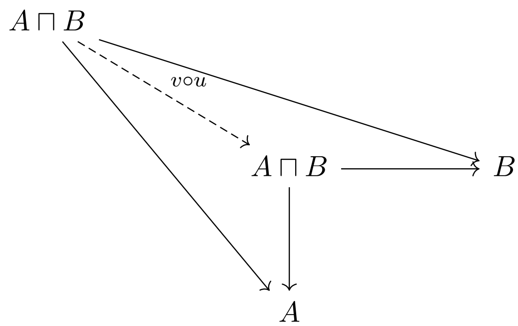


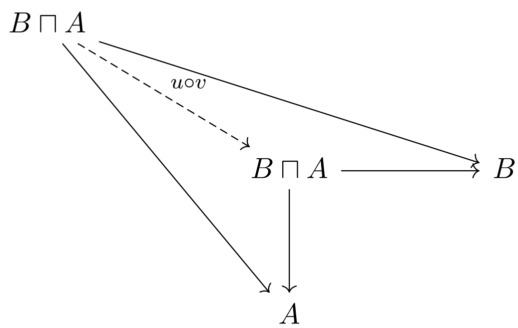
and



Will J. Merry and Berit Singer. Last modified: Sept 01, 2018.

commute. The compositions $v \circ u : A \sqcap B \to A \sqcap B$ and $u \circ v : B \sqcap A \to B \sqcap A$ make the following two diagrams commute:





Since $\mathrm{id}_{A\sqcap B}$ and $\mathrm{id}_{B\sqcap A}$ also make these diagrams commute, it follows from the uniqueness property of the universal property that $\mathrm{id}_{A\sqcap B} = v \circ u$ and $\mathrm{id}_{B\sqcap A} = u \circ v$.

Problem Q.2 (†). Let R and R' be rings (not necessarily commutative), and let M be an (R, R')-bimodule. Prove that $(\square \otimes_R M, \mathrm{Hom}_{R'}(M, \square))$ forms an adjoint pair. Prove also that $(M \otimes_{R'} \square, \mathrm{Hom}_R(M, \square))$ forms an adjoint pair.

Solution. We need to show that there is a natural isomorphism :

$$\Psi : \mathrm{Hom}_{R'}(\square \otimes_R M, \square) \to \mathrm{Hom}_R(\square, \mathrm{Hom}_{R'}(M, \square)).$$

Given a right R-module A and a left R' module B, define

$$\Psi(A, B) : \mathrm{Hom}_{R'}(A \otimes_R M, B) \to \mathrm{Hom}_R(A, \mathrm{Hom}_{R'}(M, B))$$

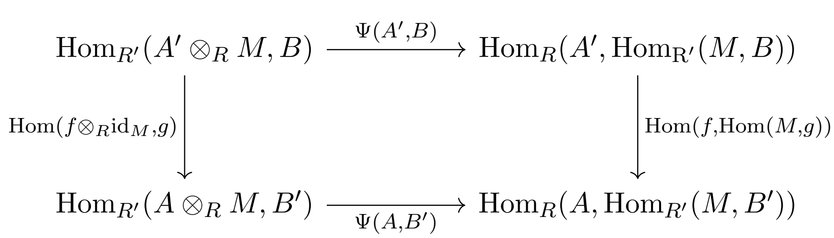
by

$$\alpha \mapsto (\Psi(A, B)(\alpha) : a \mapsto \alpha(a \otimes \cdot)).$$

This is bijective, as the inverse is

$$\beta \mapsto (a \otimes m \mapsto \beta(a)(m)).$$

If $f : A \to A'$ and $g : B \to B'$ are module homomorphism the diagram



(Q.1)

commutes. Indeed, start with a morphism α in the top left-hand corner and go around the diagram in clockwise direction. This yields

$$\alpha \longrightarrow (a' \mapsto \alpha(a' \otimes \cdot)) \longrightarrow (a \mapsto g \circ \alpha(f(a) \otimes \cdot))$$

Going anticlockwise yields

$$\alpha \longrightarrow (a \otimes m \mapsto g \circ \alpha(f(a) \otimes m)) \longrightarrow (a \mapsto g \circ \alpha(f(a) \otimes \cdot)).$$

Hence, the diagram commutes and Ψ is a natural isomorphism, which proves the statement.

The proof that $(M \otimes_{R'} \square, \mathrm{Hom}_R(M, \square))$ forms an adjoint pair is very similar. The natural isomorphism

$$\Theta(A, B) : \mathrm{Hom}_R(M \otimes_{R'} \square, \square) \to \mathrm{Hom}_{'R}(\square, \mathrm{Hom}_R(M, \square)),$$

is defined by

$$\alpha \mapsto (\Theta(A, B)(\alpha) : a \mapsto \alpha(\cdot \otimes a))$$

for every left R'-module A and right R-module B.

Problem Q.3 (†). Let A be a set, and consider the functor $\mathrm{Hom}(A, \square)\colon \mathsf{Sets} \to \mathsf{Sets}$. Construct a functor $T\colon \mathsf{Sets} \to \mathsf{Sets}$ such that $(T, \mathrm{Hom}(A, \square))$ forms an adjoint pair. Does there exist a functor $S\colon \mathsf{Sets} \to \mathsf{Sets}$ such that $(\mathrm{Hom}(A, \square), S)$ forms an adjoint pair?

Solution. Let $T : \mathsf{Sets} \to \mathsf{Sets}$ be the functor defined by

$$T(C) := A \times C$$

and

$$T(f) := Id_A \times f$$

for $C \in \mathsf{Sets}$ and f a morphism in the category of sets. We claim that there exists a natural isomorphism $\Psi : \mathrm{Hom}(A \times \square, \square) \to \mathrm{Hom}(\square, \mathrm{Hom}(A, \square))$. Given $C, D \in \mathsf{Sets}$ define

$$\Psi(C, D) : \mathrm{Hom}(A \times C, D) \to \mathrm{Hom}(C, \mathrm{Hom}(A, D)),$$

by

$$\alpha \mapsto (\Psi(C, D)(\alpha) : c \mapsto \alpha(\cdot, c)).$$

This is a bijection as

$$\beta \mapsto ((a, c) \mapsto \beta(c)(a))$$

is the inverse. It is left to show that $\forall C, C', D, D' \in \mathsf{Sets}$ and $\forall f : C \to C', g : D \to D'$ the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathsf{Sets}}(A \times C', D) & \xrightarrow{\Psi(C', D)} & \mathrm{Hom}_{\mathsf{Sets}}(C', \mathrm{Hom}(A, D)) \\
\Big\downarrow {\scriptstyle \mathrm{Hom}(Id_A \times f, g)} & & \Big\downarrow {\scriptstyle \mathrm{Hom}(f, Hom(A, g))} \\
\mathrm{Hom}_{\mathsf{Sets}}(A \times C, D') & \xrightarrow[\Psi(C, D')]{} & \mathrm{Hom}_{\mathsf{Sets}}(C, \mathrm{Hom}(A, D'))
\end{array} \tag{Q.2}$$

commutes. Recall that

$$\mathrm{Hom}(Id_A \times f, g) : \alpha \mapsto g \circ \alpha \circ Id_A \times f$$

and

$$\mathrm{Hom}(f, \mathrm{Hom}(A, g)) : \beta \mapsto g \circ \beta \circ f.$$

Starting with α on the top left of the diagram and going around clockwise yields the morphism:

$$g \circ \Psi(C', D)(\alpha) \circ f : c \mapsto g \circ \alpha(\cdot, f(c)),$$

where $c \in C$. Counterclockwise yields the morphism:

$$g \circ \alpha \circ Id_A \times f : c \mapsto g \circ \alpha(\cdot, f(c)).$$

Thus the diagram commutes and Ψ is a natural isomorphism.

Let $S = \{a, b\}$ be a set with two elements. Then $S = \{a\} \sqcup \{b\}$ is the coproduct of $\{a\}$ and $\{b\}$ in the category of Sets. But $\mathrm{Hom}(A, S)$ is not the coproduct of $\mathrm{Hom}(A, \{a\})$ and $\mathrm{Hom}(A, \{b\})$. Hence, the functor $\mathrm{Hom}(A, \Box)$ does not preserve colimits. Hence, $\mathrm{Hom}(A, \Box)$ does not have a right-adjoint, since if it did, it would preserve colimits. (This is the content of Theorem 40.22.)

Problem Q.4 (†). Let C be a category. Suppose $A_1, A_2, B_1, B_2 \in \mathrm{obj}(\mathsf{C})$ are four objects and $f_i \colon A_i \to B_i$ are morphisms for $i = 1, 2$.

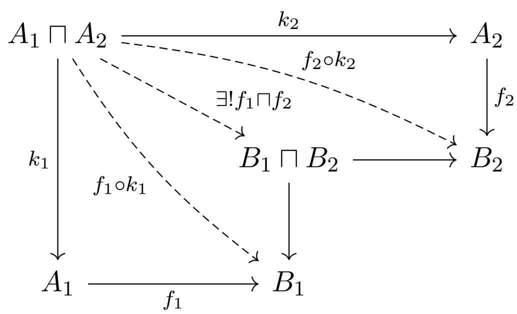
1. Prove that if the products $A_1 \sqcap A_2$ and $B_1 \sqcap B_2$ exist then there is a unique morphism $f_1 \sqcap f_2 \colon A_1 \sqcap A_2 \to B_1 \sqcap B_2$ such that the following diagram commutes for $i = 1, 2$, where the vertical maps are those induced from the colimit:

$$\begin{array}{ccc} A_1 \sqcap A_2 & \xrightarrow{f_1 \sqcap f_2} & B_1 \sqcap B_2 \\ \downarrow & & \downarrow \\ A_i & \xrightarrow[f_i]{} & B_i \end{array}$$

2. Prove that if the coproducts $A_1 \sqcup A_2$ and $B_1 \sqcup B_2$ exist then there is a unique morphism $f_1 \sqcup f_2 \colon A_1 \sqcup A_2 \to B_1 \sqcup B_2$ such that the following diagram commutes for $i = 1, 2$, where the vertical maps are those induced from the limit:

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & B_i \\ \downarrow & & \downarrow \\ A_1 \sqcup A_2 & \xrightarrow[f_1 \sqcup f_2]{} & B_1 \sqcup B_2 \end{array}$$

Solution. 1. Using Definition 40.1 we obtain the following commutative diagram, where k_1, k_2 are the induced maps of the limit $A_1 \sqcup A_2$:



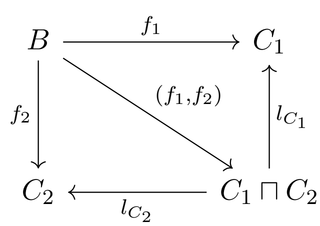
This immediately yields the desired commutative diagrams.

PROBLEM Q.5 (†). Suppose C is a category and $B, C_1, C_2, D \in \mathrm{obj}(\mathsf{C})$.

1. Assume the products $B \sqcap B$ and $C_1 \sqcap C_2$ exist. Suppose $f_i \in \mathrm{Hom}(B, C_i)$ for $i = 1, 2$. Prove that
$$(f_1, f_2) = (f_1 \sqcap f_2) \circ \Delta_C.$$
2. Assume the coproducts $D \sqcup D$ and $C_1 \sqcup C_2$ exist. Suppose $g_i \in \mathrm{Hom}(C_i, D)$ for $i = 1, 2$. Prove that
$$(g_1, g_2) = \nabla_D \circ (g_1 \sqcup g_2).$$

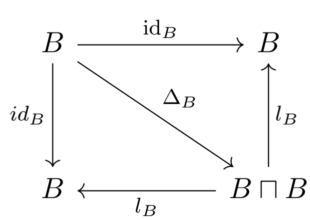
SOLUTION.

1. By Proposition 41.1 $(f_1, f_2) \in \mathrm{Hom}(B, C_1 \sqcap C_2)$ is the unique morphism such that the following diagram commutes:



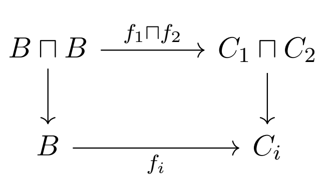
(Q.3)

Similarly, $\Delta_B \in \mathrm{Hom}(B, B \sqcap B)$ is the unique morphism such that



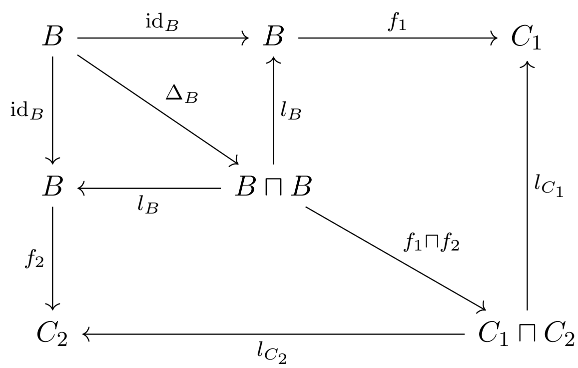
(Q.4)

commutes. Recall also from Problem Q.4, that $f_1 \sqcap f_2 \in \mathrm{Hom}(B \sqcap B, C_1 \sqcap C_2)$ is the unique morphism such that the diagrams



(Q.5)

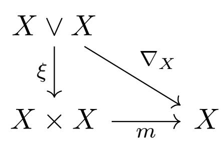
commute for $i = 1, 2$. Combining the commutative diagrams (Q.4) and (Q.5) we obtain the following commutative diagram:



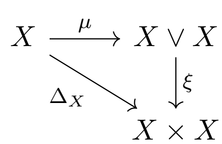
By the uniqueness of the morphism (f_1, f_2) in (Q.3) we conclude that $(f_1, f_2) = f_1 \sqcap f_2 \circ \Delta_B$.

Problem Q.6 (†). Let (X, x_0) denote a pointed space. Let $\xi \colon X \vee X \hookrightarrow X \times X$ denote the inclusion, where as usual we think of $X \vee X$ as the subspace $(X \times \{x_0\}) \cup (\{x_0\} \times X)$ of $X \times X$.

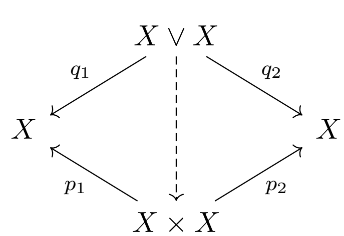
1. Assume X is an H-group with multiplication m. Prove that the following diagram commutes up to homotopy:



2. Assume X is an H-cogroup with comultiplication μ. Prove that the following diagram commutes up to homotopy:



Proof. Both parts follow immediately from the definition of an H-(co)group, once one realises that the inclusion $\xi \colon X \vee X \hookrightarrow X \times X$ agree with the maps $(j_1, j_2) \colon X \vee X \to X \times X$ and $(q_1, q_2) \colon X \vee X \to X \times X$ (the notation is using Proposition 41.1!). For example, both ξ and (q_1, q_2) solve the dashed arrow in the following diagram:



■

PROBLEM Q.7 ($\star$). The aim of this exercise is to deduce that spheres are cogroup objects in hTop_*. Given pointed spaces X and Y, define the **smash product** $X \wedge Y$ to be the pointed space[1] $X \times Y/(X \vee Y)$ (where as usual we think of $X \vee Y$ as a subspace of $X \times Y$.)

1. Prove that if X is a locally compact Hausdorff space then ΣX is homeomorphic to $X \wedge S^1$ (as pointed spaces).
2. Given a locally compact Hausdorff space X, let $X^\infty = X \cup \infty$ denote the one-point compactification of X, which we think of as a pointed space where ∞ is chosen as the basepoint. Prove that if X and Y are locally compact Hausdorff spaces then $X^\infty \wedge Y^\infty$ is homeomorphic to $(X \times Y)^\infty$.
3. Deduce that ΣS^n is homeomorphic to S^{n+1} for all $n \geq 0$, and thus that S^n is a cogroup object in hTop_* for all $n \geq 1$.

SOLUTION.

1. Since X is locally compact and Hausdorff the map $\mathrm{id}_X \times \exp\colon X \times I \to X \times S^1$ is a quotient map. Thus the composition $p\colon X \times I \to X \times S^1 \to X \wedge S^1$ is a quotient map. Since $(X \times I)\big/\ker p = \Sigma X$, the claim follows from the definition of the quotient topology.
2. For Z a compact Hausdorff space and $Z' \subset Z$ a closed subspace, one has $Z/Z' \cong (Z \setminus Z')^\infty$ (as pointed spaces) by properties of the one-point compactification. Since $X^\infty \wedge Y^\infty = (X^\infty \times Y^\infty)/(X^\infty \vee Y^\infty)$ is the quotient of a compact Hausdorff space by a closed subspace, it follows that $X^\infty \wedge Y^\infty$ is the one-point compactification of $(X^\infty \times Y^\infty) \setminus (X^\infty \vee Y^\infty)$. But
$$(X^\infty \times Y^\infty) = (X \times Y) \cup (\{\infty\} \times Y^\infty) \cup (X^\infty \times \{\infty\}),$$
and
$$(X^\infty \vee Y^\infty) = (\{\infty\} \times Y^\infty) \cup (X^\infty \times \{\infty\}),$$
so their difference is $X \times Y$ as required.
3. For $n = 0$ this is obvious. For $n \geq 1$, since the one-point compactification of $\mathbb{R}^n$ is S^n, one has
$$\Sigma S^n \cong S^n \wedge S^1 \cong (\mathbb{R}^n)^\infty \wedge \mathbb{R}^\infty \cong (\mathbb{R}^n \times \mathbb{R})^\infty = (\mathbb{R}^{n+1})^\infty \cong S^{n+1}.$$

[1] **Warning:** The smash product really does depend on the basepoint. For instance, $I \wedge I$ is homeomorphic to $I \times I$ if we choose $0 \in I$ as the basepoint in both factors, but if we choose $1/2$ as the basepoint then it is homeomorphic to the wedge of four copies of $I \times I$.

Problem Sheet R

This Problem Sheet is based on Lectures 43, 44 and 45. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

Problem R.1 (†). Let A be a set, and assume A is equipped with two binary operations $*$ and $\circ$ such that:

1. $*$ and $\circ$ have a common two-sided unit,
2. $*$ and $\circ$ are mutually distributive, that is,

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d), \qquad \forall\, a, b, c, d \in A.$$

Prove that $*$ and $\circ$ coincide and that each is commutative and associative.

Problem R.2.

1. Let $f\colon (X, x_0) \to (Y, y_0)$ be a pointed continuous map. Define a space Ff by:

   $$Ff := \left\{ (x, w) \in X \times Y^I \mid w(1) = f(x) \right\}.$$

   Let $p\colon Ff \to Y$ be the map $(x, w) \mapsto w(0)$. Prove that p is a fibration with fibre the mapping fibre Mf of f.
2. Assume X is path connected. Let $g\colon X \to Y$ be any continuous (not necessarily pointed) map. Prove that one can write $g = p \circ h$ where p is a fibration and h is a homotopy equivalence.

Problem R.3. Consider the Hopf fibration $S^1 \to S^3 \to S^2$ from Proposition 33.7. Use the long exact sequence of homotopy groups to prove that[1] $\pi_2(S^2) = \mathbb{Z}$ and that $\pi_n(S^3) \cong \pi_n(S^2)$ for all $n \geq 3$. Deduce that $\pi_3(S^2) \neq 0$. Try to visualise this.

Problem R.4. Let $p\colon E \to X$ be a weak fibration with fibre F. Prove that $\pi_2(E, F)$ is abelian. If F is simply connected, prove that $\pi_1(p)\colon \pi_1(E) \to \pi_1(X)$ is an isomorphism.

Problem R.5 (⋆). Let $X = S^2 \vee S^4$ and let $Y = \mathbb{C}P^2$. Prove that X and Y have the same homology groups. Prove however that $\pi_4(X) \ncong \pi_4(Y)$ and hence that X and Y are not homotopy equivalent.

[1]No, you may *not* simply quote the Hurewicz Theorem 46.1!

Solutions to Problem Sheet R

This Problem Sheet is based on Lectures 43-46. A (†) means I will use the problem in lectures; a (⋆) means I think the problem is challenging.

PROBLEM R.1 (†). Let A be a set, and assume A is equipped with two binary operations $*$ and $\circ$ such that:

1. $*$ and $\circ$ have a common two-sided unit,
2. $*$ and $\circ$ are mutually distributive, that is,

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d), \qquad \forall\, a, b, c, d \in A.$$

Prove that $*$ and $\circ$ coincide and that each is commutative and associative.

SOLUTION. Let e denote the unit. Then

$$a * b = (a \circ e) * (e \circ b) = (a * e) \circ (e * b) = a \circ b.$$

Thus $*$ and $\circ$ coincide. Next,

$$a * b = (e \circ a) * (b \circ e) = (e * b) \circ (a * e) = b \circ a = b * a,$$

which proves commutativity. Finally

$$a * (b * c) = (a \circ e) * (b \circ c) = (a * b) \circ (e * c) = (a * b) * c,$$

which proves associativity.

PROBLEM R.2 (†).

1. Let $f\colon (X, x_0) \to (Y, y_0)$ be a pointed continuous map. Define a space Ff by:

$$Ff := \left\{ (x, w) \in X \times Y^I \mid w(1) = f(x) \right\}.$$

   Let $p\colon Ff \to Y$ be the map $(x, w) \mapsto w(0)$. Prove that p is a fibration with fibre the mapping fibre Mf of f.

2. Assume X is path connected. Let $g\colon X \to Y$ be any continuous (not necessarily pointed) map. Prove that one can write $g = p \circ h$ where p is a fibration and h is a homotopy equivalence.

SOLUTION.

1. Let V be a topological space, $h_t \colon V \to Y$ be a homotopy and $g_0 \colon V \to Ff$ a map such that $p \circ g_0 = h_0$. We want to show that there exists a homotopy $g_t \colon V \to Ff$ such that $p \circ g_t = h_t$, $\forall t$. Let $\tau_{t,v} \in Y^I$ be the path $s \mapsto \tau_{t,v}(s) := h_{t(1-s)}(v)$. Writing
$$g_0 \colon V \to Ff; \quad v \mapsto (x(v), \sigma_v = (t \mapsto \sigma_v(t))$$
we define the path $I \to Y$
$$w_{t,v} := \sigma_v * \tau_{t,v}(r) = \begin{cases} \tau_{t,v}(2r), \ 0 \le r \le \frac{1}{2} \\ \sigma_v(2r-1), \ \frac{1}{2} \le r \le 1. \end{cases}$$
This is well-defined since $\tau_{t,v}(1) = h_0(v) = p \circ g_0(v) = \sigma_v(0)$. Let
$$g_t \colon V \to Ff; \quad v \mapsto (x(v), w_{t,v}).$$
Then we have $w_{t,v}(1) = \sigma_v(1) = f(x(v))$ and $p \circ g_t(v) = w_{t,v}(0) = h_t(v)$.
2. Define
$$h \colon X \to Fg; \quad x \mapsto (x, c_{g(x)})$$
where $c_{g(x)}$ is the constant path at $g(x)$ and let
$$q \colon Fg \to X; \quad (x, w) \mapsto x$$
be the projection. Then we have that $q \circ h = \mathrm{id}_X$ and moreover
$$H \colon Fg \times I \to Fg; \quad ((x, w), t) \mapsto (x, w|_{[t,1]})$$
is a homotopy between $h \circ q$ and id_{Fg}. Thus h is a homotopy equivalence and $p \circ h(x) = c_{g(x)}(1) = g(x)$.

PROBLEM R.3. Consider the Hopf fibration $S^1 \to S^3 \to S^2$ from Proposition 33.7. Use the long exact sequence of homotopy groups to prove that[1] $\pi_2(S^2) = \mathbb{Z}$ and that $\pi_n(S^3) \cong \pi_n(S^2)$ for all $n \ge 3$. Deduce that $\pi_3(S^2) \ne 0$. Try to visualise this.

SOLUTION. By Theorem 45.5 and Theorem 43.18 we have a long exact sequence as follows:

$$\ldots \to \underbrace{\pi_3(S^1)}_{=0} \to \pi_3(S^3) \to \pi_3(S^2) \to \underbrace{\pi_2(S^1)}_{=0} \to \underbrace{\pi_2(S^3)}_{=0} \to \pi_2(S^2) \to \underbrace{\pi_1(S^1)}_{\cong \mathbb{Z}} \to \underbrace{\pi_1(S^3)}_{=0} \to \ldots$$

Therefore (again by Theorem 43.18) we get (by induction) that $\pi_n(S^3) \cong \pi_n(S^2)$ for all $n \ge 3$ and $\pi_2(S^2) \cong \mathbb{Z}$. Since $\pi_3(S^3) \ne 0$ by Theorem 43.18, we also have $\pi_3(S^2) \ne 0$.

PROBLEM R.4. Let $p \colon E \to X$ be a weak fibration with fibre F. Prove that $\pi_2(E, F)$ is abelian. If F is simply connected, prove that $\pi_1(p) \colon \pi_1(E) \to \pi_1(X)$ is an isomorphism.

[1]No, you may *not* simply quote the Hurewicz Theorem 46.1!

SOLUTION. By Theorem 45.16 (Serre's Theorem) we have that $\pi_2(p')\colon \pi_2(E,F) \to \pi_2(X,x_0)$ is an isomorphism. By Corollary 43.5 $\pi_2(X,x_0)$ is abelian, thus $\pi_2(E,F)$ is abelian as well. If F is simply connected, then $\pi_0(F) = \pi_1(F) = 0$ and the long exact sequence of Corollary 45.18 takes the form

$$\ldots \to \underbrace{\pi_1(F)}_{=0} \to \pi_1(E) \overset{\pi_1(p)}{\to} \pi_1(X) \to \underbrace{\pi_0(F)}_{=0} \to \ldots$$

which proves that $\pi_1(p)\colon \pi_1(E) \to \pi_1(X)$ is an isomorphism.

PROBLEM R.5 ($\star$). Let $X = S^2 \vee S^4$ and let $Y = \mathbb{C}P^2$. Prove that X and Y have the same homology groups. Prove however that $\pi_4(X) \ncong \pi_4(Y)$ and hence that X and Y are not homotopy equivalent.

SOLUTION. Let $X \subset Y$ be a subspace pf the topological space Y. Recall that X is a *retract* of Y is and only if $\exists$ a continuous map $r\colon Y \to X$ such that $r|_X = f_X$ (r is called *retraction*). S^4 is a retract of $S^2 \vee S^4$. Indeed, define

$$r\colon S^2 \times \{p_2\} \sqcup \{p_1\} \times S^4 \to \{p_1\} \times S^4; \quad (x,y) \mapsto \begin{cases} 0, & \text{if } x \neq p_1 \\ (x,y), & \text{if } x = p_1. \end{cases}$$

In particular

$$\begin{array}{ccccc} & & S^2 \vee S^4 & & \\ & \overset{i}{\nearrow} & & \overset{r}{\searrow} & \\ S^4 & & \xrightarrow[\text{id}]{\hspace{6em}} & & S^4 \end{array}$$

commutes and thus we get the following commutative diagram

$$\begin{array}{ccccc} & & \pi_4(S^2 \vee S^4) & & \\ & \overset{\pi_4(i)}{\nearrow} & & \overset{\pi_4(r)}{\searrow} & \\ \pi_4(S^4) & & \xrightarrow[\text{id}]{\hspace{6em}} & & \pi_4(S^4). \end{array}$$

By Theorem 43.18 we know that $\pi_4(S^4) \neq 0$ and hence $\pi_4(S^2 \vee S^4)$ cannot vanish.

Now consider the fibre bundle $S^1 \to S^{2n+1} \overset{p}{\to} \mathbb{C}P^n \cong S^{2n+1}/\sim$ (here $n = 2$). By Corollary 45.14 $p\colon S^{2n+1} \to \mathbb{C}P^n$ is a weak fibration and using the long exact sequence for weak fibrations (Corollary 45.18) we get

$$\ldots \to \pi_4(S^1) \to \pi_4(S^5) \to \pi_4(\mathbb{C}P^2) \to \pi_3(S^1) \to \ldots$$

By Theorem 43.18 we have that $\pi_4(S^1) = \pi_3(S^1) = 0$ and $\pi_4(S^5) = 0$, therefore $\pi_4(\mathbb{C}P^2) = 0$. Hence $\pi_4(\mathbb{C}P^2) \neq \pi_4(S^2 \vee S^4)$ and we conclude that $\mathbb{C}P^2$ is not homotopy equivalent to $S^2 \vee S^4$. (Note that $H_i(\mathbb{C}P^2) \cong H_i(S^2 \vee S^4)$, $\forall i$.)